Confluence of Graph Rewriting with Interfaces

Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Pawel Sobocinski and Fabio Zanasi

IFIP WG 1.3
9-12/01/2017, Binz

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To appear at ESOP 2017

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Plan of the Talk

1) Confluence for Term Rewriting
2) Confluence for DPO Rewriting
3) Confluence for DPO Rewriting with Interfaces
4) Confluence for PROP Rewriting

Plan of the Talk

1) Confluence for Term Rewriting

2) Confluence for DPO Rewriting

3) Confluence for DPO Rewriting with Interfaces

4) Confluence for PROP Rewriting

Confluence and Termination

Confluence and Termination

Rewriting relation $\Rightarrow$

Confluence and Termination

Rewriting relation $\Longrightarrow$

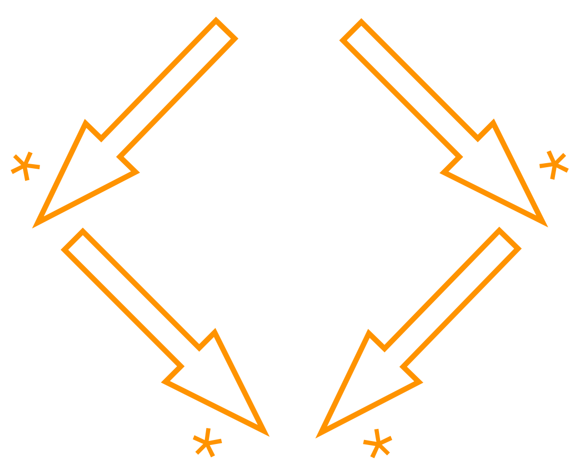
Its reflexive and transitive closure $\overset{*}{\Longrightarrow}$

Confluence and Termination

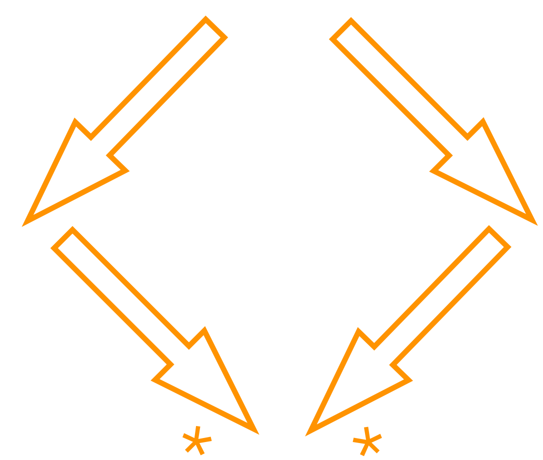
Rewriting relation $\Rightarrow$

Its reflexive and transitive closure $\overset{*}{\Rightarrow}$

Confluence



Local Confluence

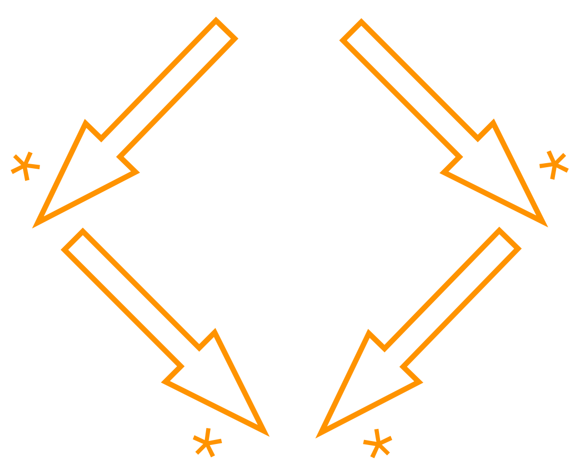


Confluence and Termination

Rewriting relation $\Rightarrow$

Its reflexive and transitive closure $\overset{*}{\Rightarrow}$

Confluence



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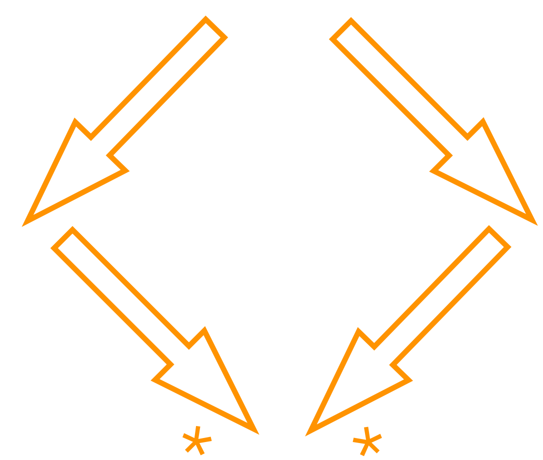

Newman's Lemma

In a terminating rewriting system,
local confluence implies confluence

Confluence for Term Rewriting

Knuth-Bendix 1970

$\boxed{\text{Confluence of a terminating rewriting system is decidable}}$

Confluence for Term Rewriting

Knuth-Bendix 1970

Confluence of a terminating rewriting system is decidable

If all the critical pairs are joinable, then the system is locally confluent

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The converse implication is trivial

Confluence for Term Rewriting

Knuth-Bendix 1970

Confluence of a terminating rewriting system is decidable

If all the critical pairs are joinable,
then the system is locally confluent

The converse implication is trivial

In a terminating system,
checking joinability of critical pairs is easy

Example

Two unary symbols f,g:1-->1

$f(f(x)) \xRightarrow{(1)} f(x)$ $\quad\quad$ $f(g(x)) \xRightarrow{(2)} g(x)$

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Critical pair

$$\begin{tikzcd}
 & f(f(g(x))) \arrow[dl, Rightarrow, "(1)"'] \arrow[dr, Rightarrow, "(2)"] & \\
f(g(x)) \arrow[r, Rightarrow, "*"] & {} & f(g(x)) \arrow[l, Rightarrow, "*"']
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For all terms, rewriting is confluent

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For all terms, rewriting is confluent

Every term has a unique normal form

Ground Confluence

Two unary symbols f,g:1-->1 and one constant c:0-->1

$f(g(f(x))) \Rightarrow x \qquad f(c) \Rightarrow c \qquad g(c) \Rightarrow c$

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So, the system is not confluent

But it is ground confluent:

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But it is ground confluent:

it is confluent for all the ground terms

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Kapur et al. 1990

For a terminating Term Rewriting System, ground confluence is not decidable

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The trivial implication is not trivial anymore

Plan of the Talk

1) Confluence for Term Rewriting

2) Confluence for DPO Rewriting

3) Confluence for DPO Rewriting with Interfaces

4) Confluence for PROP Rewriting

Graph Rewriting (the DPO approach)

We work in an arbitrary adhesive category,
typically the category of hypergraphs and their morphisms

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$$G \Rightarrow H$$

Confluence for DPO rewriting

Plump 1993

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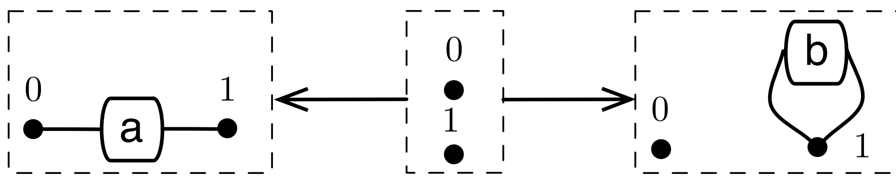
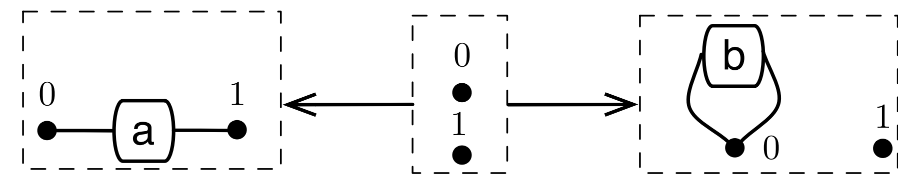
Critical Pairs

$$
\begin{tikzcd}
R_1 \arrow[d] & K_1 \arrow[l] \arrow[r] \arrow[d] & L_1 \arrow[dr, "f_1"] & & L_2 \arrow[dl, "f_2"'] & K_2 \arrow[l] \arrow[r] \arrow[d] & R_2 \arrow[d] \\
H_1 & C_1 \arrow[l] \arrow[rr] & & G & & C_2 \arrow[ll] \arrow[r] & H_2
\end{tikzcd}
$$

f_1 and f_2 are jointly epi

Counter-example





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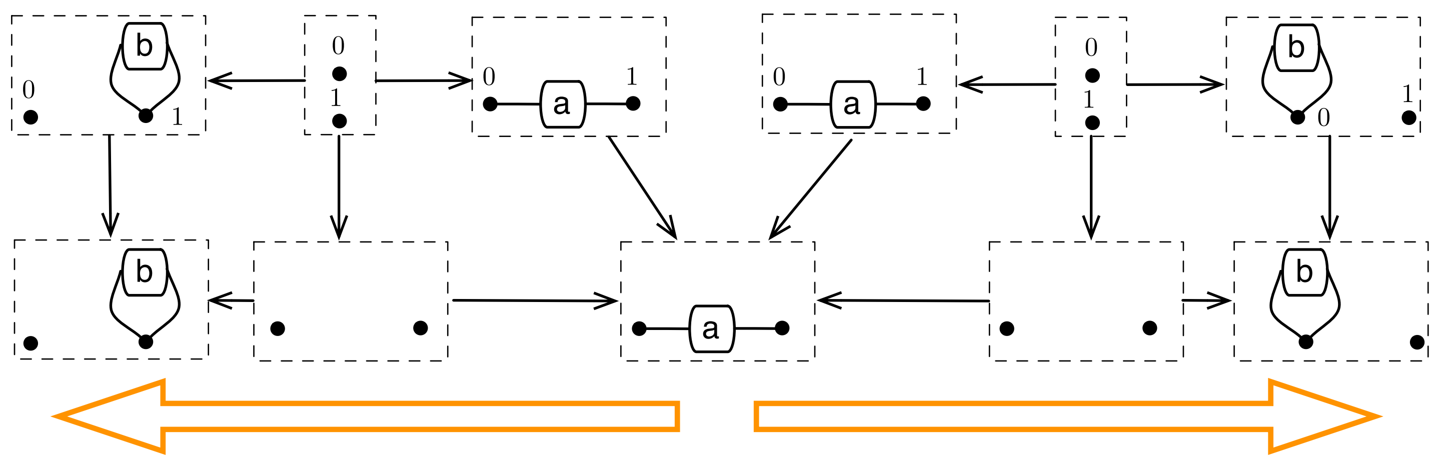

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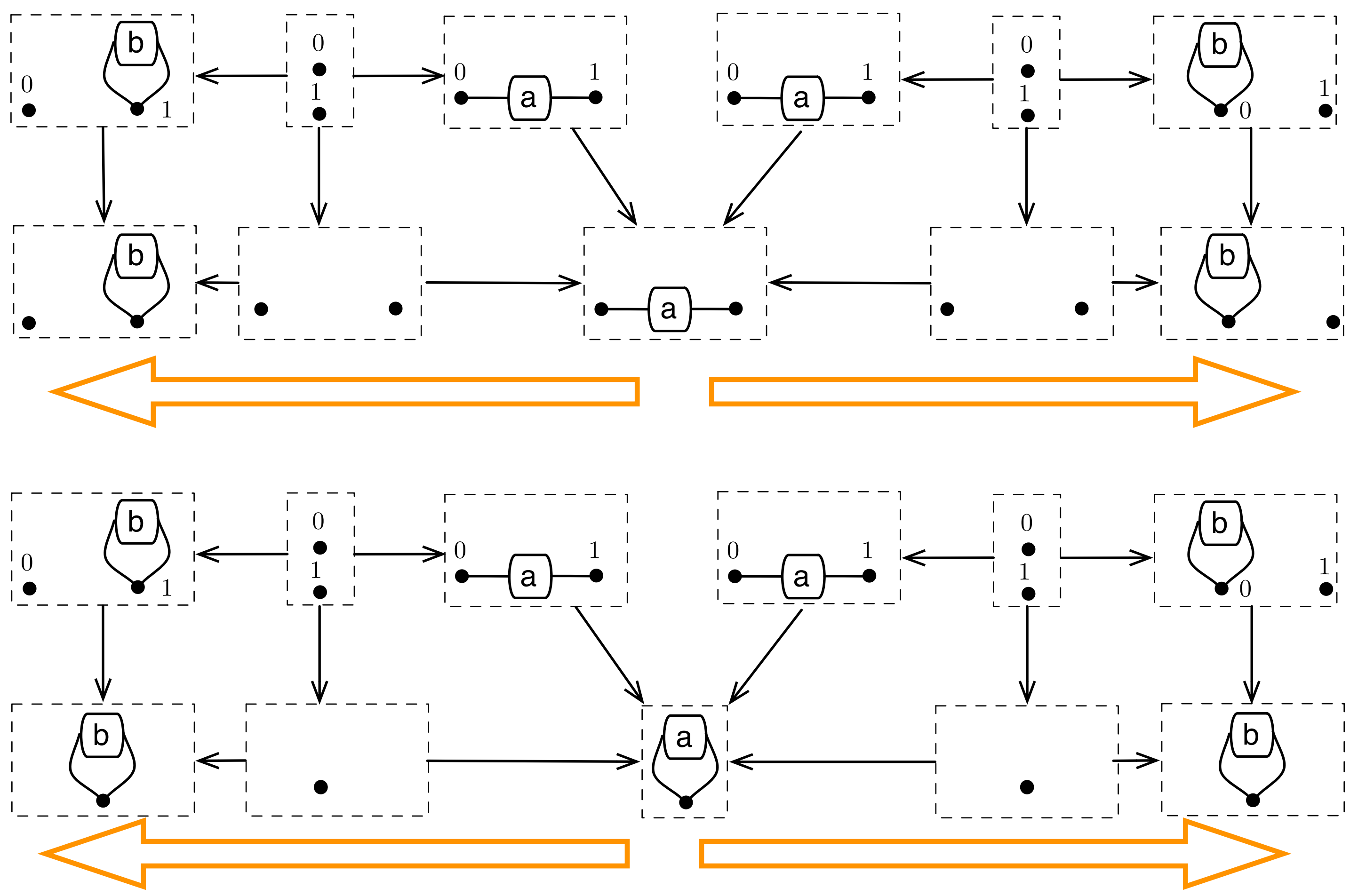

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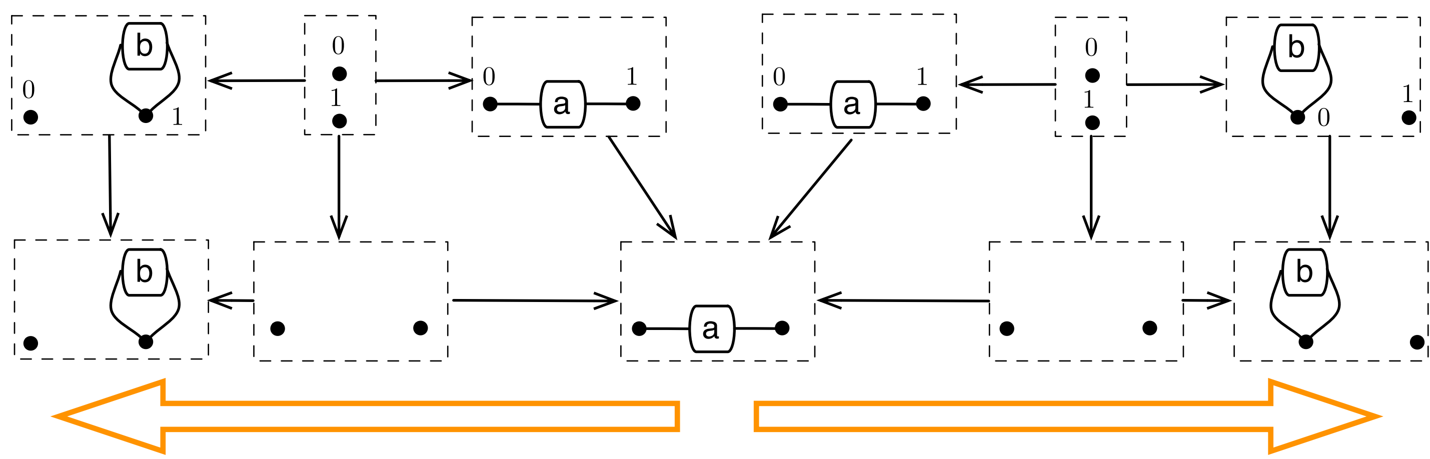

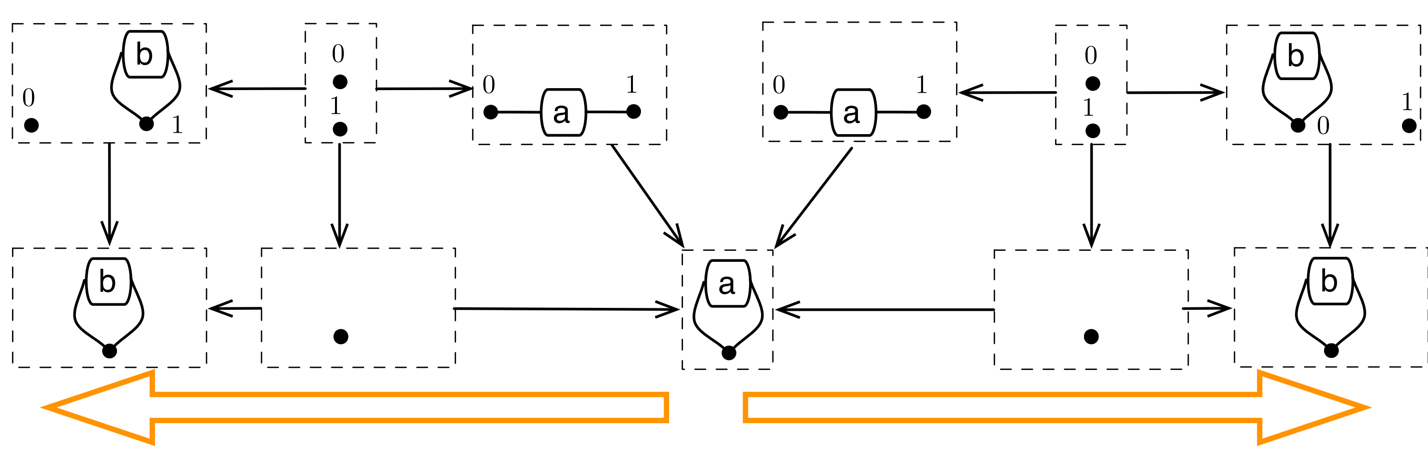

Both are trivially joinable
but the system is not confluent

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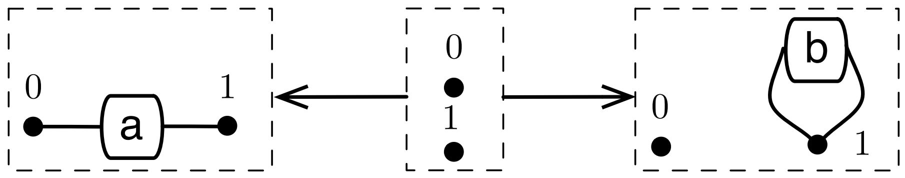

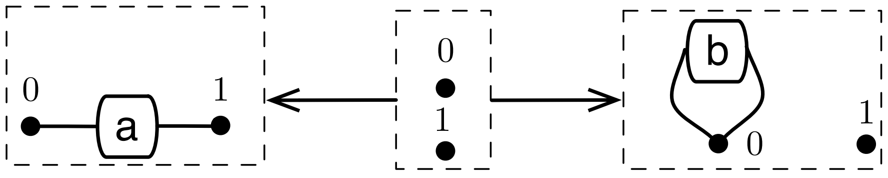

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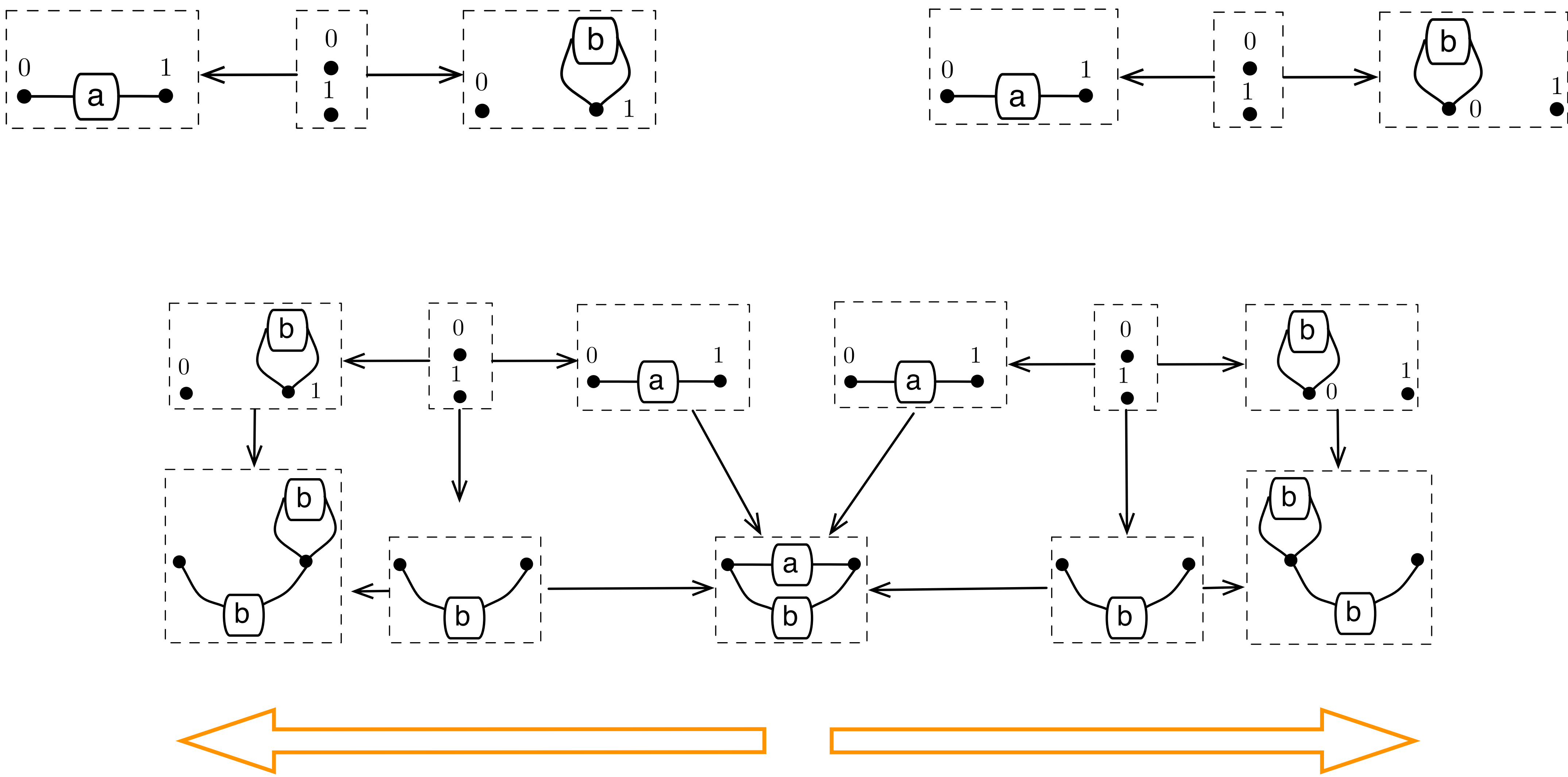

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DPO rewriting
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DPO Rewriting with Borrowed Contexts:
Ehrig and Koenig 2004

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Graphical Encoding of Process Calculi
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Foundational studies of computads in cospans categories
Gadducci, Heckel 1997 - Sassone, Sobocinski 2005

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Rather than rewriting graphs,
we rewrite graphs with interfaces

$G \leftarrow J$

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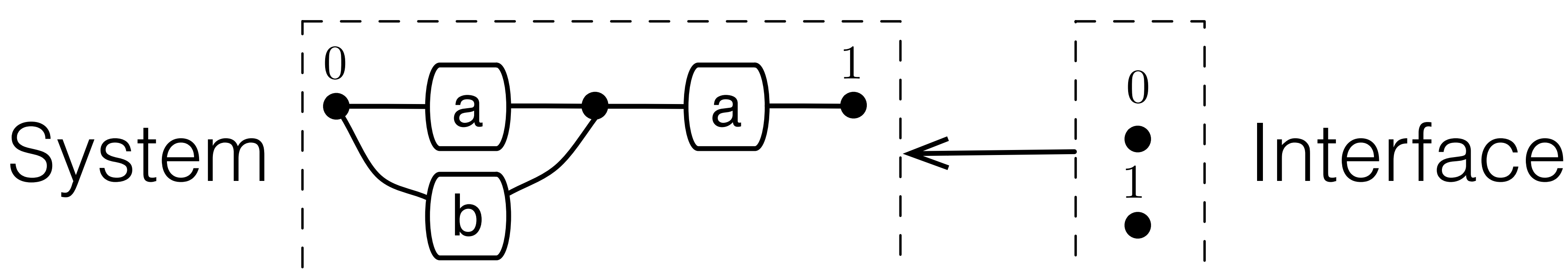

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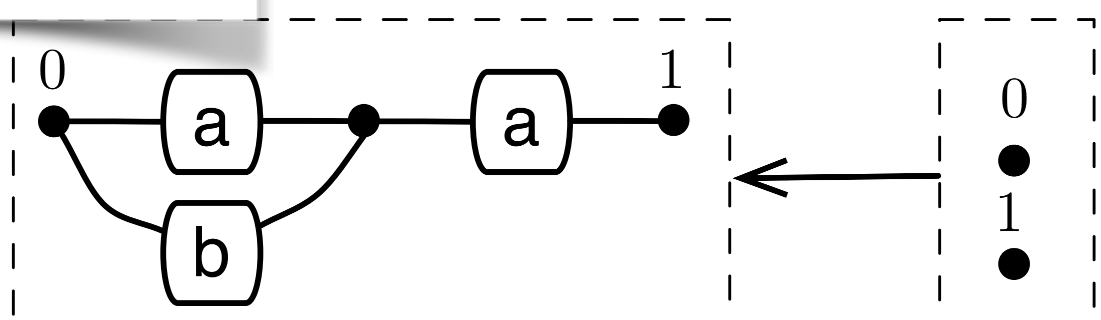
Ubiquitous in computer science:
Queries in Databases,
Kleene Algebra,
etc...

Rather than rewriting graphs,
rewrite graphs with interfaces

$$G \longleftarrow J$$

System  Interface

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& J \arrow[ul] \arrow[u] \arrow[ur] &
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where the two squares are pushouts

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$$(G \leftarrow J) \Longrightarrow (H \leftarrow J)$$

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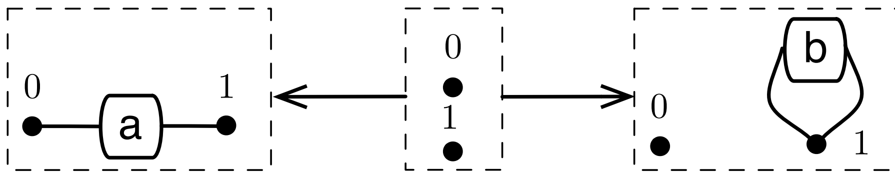
& J \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

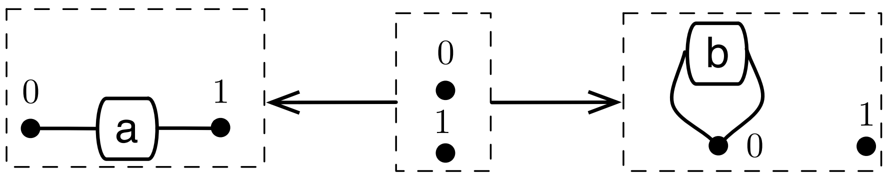
standard DPO is an instance when J is the initial object 0

where the two squares are pushouts

$(G \leftarrow J) \Rightarrow (H \leftarrow J)$

Example





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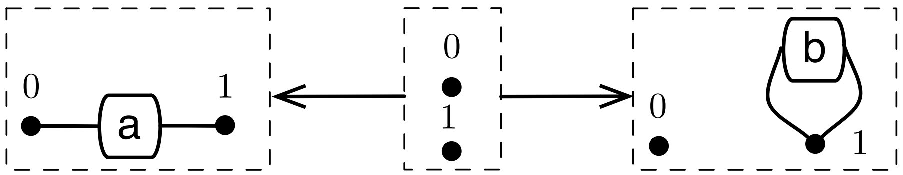

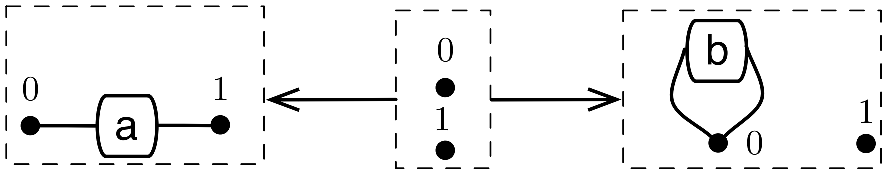

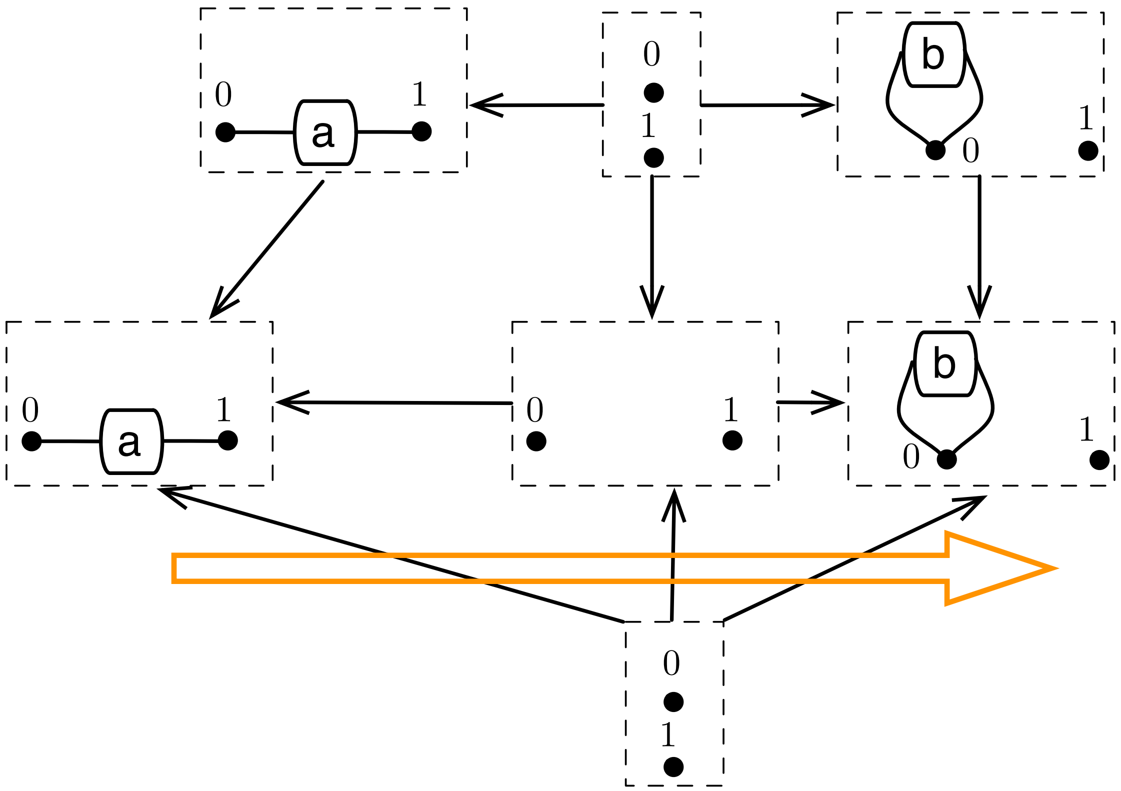

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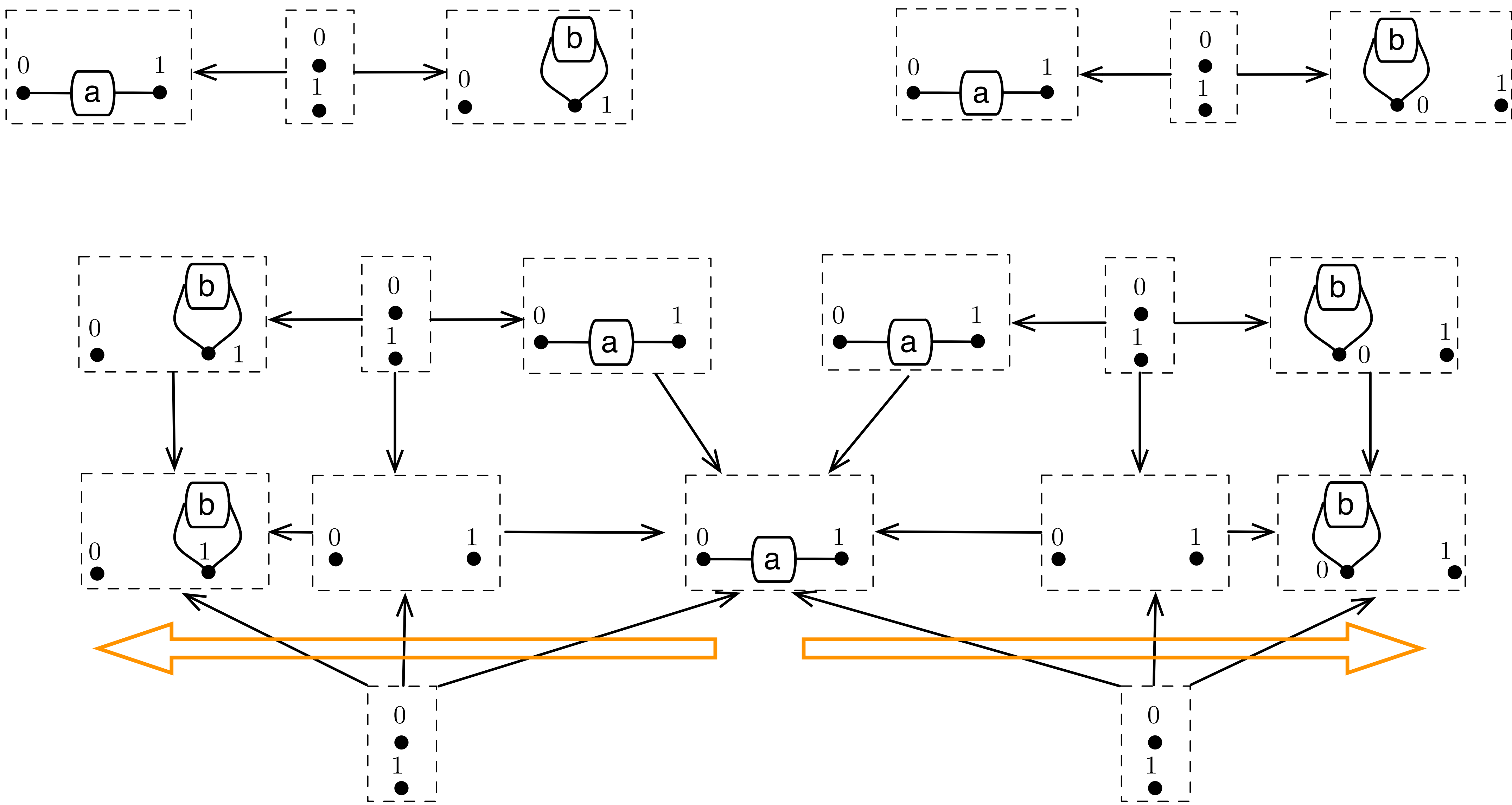

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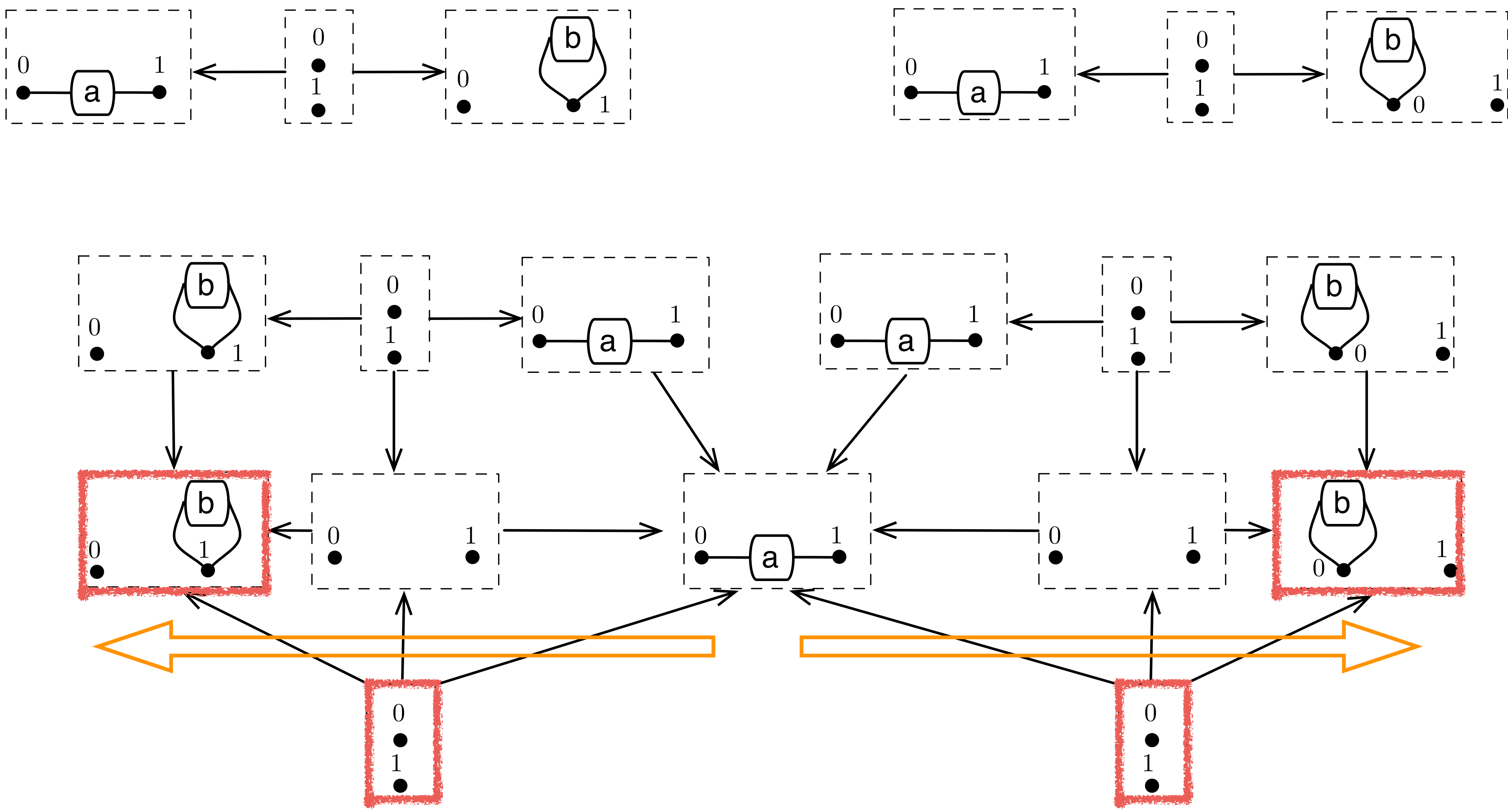

By adding the interface,
the arriving states are distinguished

Critical Pairs
with interfaces

Critical Pairs with interfaces

f_1 and f_2 are jointly epi

$$
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R_1 \arrow[d] & K_1 \arrow[l] \arrow[r] \arrow[d] & L_1 \arrow[dr, "f_1"'] & & L_2 \arrow[dl, "f_2"] & K_2 \arrow[l] \arrow[r] \arrow[d] & R_2 \arrow[d] \\
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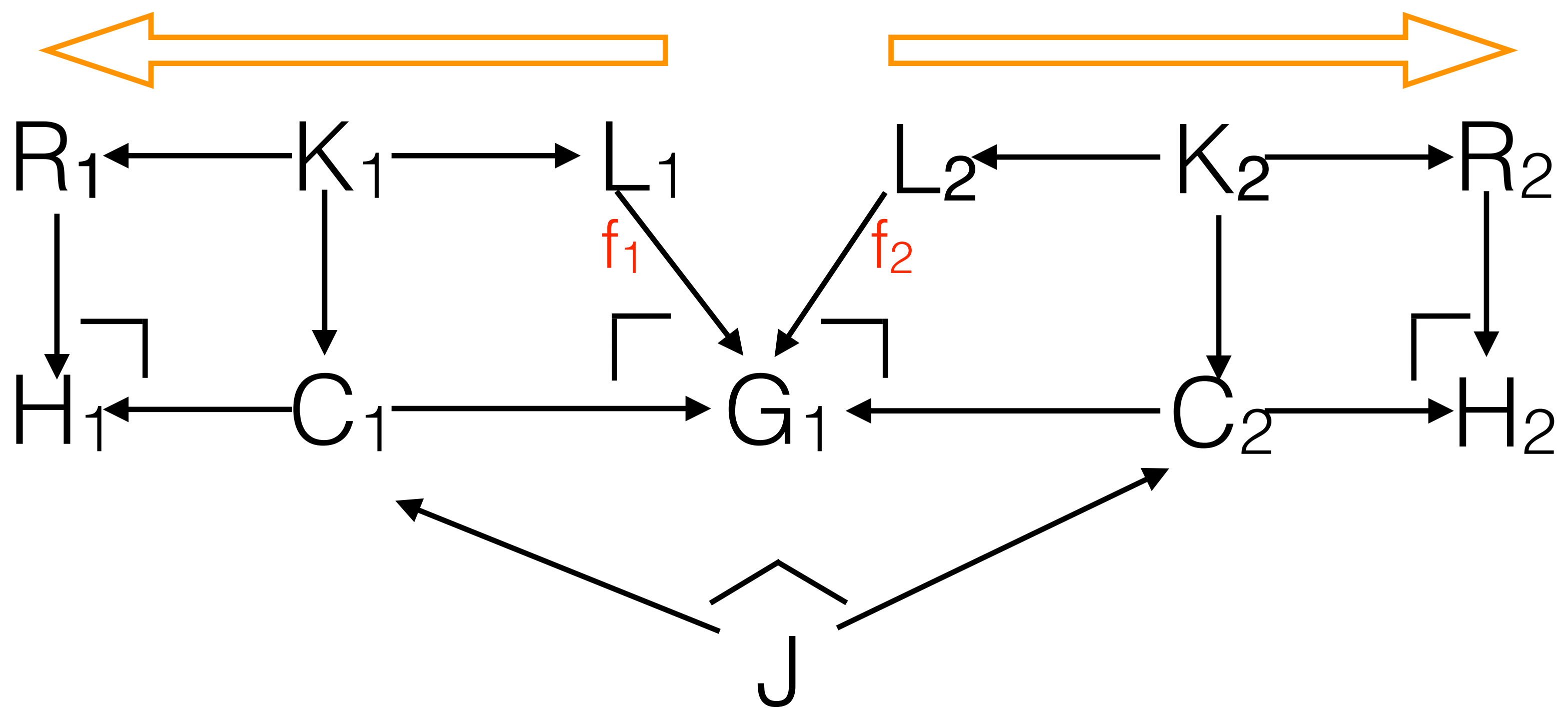

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& & & J \arrow[ull] \arrow[urr] & & &
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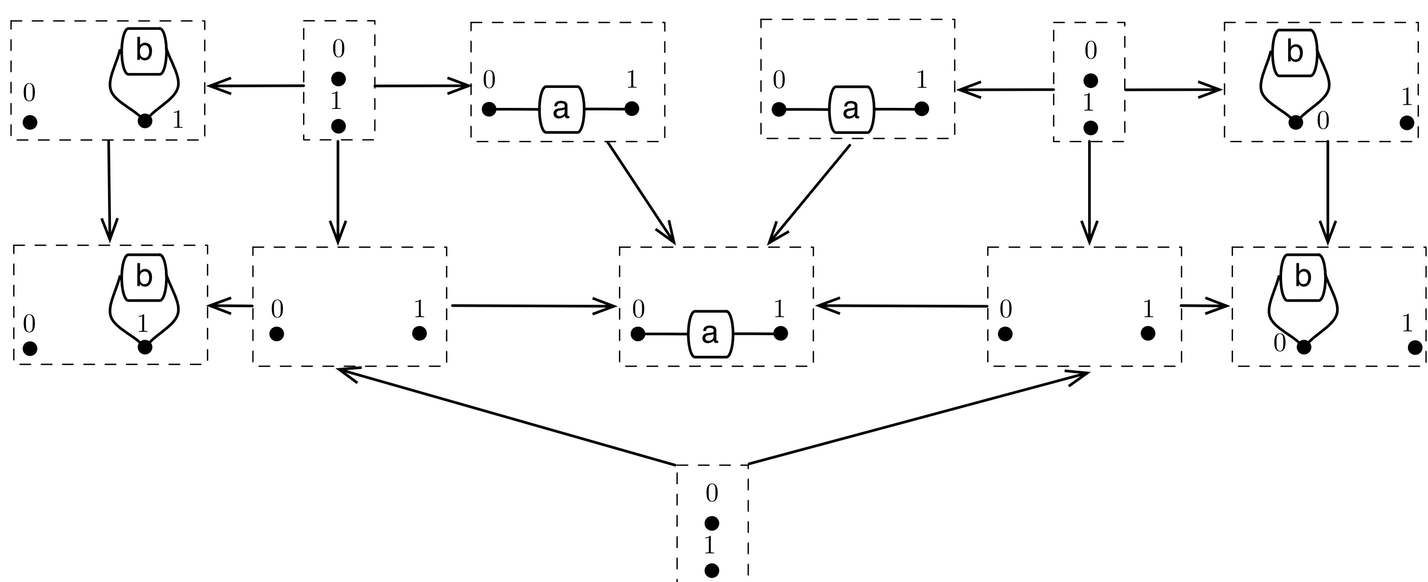

Confluence for
DPO with Interfaces

Confluence for DPO with Interfaces

Theorem

In a DPO rewriting system with interfaces,
if all critical pairs are joinable,
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Corollary

In a terminating DPO rewriting system
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Plump's result concerns all graphs with interfaces G <- 0

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A nice analogy

	Terminating Term Rewriting	Terminating DPO Rewriting
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Confluence	Decidable (Knuth-Bendix 1970)	Decidable (This talk)

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Plump's notion of "strongly joinable"

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Koenig et al. 2011

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Freely Generated PROPs

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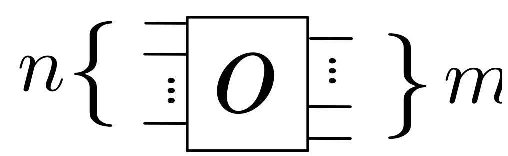
A signature Σ is a set of gates with arity and coarity



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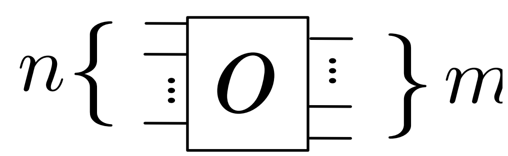

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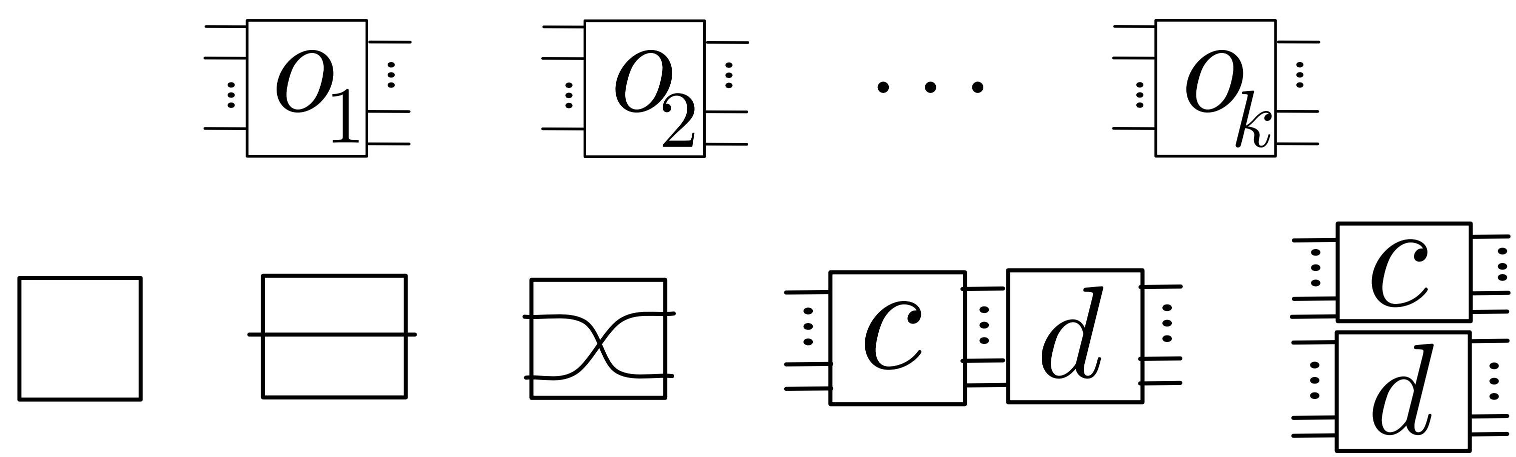

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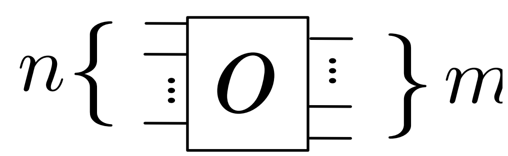

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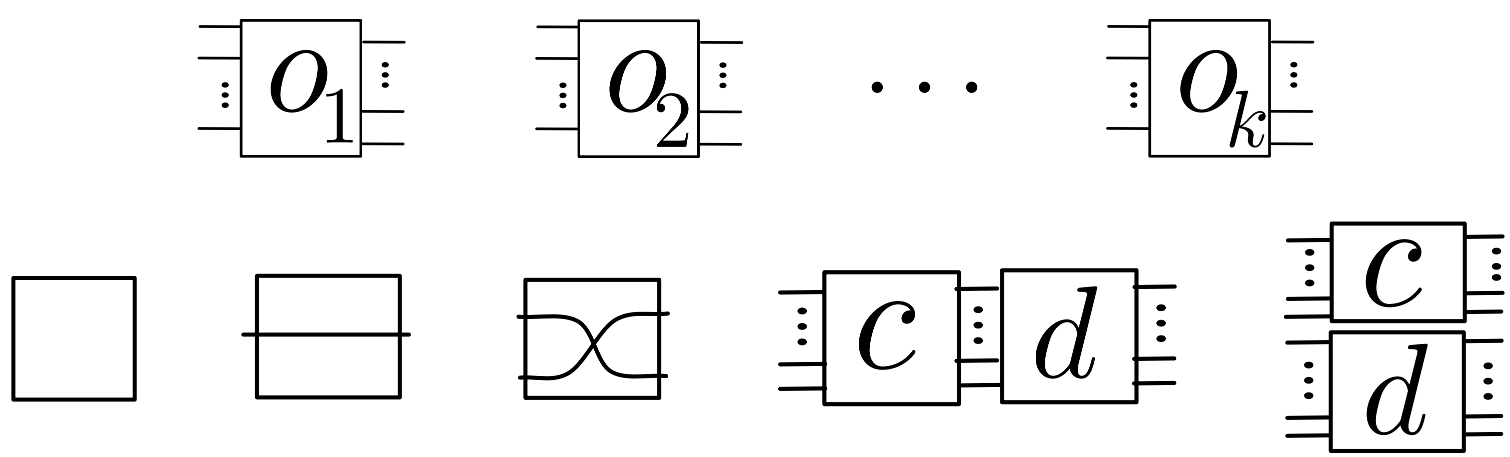

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The PROP freely generated by Σ, $\mathbf{T}_\Sigma$, has
as arrows the Σ-diagrams
modulo the laws of strict symmetric monoidal categories

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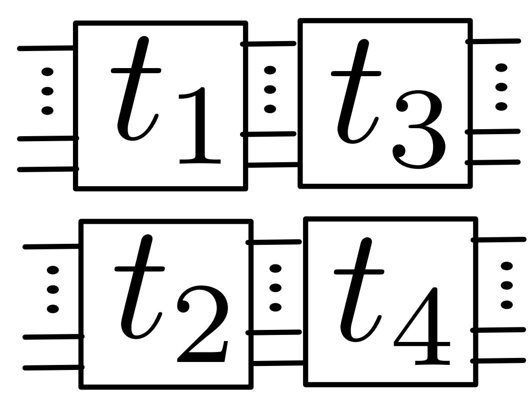

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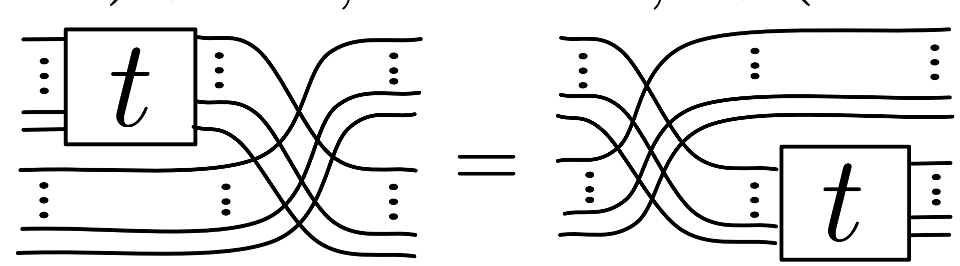

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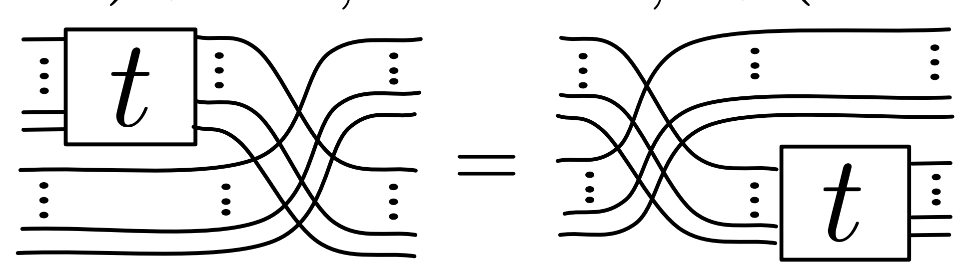

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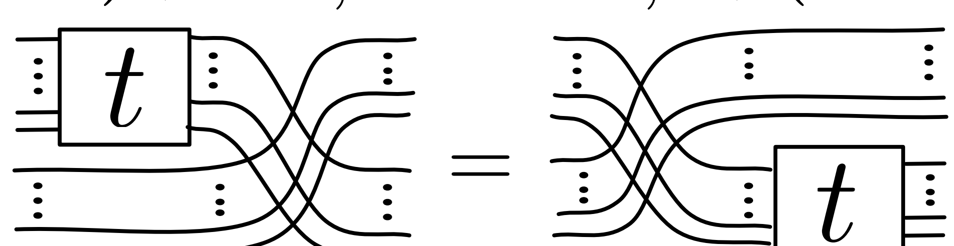

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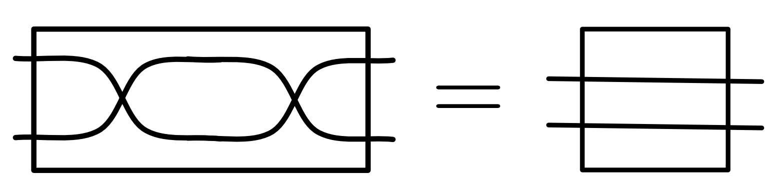

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An SMT is a pair (Σ,E) where

- Σ is a signature and
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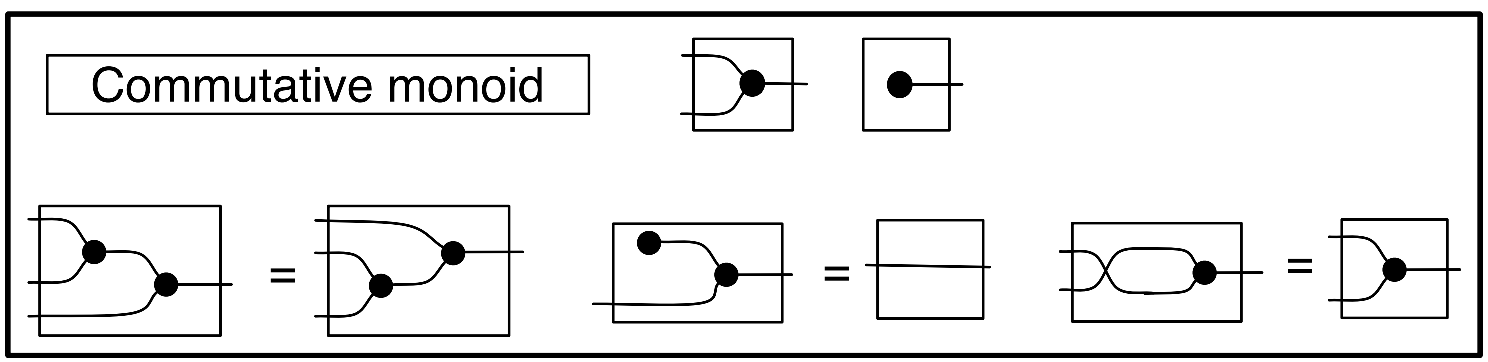

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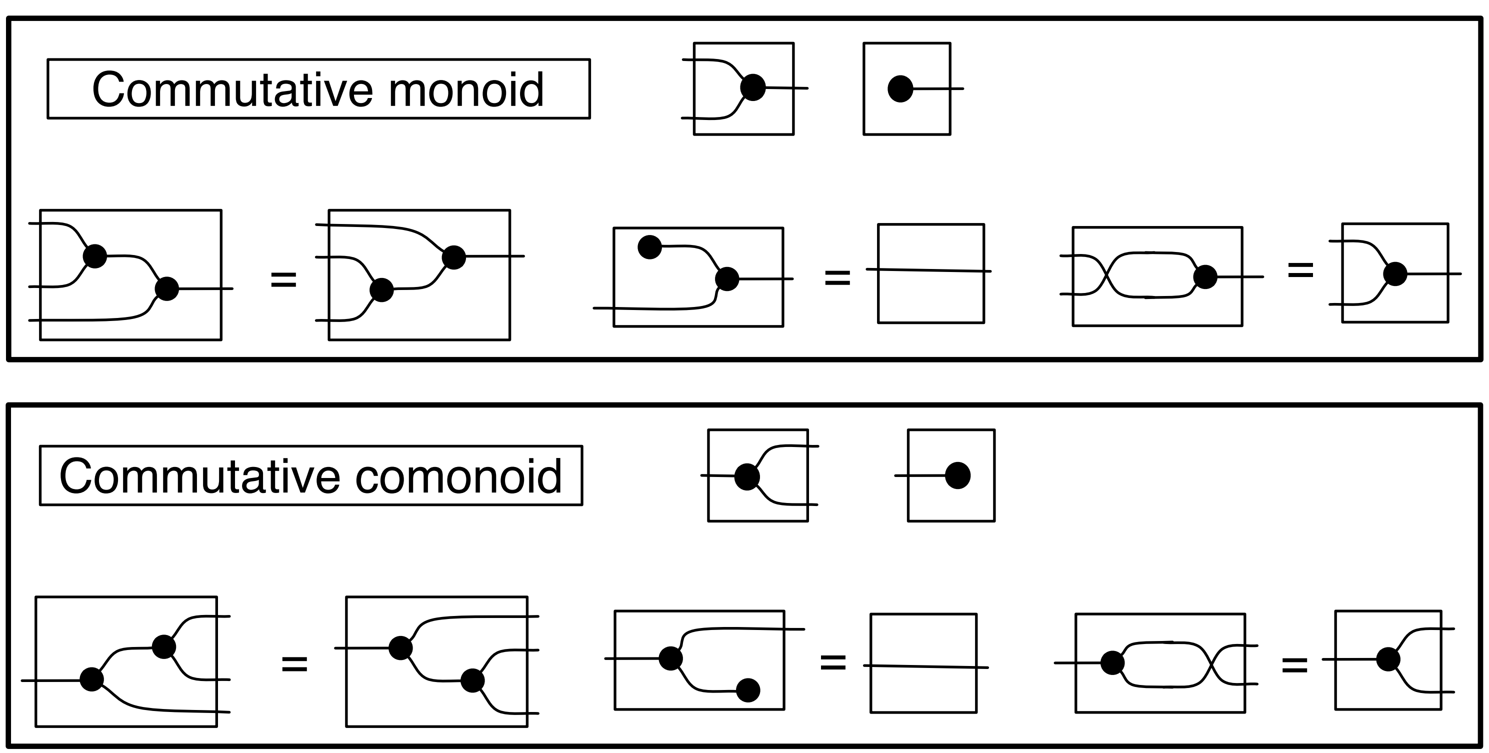

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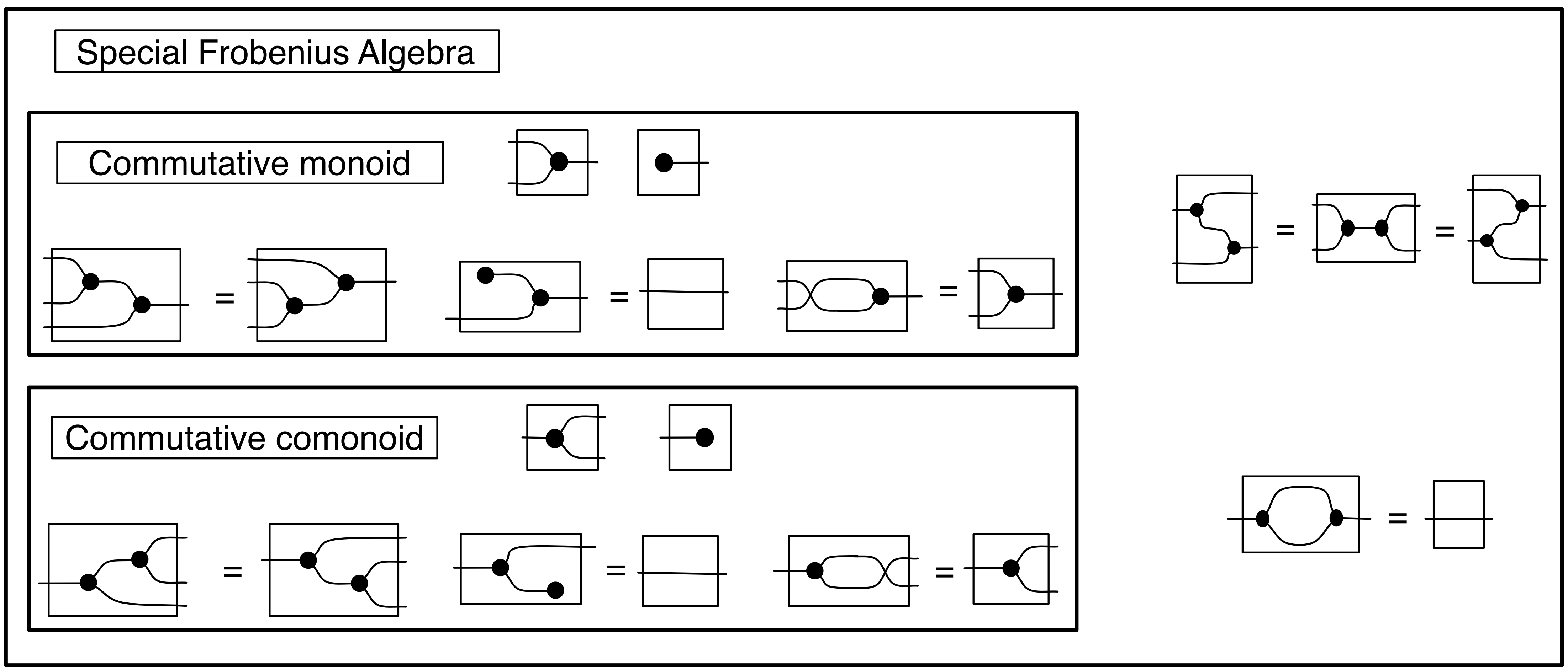

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The celebrated theoretical physicist John Baez "reinvented" DPO rewriting

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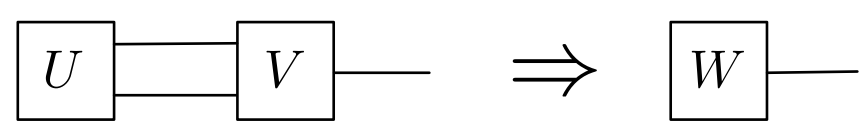

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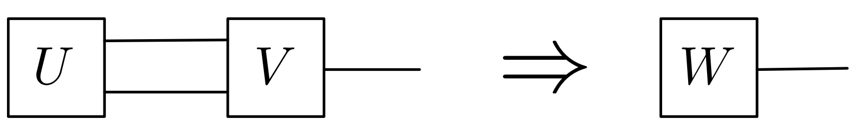

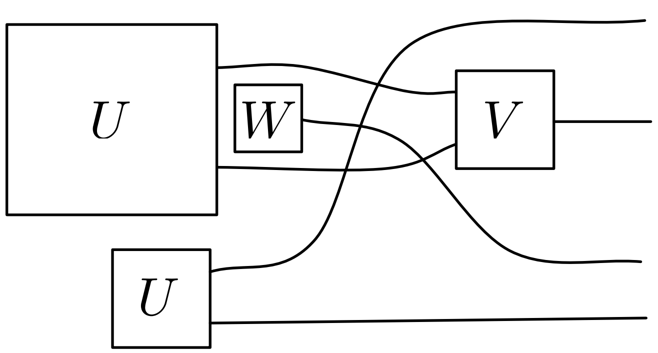

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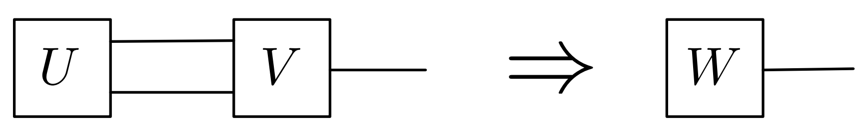

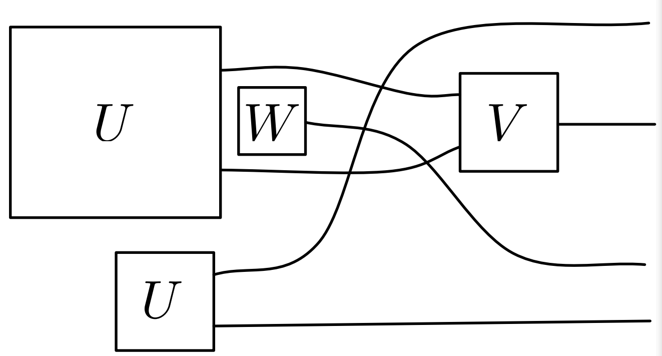

French School of Rewriting
(Yves la Font,
Samuel Mimram,
Philippe Malboss, ...)

Confluence for PROP Rewriting

Lafont 2003 - Mimram 2014

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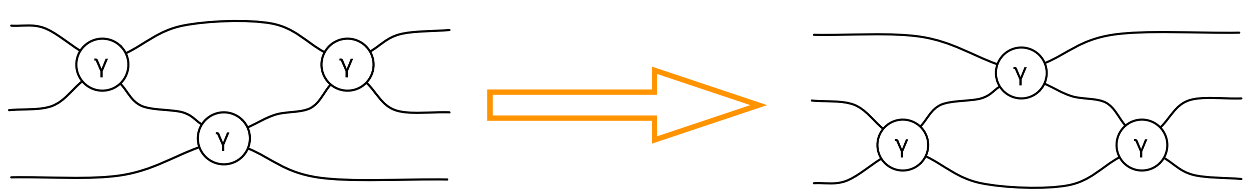

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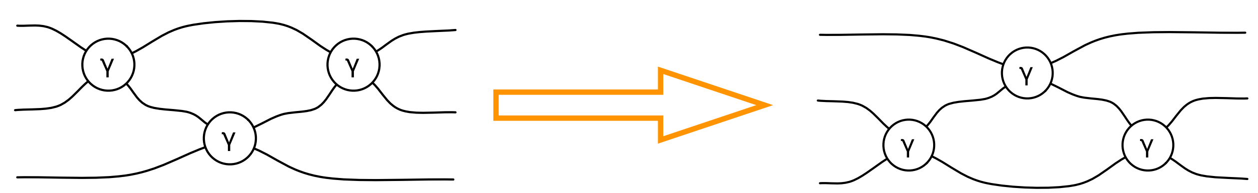
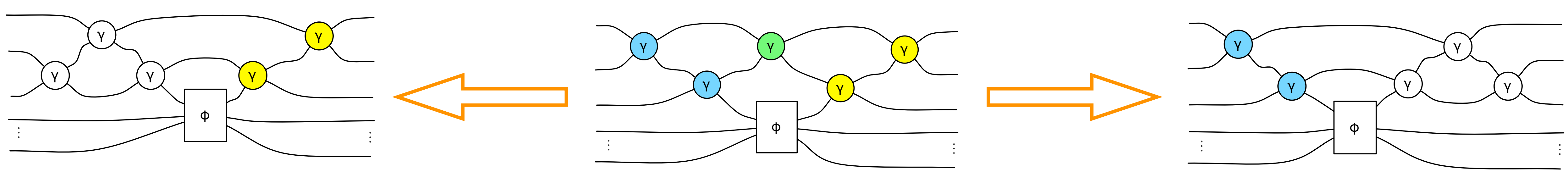


Infinitely many critical pairs: one for each diagram **ϕ**



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A functorial semantics for them is still not understood

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We need tools for supporting combinatorial reasoning

1) Implementing rewriting with Interfaces (for arbitrary matches and rules)
2) Automatically proving confluence
3) (Semi-)Automatically check equivalence