Continuous components, monadically

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Eindhoven, 1 April 2016

Components as coalgebras

The *motto* of a component calculus

- coalgebraic: standard notions of observational equivalence and refinement;
- typed: components are arrows in a suitable (bi)category;
- point free: reasoning is driven by composition (rather than by application)

Lifting AoP (Bird & Moor, *Algebra of programming*, 97)
to the architectural level

Components as coalgebras

Components as coalgebras with transition 'shape'

$$\mathcal{F}X \;=\; \mathcal{B}(X \times O)^I$$

i.e., generalised Mealy machines where $\mathcal{B}$ is a strong monad, e.g.

- Partiality: $\mathcal{B} = \mathbf{Id} + \mathbf{1}$
- Non determinism: $\mathcal{B} = \mathcal{P}$
- Monoidal labelling: $\mathcal{B} = \mathbf{Id} \times M$, with M a monoid.
- Probabilistic evolution: $\mathcal{B} = \mathcal{D}$ with

$$\mathcal{D} \;=\; \{\mu \in [0,1]^B \mid \sum_{b \in B} \mu b = 1\}$$

Components as coalgebras

The calculus

Bicategory Cp, parametric on a behaviour monad $\mathcal{B}$

- Objects: sets (interfaces)
- Arrows: seeded $\mathcal{F}$-coalgebras

$$p = (u \in U, a_p : U \longrightarrow \mathcal{B}(U \times O)^I)$$

- For each pair (I, O) of objects, a category $\mathrm{Cp}(I, O)$ whose arrows

$$h : (u_p, \overline{a}_p) \longrightarrow (u_q, \overline{a}_q) \qquad \begin{array}{ccc} U_p & \xrightarrow{\overline{a}_p} & \mathcal{F}\, U_p \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{F}\, h} \\ U_q & \xrightarrow[\overline{a}_q]{} & \mathcal{F}\, U_q \end{array}$$

are seed preserving morphisms.

Continuous systems

Can continuous evolution be encoded as a computational effect?

... for each input, outputs a (continuous) evolution over time; *i.e.*, an arrow typed as

$$I \longrightarrow \coprod_{d \in [0,\infty]} O^{[0,d]}$$

where I, O are input and output universes, respectively.

$$\boxed{I \longrightarrow \{\, (f, d) \in O^{\mathsf{T}} \times [0, \infty] \mid f \cdot \curlywedge_d = f\}}$$

where $\curlywedge_d \mathrel{\widehat{=}} id \lhd (\leq_d) \rhd \underline{d}$ which can be turned into a monad in **Top**.

$\mathcal{H}$: A monad for continuous evolution

$$\mathcal{H} : \textbf{Top} \rightarrow \textbf{Top}$$

$$\begin{aligned} \mathcal{H}X &\mathrel{\widehat{=}} \{\, (f, d) \in X^{T} \times \mathrm{D} \mid f \cdot \lambda_d = f \,\} \\ \mathcal{H}g &\mathrel{\widehat{=}} g^{T} \times id \end{aligned}$$

where $\mathrm{D} = [0, \infty]$ and $g^{\mathsf{T}}\, f = g \cdot f$

$\mathcal{H}$: A monad for continuous evolution

Intuitively,

- η (= *copy*) produces trivial evolutions, *i.e.*, with duration 0
- μ concatenates evolutions

Unit $\eta \;:\; Id \rightarrow \mathcal{H}$

$$\boxed{\eta_X\, x \mathrel{\widehat{=}} (\underline{x},\, 0)}$$

$\mathcal{H}$: A monad for continuous evolution

Multiplication $\mu : \mathcal{H}\mathcal{H} \to \mathcal{H}$

$$\mu_X\,(f,d) \mathrel{\widehat{=}} \begin{cases} (\theta \cdot f, d) +\!\!+ (f\ d) & \text{if } d \neq \infty \\ (\theta \cdot f,\ \infty) & \text{otherwise} \end{cases}$$

where

- $\theta : \mathcal{H} \to Id$ is such that $\theta\,(f,d) \mathrel{\widehat{=}} f\,0$
- $(f,d) +\!\!+ (g,e) \mathrel{\widehat{=}} (f +\!\!+_d\ g, d+e)$
  where $f +\!\!+_d\ g \mathrel{\widehat{=}} f \lhd (\leq_d) \rhd g\,(\_ - d)$

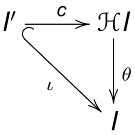
Kleisli composition

$$
\begin{aligned}
& (c_2 \bullet c_1)\ x \\
= & \quad \{ \text{ Kleisli composition } \} \\
& \mu \cdot \mathcal{H} c_2 \cdot c_1\ \ x \\
= & \quad \{ \text{ Definition of } \mathcal{H}, \text{ let } d = \pi_2 \cdot c_1\ x \ \} \\
& \mu\,(c_2 \cdot (f_{c_1}\ x), d) \\
= & \quad \{ \text{ Definition of } \mu \ \} \\
& (\theta \cdot c_2 \cdot (f_{c_1}\ x) + (f_{c_2}\ (f_{c_1}\ x\ d)), d + \pi_2(\ c_2 \cdot (f_{c_1}\ x)\ \ d\ )) \\
= & \quad \{ \ \text{ Definition of } + \text{ and composition } \ \} \\
& (\boxed{\theta \cdot c_2\ (f_{c_1}\ x\ \ \_\ )\ \lhd\ (\leq_d)\ \rhd\ f_{c_2}\ (f_{c_1}\ x\ d)\ (\ \_\ - d)}, \boxed{\mathrm{d} + \pi_2\ (\ c_2\ (f_{c_1}\ x\ d)\ )})
\end{aligned}
$$

where $c_1 : I \rightarrowtail K, \quad c_2 : K \rightarrowtail O$.

Kleisli composition

If c_2 in$(c_2 \bullet c_1)$ is pre-dynamical in the sense that



than composition yields sequencing: for the duration of $c_1\ x$, $c_2 \bullet c_1\ x$ evolves first according to c_1, and then, on its termination, according to c_2 which receives as input the endpoint of $f_{c_1}\ x$.

Otherwise yields a form of superposition.

Note:
θ is an Eilenberg-Moore $\mathcal{H}$-algebra: $\theta \cdot \eta_X = id$ and $\theta \cdot \mu = \theta \cdot \mathcal{H}\theta$.

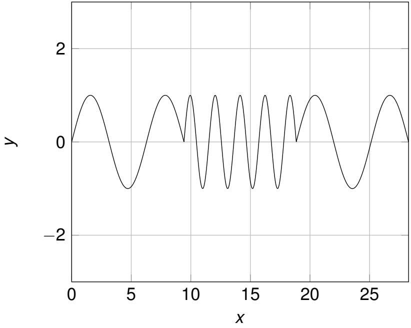
Kleisli composition

Example

Let $c_1, c_2 : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as

$$c_1\ r \mathrel{\widehat{=}} (r + (sin\ \_), 3\pi), \quad c_2\ r \mathrel{\widehat{=}} (r + sin\,(3 \times\ \_), 3\pi)$$

$$c_1 \bullet (c_2 \bullet c_1)\ 0$$

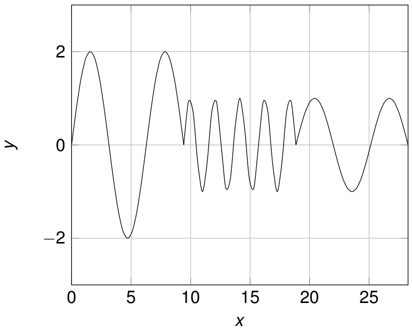


Kleisli composition

Example

Introduce an amplifier $a : \mathbb{R} \rightarrowtail \mathbb{R}$, where $a\, r \mathrel{\widehat{=}} (\underline{r \times 2}, 0)$ (note that a is not pre-dynamical)

$c_1 \bullet (c_2 \bullet (a \bullet c_1))\ 0$



Kleisli composition

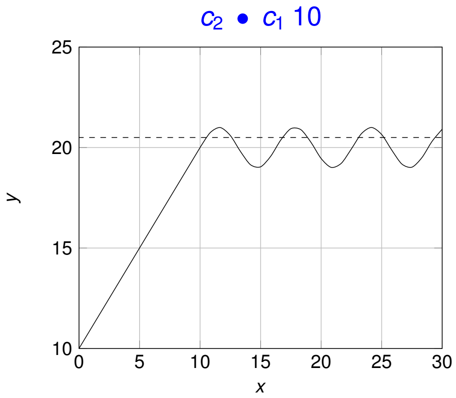
Example: temperature regulator

Objective: starting at 10 °C, seek to reach and maintain 20 °C, but in no case surpass 20.5 °C.

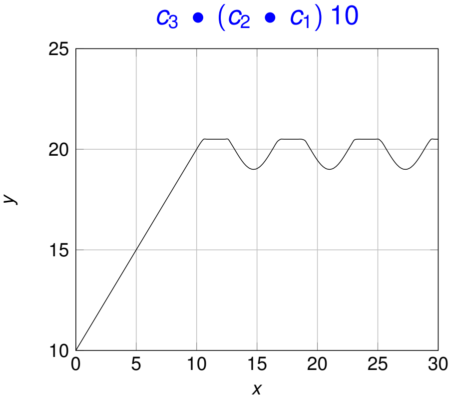
Components:

$$
\begin{aligned}
c_1\, x &= (\,(x + \_),\, 20 \ominus x\,) \\
c_2\, x &= (\,x + (\sin\, \_),\, \infty\,) \\
c_3\, x &= (\,\underline{x} \lhd (x \leq 20.5) \rhd \underline{20.5}\,, 0\,)
\end{aligned}
$$

Kleisli composition



Kleisli composition



Note that c_3 is able to play a supervisory role precisely because it is not pre-dynamical.

Composition

- $\mathcal{H}$-Kleisli composition provides the basic composition mechanism for continuous components
- the structure of $\mathbf{Top}_{\mathcal{H}}$ yields its basic laws.

$$copy \bullet c_1 = c_1 \tag{1}$$

$$c_1 \bullet copy = c_1 \tag{2}$$

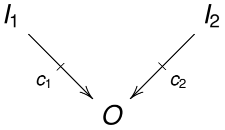
$$(c_3 \bullet c_2) \bullet c_1 = c_3 \bullet (c_2 \bullet c_1) \tag{3}$$

$$c_2 \bullet c_1 = c_1 \tag{4}$$

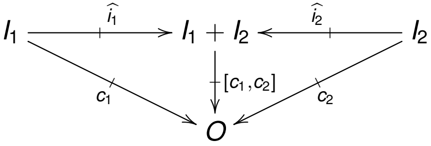
if c_2 pre-dynamical and c_1 has an infinite duration

Choice

Given two components



define component $[c_1, c_2] : I_1 + I_2 \rightarrow\!\!\!\!\!+ \; O$ which makes the following diagram to commute,



where $[-,-]$ is inherited through the canonical adjunction between **Top** and $\mathbf{Top}_{\mathcal{H}}$.

Sum: $c_1 \boxplus c_2 : I_1 + I_2 \leftrightarrow O_1 + O_2$

$$c_1 \boxplus c_2 \mathrel{\widehat{=}} [\widehat{i_1} \bullet c_1, \widehat{i_2} \bullet c_2]$$

Laws

$$c_3 \bullet [c_1, c_2] = [c_3 \bullet c_1, c_3 \bullet c_2] \tag{5}$$

$$(c_1 \boxplus c_2) \bullet \widehat{i_1} = \widehat{i_1} \bullet c_1 \tag{6}$$

$$(c_1 \boxplus c_2) \bullet \widehat{i_2} = \widehat{i_2} \bullet c_2 \tag{7}$$

$$copy_X \boxplus copy_Y = copy_{X+Y} \tag{8}$$

$$(d_1 \boxplus d_2) \bullet (c_1 \boxplus c_2) = (d_1 \bullet c_1) \boxplus (d_2 \bullet c_2) \tag{9}$$

$$[d_1, d_2] \bullet (c_1 \boxplus c_2) = [d_1 \bullet c_1, d_2 \bullet c_2] \tag{10}$$

$$\widehat{f} \boxplus \widehat{g} = \widehat{f + g} \tag{11}$$

for any continuous functions $f : X_1 \to Y_1, g : X_2 \to Y_2$

Wiring

Wires are functions lifted to $\mathbf{Top}_{\mathcal{H}}$:

- $\widehat{\Delta} : X \nrightarrow X \times X$ (duplicates evolutions)
- $\widehat{*_s} : \mathbb{R} \nrightarrow \mathbb{R}$ (amplifies signals)
- $\widehat{\pi_1} : X \times Y \nrightarrow X$ which eliminates the right side of 'paired' evolutions.
- $\widehat{sw} : X \times Y \nrightarrow Y \times X$ (wire swapping)

As expected,

$$\widehat{id} = copy \tag{12}$$

$$\widehat{g} \bullet \widehat{f} = \widehat{g \cdot f} \tag{13}$$

Parallel

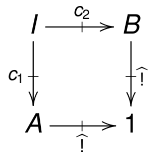
Finding limits in a Kleisli category is often more difficult, but

- under the Kleisli adjunction, $L \dashv R$, of a given monad, L preserves whatever limits the monad does,
- and $\mathcal{H}$ preserves pullbacks.

Thus, whenever two systems are compatible – in the sense that for any input they produce evolutions with equal duration – their parallel composition can be defined.

Parallel

i.e., two systems are compatible when the diagram



commutes. Then, $\langle\langle c_1, c_2\rangle\rangle : I \leftrightarrow (A \times_1 B)$ comes through the mediating arrow (of the pullback):

$$\langle\langle c_1, c_2\rangle\rangle \mathrel{\widehat{=}} \gamma \cdot \langle c_1, c_2\rangle$$

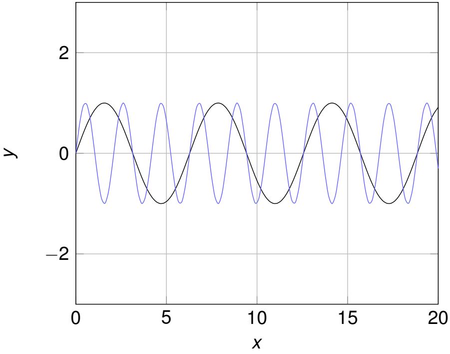
$$I \xrightarrow{\langle c_1, c_2\rangle} \{\, ((f,d),(g,e)) \in \mathcal{H}A \times \mathcal{H}B \mid d = e \,\} \xrightarrow{\gamma} \mathcal{H}(A \times_1 B)$$

$$\gamma\,((f,d),(g,d)) \mathrel{\widehat{=}} (\langle f, g\rangle, d)$$

Example

$$c_1\ x\ =\ (\ x + (\sin\ \_\ ),\ 20\ ),\ \ c_2\ x\ =\ (\ x + \sin\ (3 \times \_\ ),\ 20\ )$$

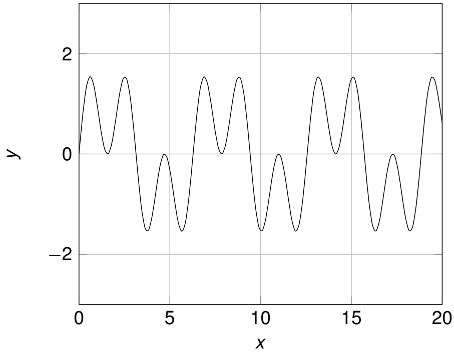
$\langle\langle c_1, c_2 \rangle\rangle\ 0$



Example

Let $\widehat{+} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ add incoming signals.

$\widehat{+} \bullet \langle\langle c_1, c_2 \rangle\rangle\ 0)$



Product: $c_1 \boxtimes c_2 : I_1 \times I_2 \rightarrowtail O_1 \times O_2$

$$c_1 \boxtimes c_2 \,\widehat{=}\, \langle\langle c_1 \bullet \widehat{\pi_1}, c_2 \bullet \widehat{\pi_2} \rangle\rangle$$

Laws

$$\langle\langle c_1, c_2 \rangle\rangle \bullet d = \langle\langle c_1 \bullet d, c_2 \bullet d \rangle\rangle \quad (14)$$

$$\widehat{\pi_1} \bullet (c_1 \boxtimes c_2) = c_1 \bullet \widehat{\pi_1} \quad (15)$$

$$\widehat{\pi_2} \bullet (c_1 \boxtimes c_2) = c_2 \bullet \widehat{\pi_2} \quad (16)$$

$$\langle\langle c_1, c_2 \rangle\rangle = (c_1 \boxtimes c_2) \bullet \widehat{\triangle} \quad (17)$$

$$copy_X \boxtimes copy_Y = copy_{X \times Y} \quad (18)$$

$$(d_1 \boxtimes d_2) \bullet (c_1 \boxtimes c_2) = (d_1 \bullet c_1) \boxtimes (d_2 \bullet c_2) \quad (19)$$

$$(d_1 \boxtimes d_2) \bullet \langle\langle c_1, c_2 \rangle\rangle = \langle\langle d_1 \bullet c_1, d_2 \bullet c_2 \rangle\rangle \quad (20)$$

$$\widehat{f} \boxtimes \widehat{g} = \widehat{f \times g} \quad (21)$$

Synchronised product

- relax the equal duration condition
- definition entails a monoidal structure to functor $\mathcal{H}$, i.e.,
  - a morphism $m : 1 \rightarrow \mathcal{H}1$: $m \,\widehat{=}\, copy$
  - a natural transformation $\delta : \mathcal{H}X \times \mathcal{H}Y \rightarrow \mathcal{H}(X \times Y)$:
    $\delta\,((f, d), (g, e)) \,\widehat{=}\, (\langle f, g\rangle, d \curlyvee e)$

subjected to the usual conditions.

Synchronised product

Based on the monoidal structure $(\mathcal{H}, \delta, m)$ define

$$(\!|c_1, c_2|\!) = \delta \cdot \langle c_1, c_2 \rangle : I \to \mathcal{H}A \times \mathcal{H}B \to \mathcal{H}(A \times B)$$

and

$$c_1 \boxed{S} c_2 \mathrel{\widehat{=}} (\!| c_1 \bullet \widehat{\pi_1}, c_2 \bullet \widehat{\pi_2} |\!)$$

Synchronised product

Laws

$$\widehat{f \times g} \bullet (\!| c_1, c_2 |\!) = (\!| \widehat{f} \bullet c_1, \widehat{g} \bullet c_2 |\!) \tag{22}$$

$$\widehat{\alpha} \bullet (\!|(\!| c_1, c_2 |\!), c_3 |\!) = (\!| c_1, (\!| c_2, c_3 |\!) |\!) \tag{23}$$

$$\widehat{\pi_1} \bullet (\!| c, copy |\!) = c \tag{24}$$

$$\widehat{\pi_2} \bullet (\!| copy, c |\!) = c \tag{25}$$

$$\widehat{sw} \bullet (c_2 \boxed{S}\, c_1) = (c_1 \boxed{S}\, c_2) \cdot sw \tag{26}$$

$$\widehat{\alpha} \bullet ((c_1 \boxed{S}\, c_2) \boxed{S}\, c_3) = (c_1 \boxed{S}\, (c_2 \boxed{S}\, c_3)) \cdot \alpha \tag{27}$$

$$copy_X \boxed{S}\, copy_Y = copy_{X \times Y} \tag{28}$$

$$\widehat{f} \boxed{S}\, \widehat{g} = \widehat{f \times g} \tag{29}$$

Hybrid components

$$S \times I \longrightarrow S \times \mathcal{H}O$$

- Add an (internal) state space that behaves in a discrete manner: given a state ($s \in S$) and an input ($i \in I$), the component transits (internally) into another state and presents continuous evolutions that can be directly observed.
- The definition is aligned with both the *classical notion of hybrid system*, as a family of continuous systems indexed by a state space,
- and to the components as coalgebras mentioned above.

Hybrid components

The cornerstone of this move is tensorial strength for monad $\mathcal{H}$:

$$\tau : Id \times \mathcal{H} \to \mathcal{H}(Id \times Id)$$
$$\tau_{X,Y}\,(x,(f,d)) \mathrel{\widehat{=}} (\langle \underline{x}, f\rangle, d).$$

which allows us to transport such systems to $\mathbf{Top}_{\mathcal{H}}$, through

$$\frac{c : S \times I \to S \times \mathcal{H}O}{\tau \cdot c : S \times I \to \mathcal{H}(S \times O)}$$

Example

The bouncing ball

Let a ball be dropped at some positive height and with no initial velocity. Due to the gravitational effect, it will fall into the ground but then bounce back up, losing, of course, part of its kinetic energy in the process.

- ... regarded as a hybrid component whose (continuous) observable behaviour is the evolution of its spacial position, and the internal memory records velocity, updated at each bounce.

Example

$b \mathrel{\widehat{=}} \tau \cdot \langle b_d, b_c \rangle$

- The discrete behaviour $b_d : V \times P \to V$ is given

$$\boxed{\mathrm{b}_d\,(v, p) \mathrel{\widehat{=}} vel_g(v, zpos_g(v, p)) \times -0.5}$$

where 0.5 is the dampening coefficient, and

$$zpos_a\,(v, p) = \frac{\sqrt{2ap + v^2} + v}{a}, \quad vel_a\,(v, t) = v - at$$

- The continuous behaviour $b_c : V \times P \to \mathcal{H}P$ is

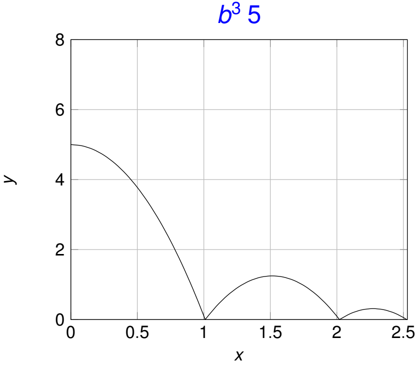
$$\boxed{\mathrm{b}_c \mathrel{\widehat{=}} (pos_g, zpos_g)}$$

where

$$pos_a\,(v, p)t = p + vt - \tfrac{1}{2}at^2$$

Example

The bouncing ball: $b \mathrel{\widehat{=}} \tau \cdot \langle b_d, b_c \rangle$

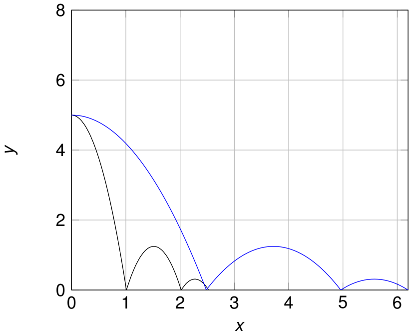


Example

Bouncing on the Earth and the Moon

... putting both components in parallel, with the same initial state 0, allows us to compare their behaviour

$(b^3 \boxed{s} c^3)\ (5, 5)$



Related work

- Hybrid automata as the *de facto* formalism for the specification of hybrid systems, and extensions
  - (Bornot,Sifakis,98): new synchronisation mechanisms
  - (Pnueli,10): taking *reaction times* into consideration
- SIMULINK (the *de facto* standard tool)
- P. Höfner's (2009) algebra of hybrid systems
- E. Haghverdi, P. Tabuada, G. Pappas (2005)
- B. Jacos (2000): coalgebras define the discrete transitions, and monoid actions the continuous evolutions.
- ...

Current work

- a calculus of hybrid components based on monad $\mathcal{H}$ and its Kleisli category.
- alternative notions of (bi)simulation for continuous and hybrid systems.
- a taxonomy of continuous, and hybrid systems living in $\mathbf{Top}_{\mathcal{H}}$.

  e.g., the notion of robustness (a system is robust if small changes in the input lead to very similar evolutions) can be expressed by varying the topology in its source object.