A General Account of Coinduction Up-To

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A Fruitful Approach

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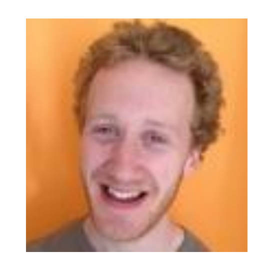
10-15/07/2016
Shangai



Coinduction (lattice theoretic)

<u>Knaster-Tarski fixed point:</u>

L a complete lattice and B:**L**-->**L** a monotone map

$\nu B = \bigcup \{x \mid x \subseteq B(x)\}$

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The post fixed points of B are called *invariants* or *bisimulations*

Language Equivalence

A Deterministic Automaton (DA) is a triple (X,o,t)

- X is the set of states
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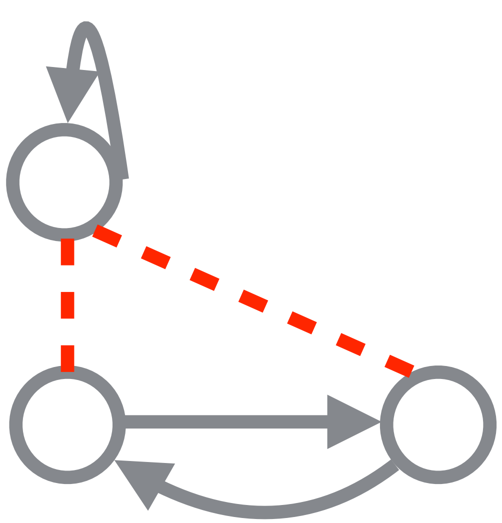

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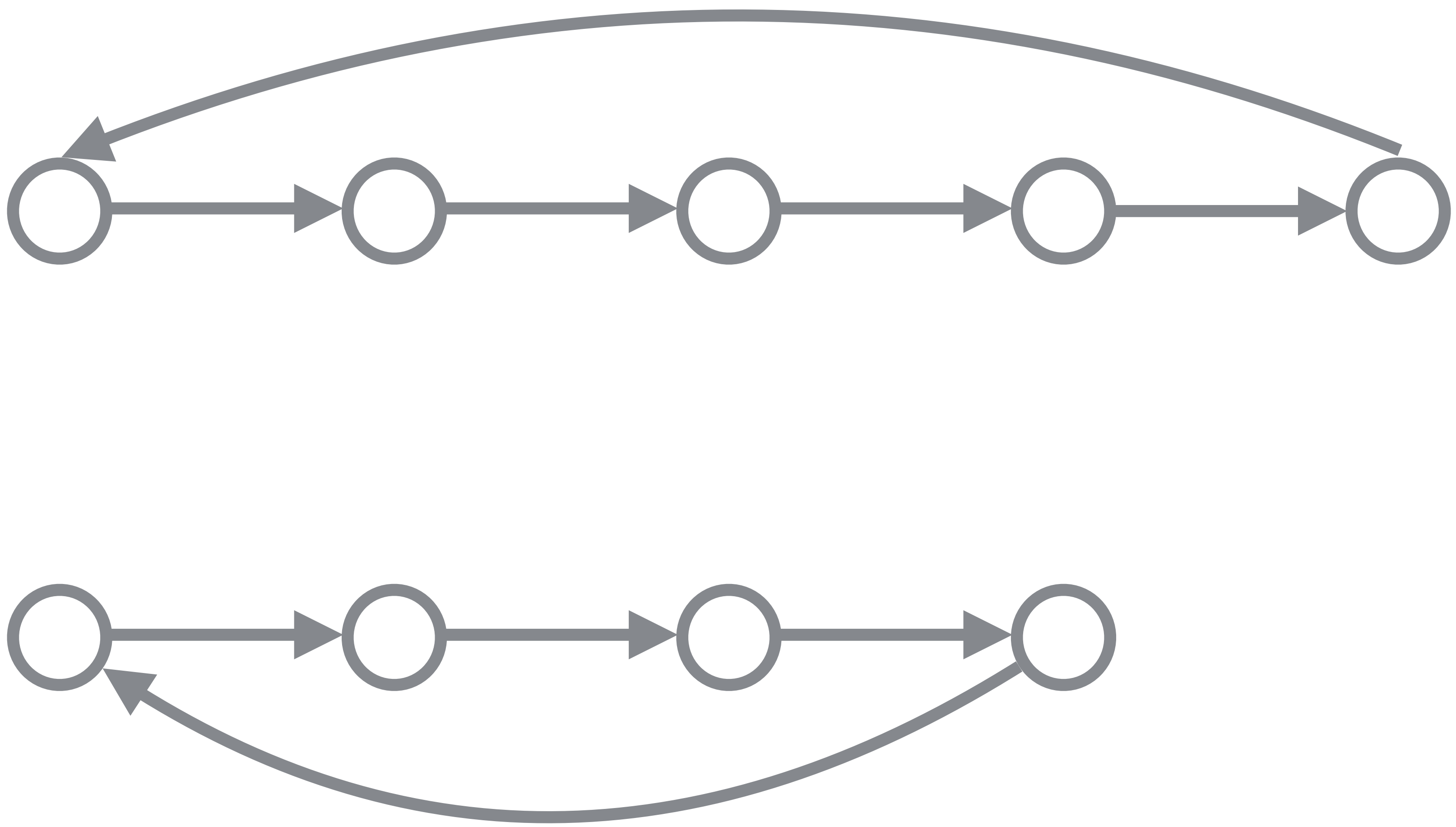

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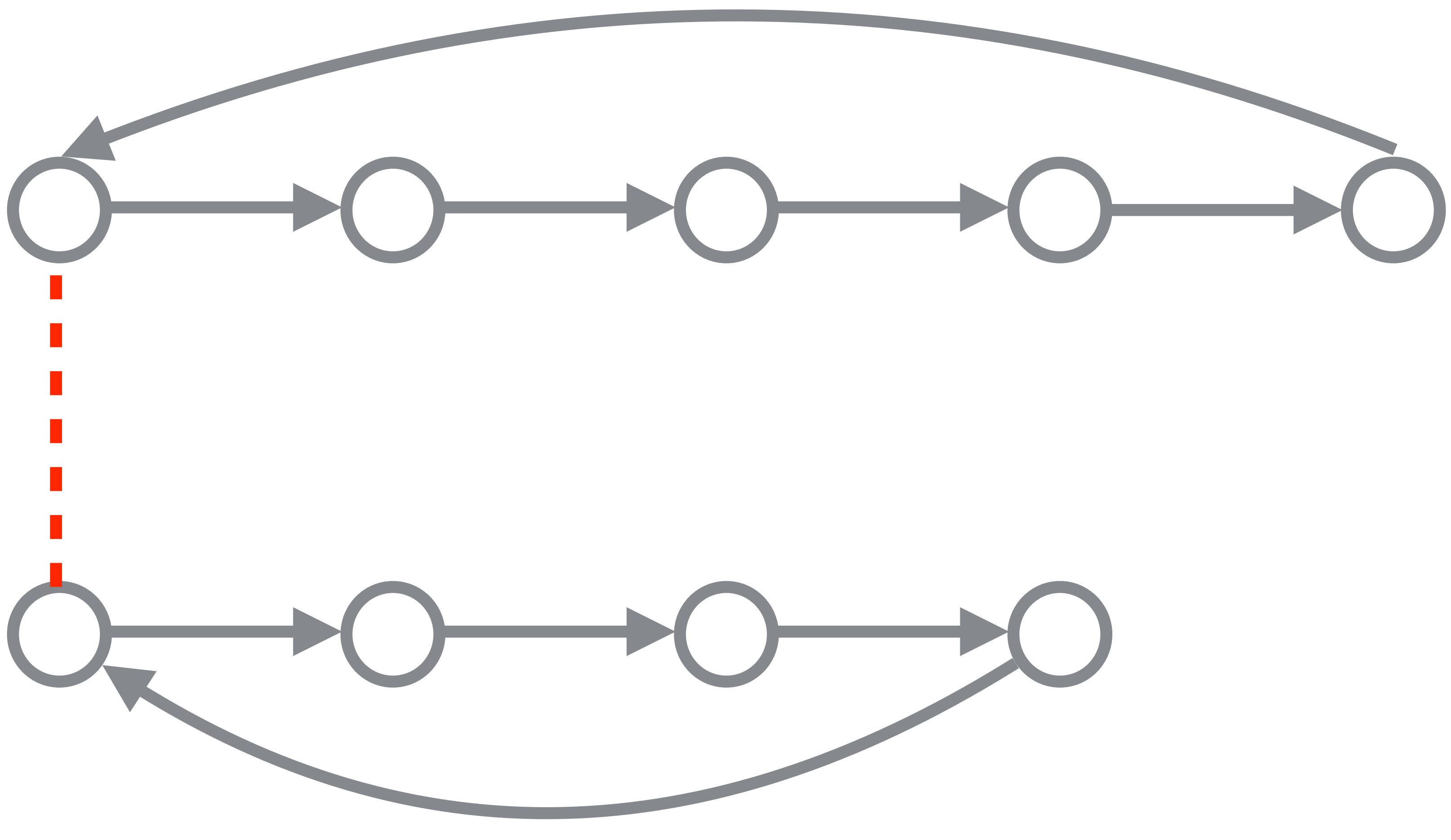

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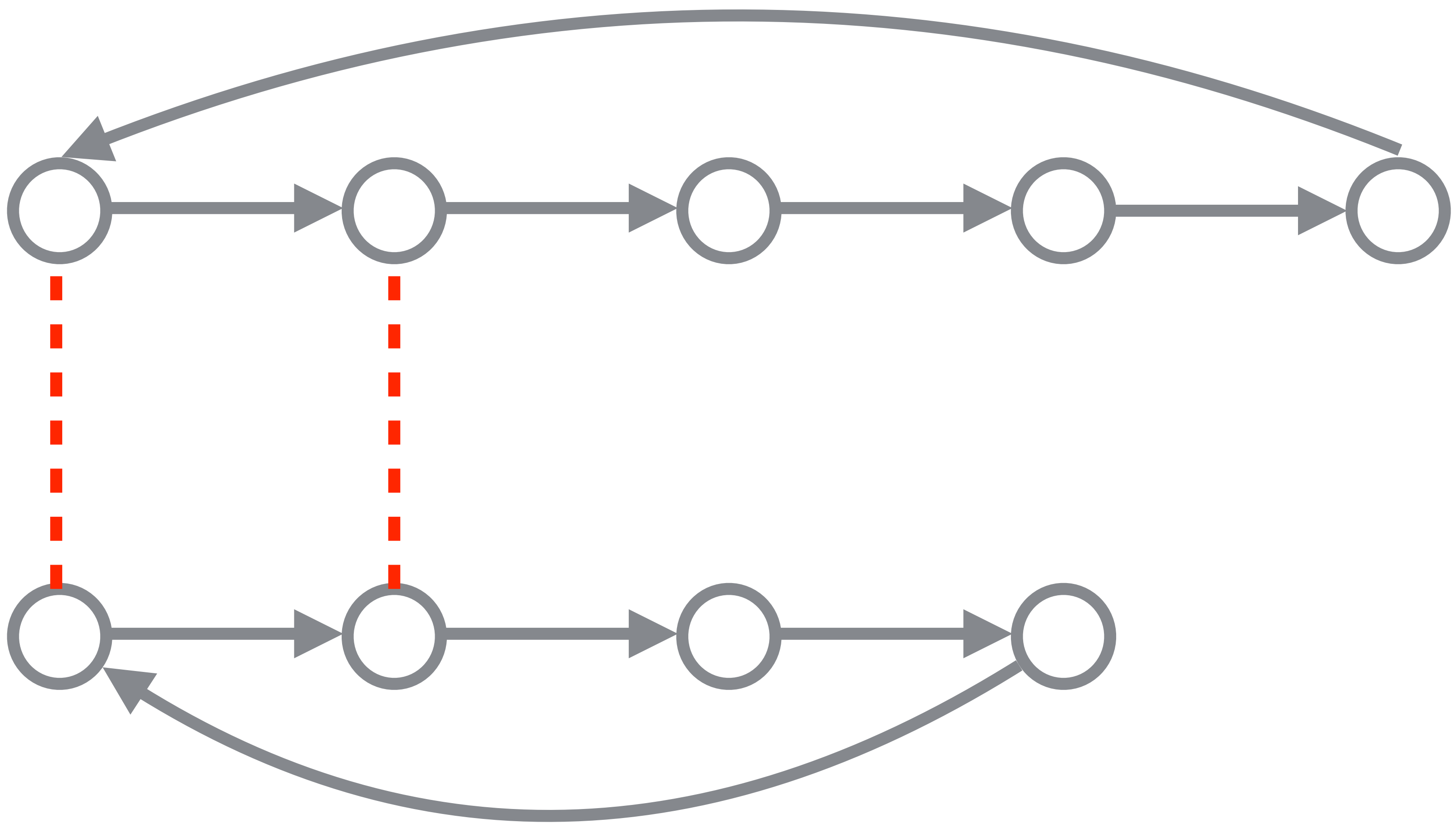

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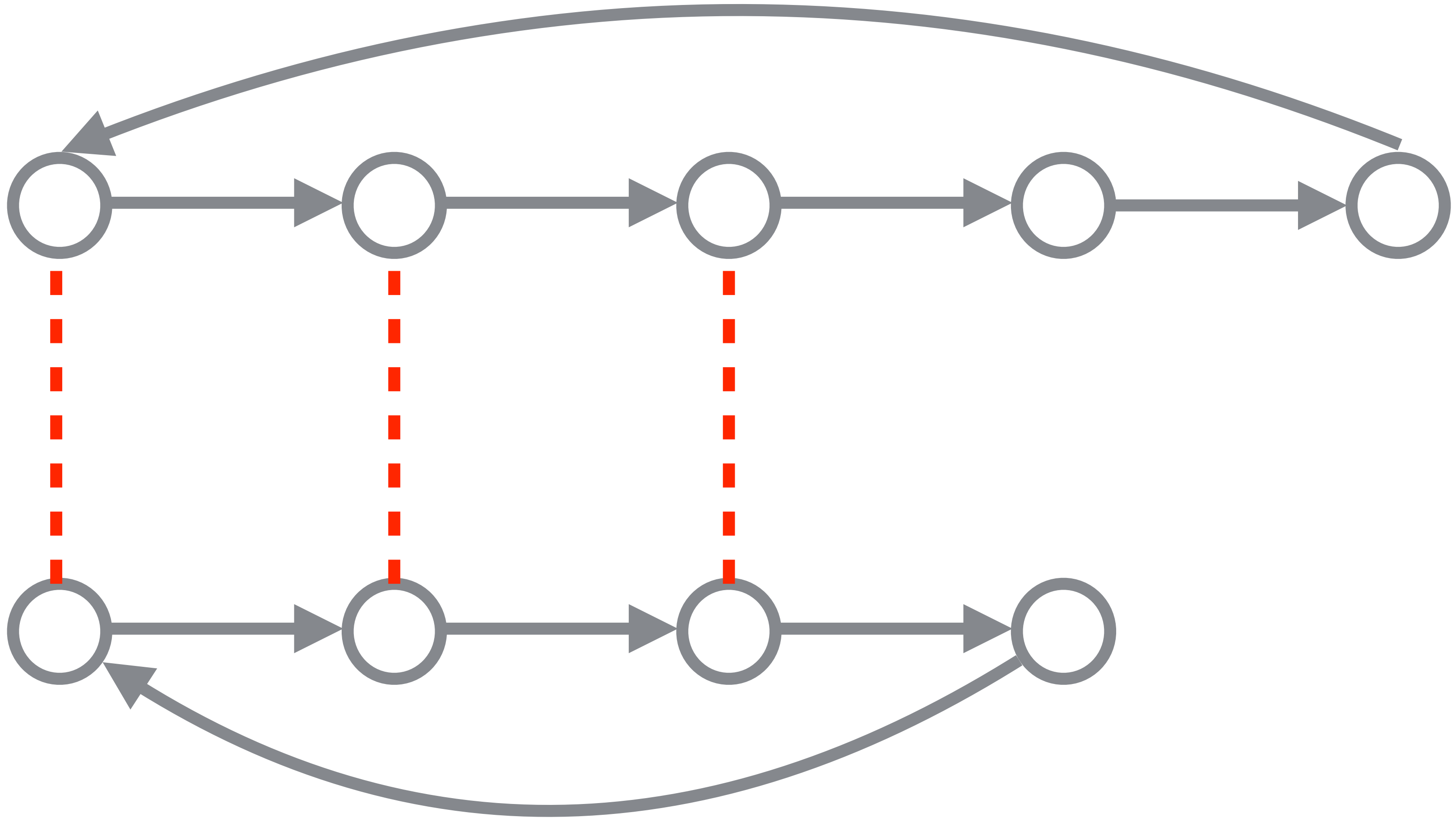

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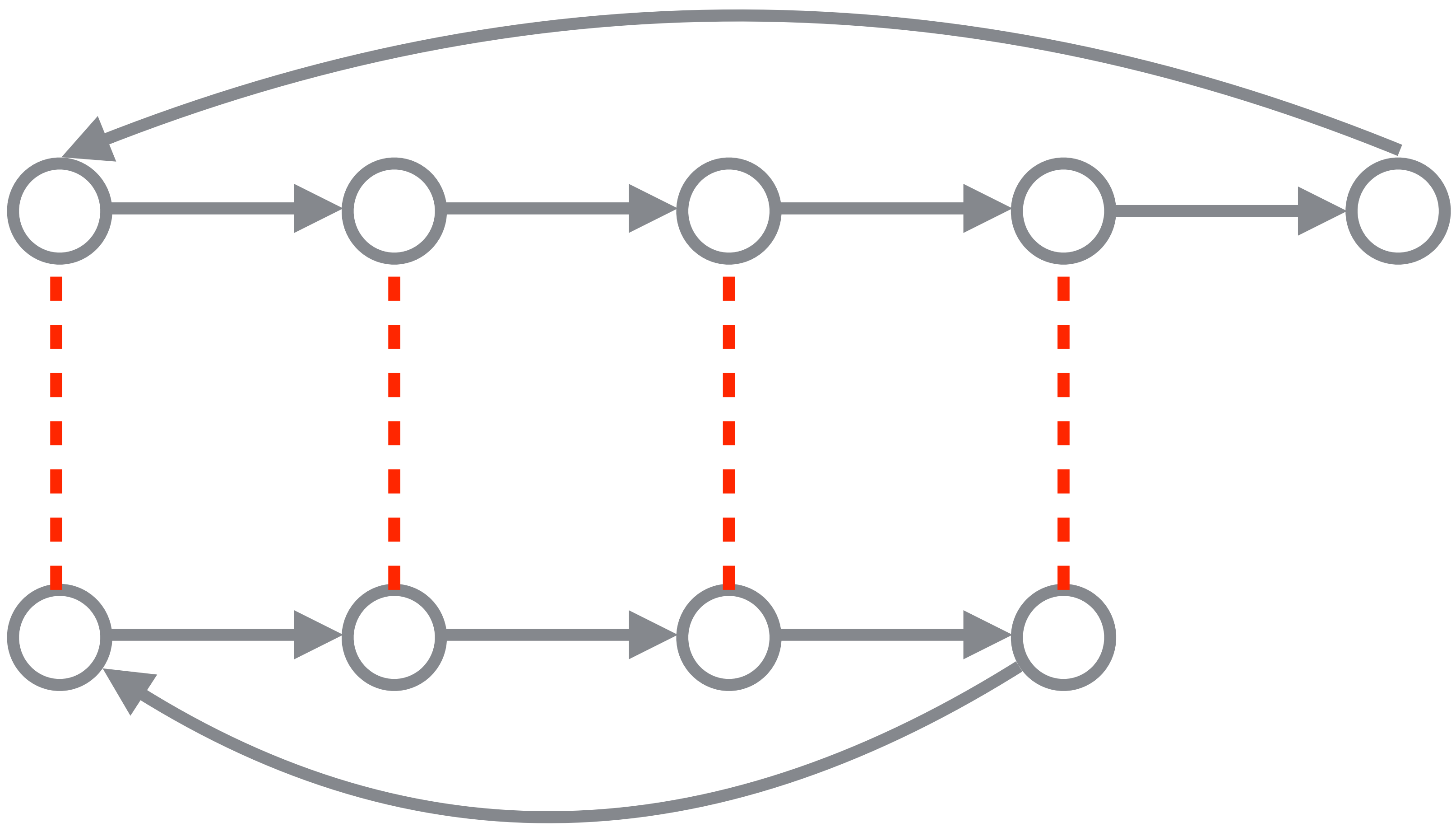

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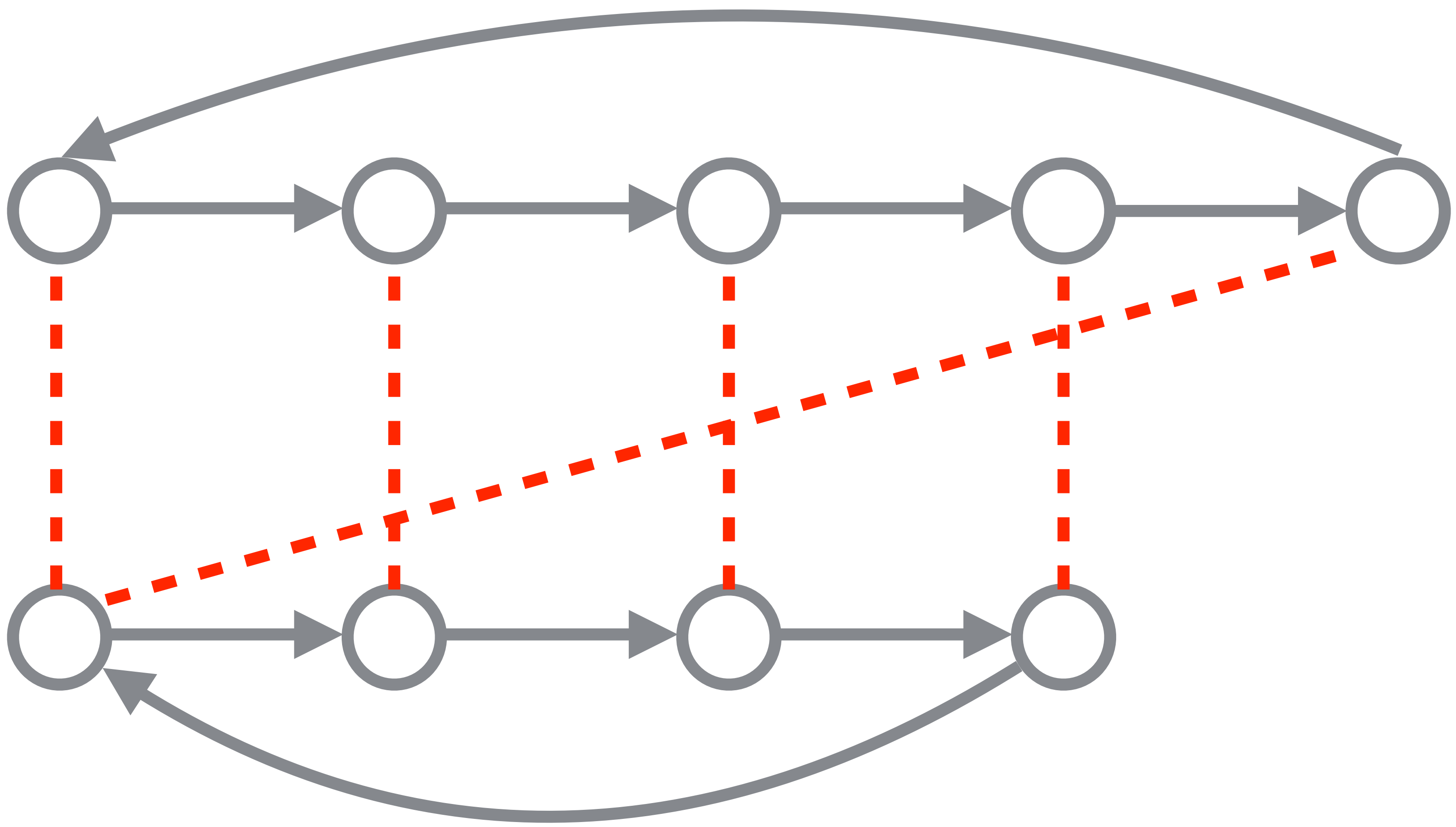

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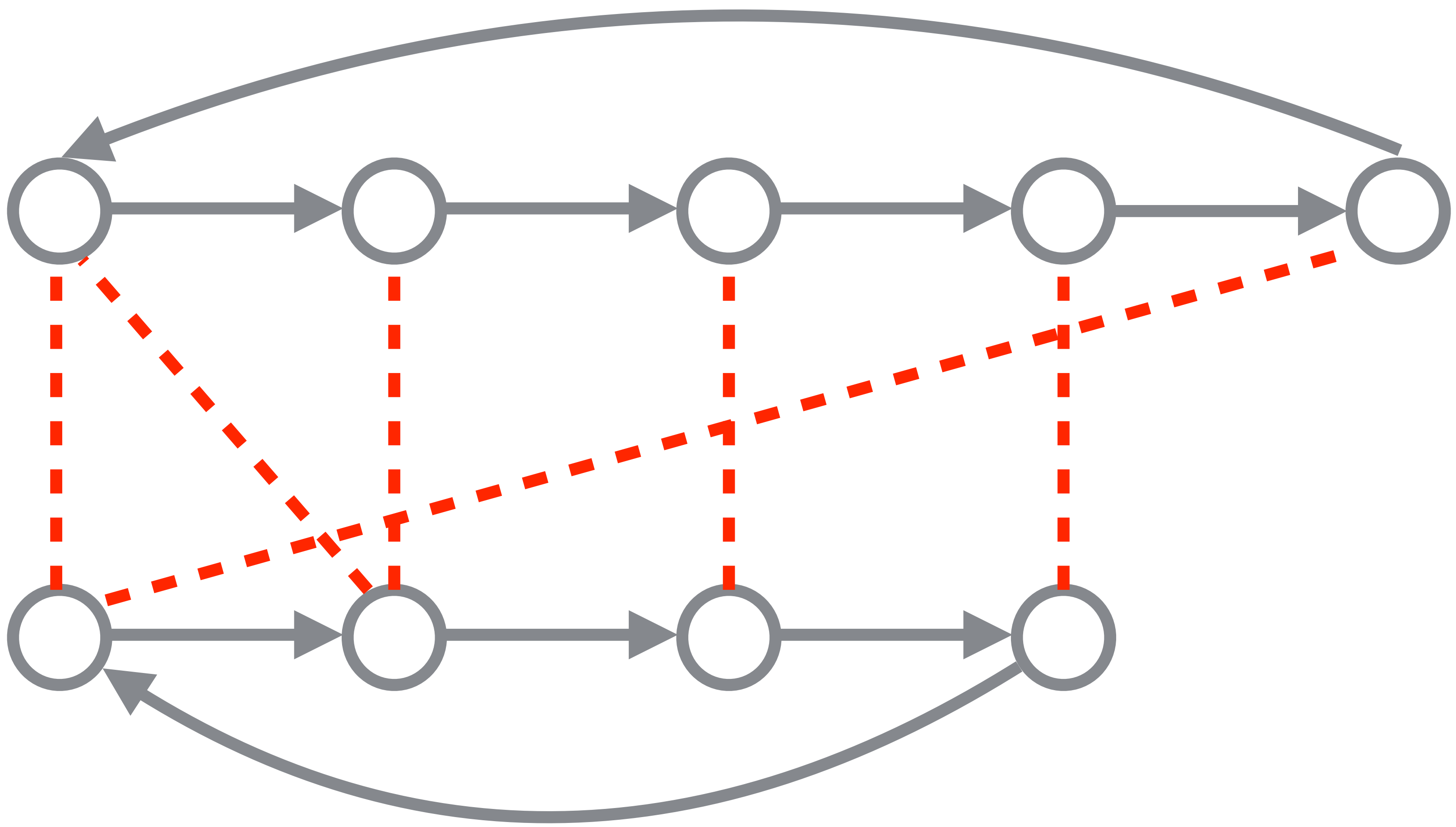

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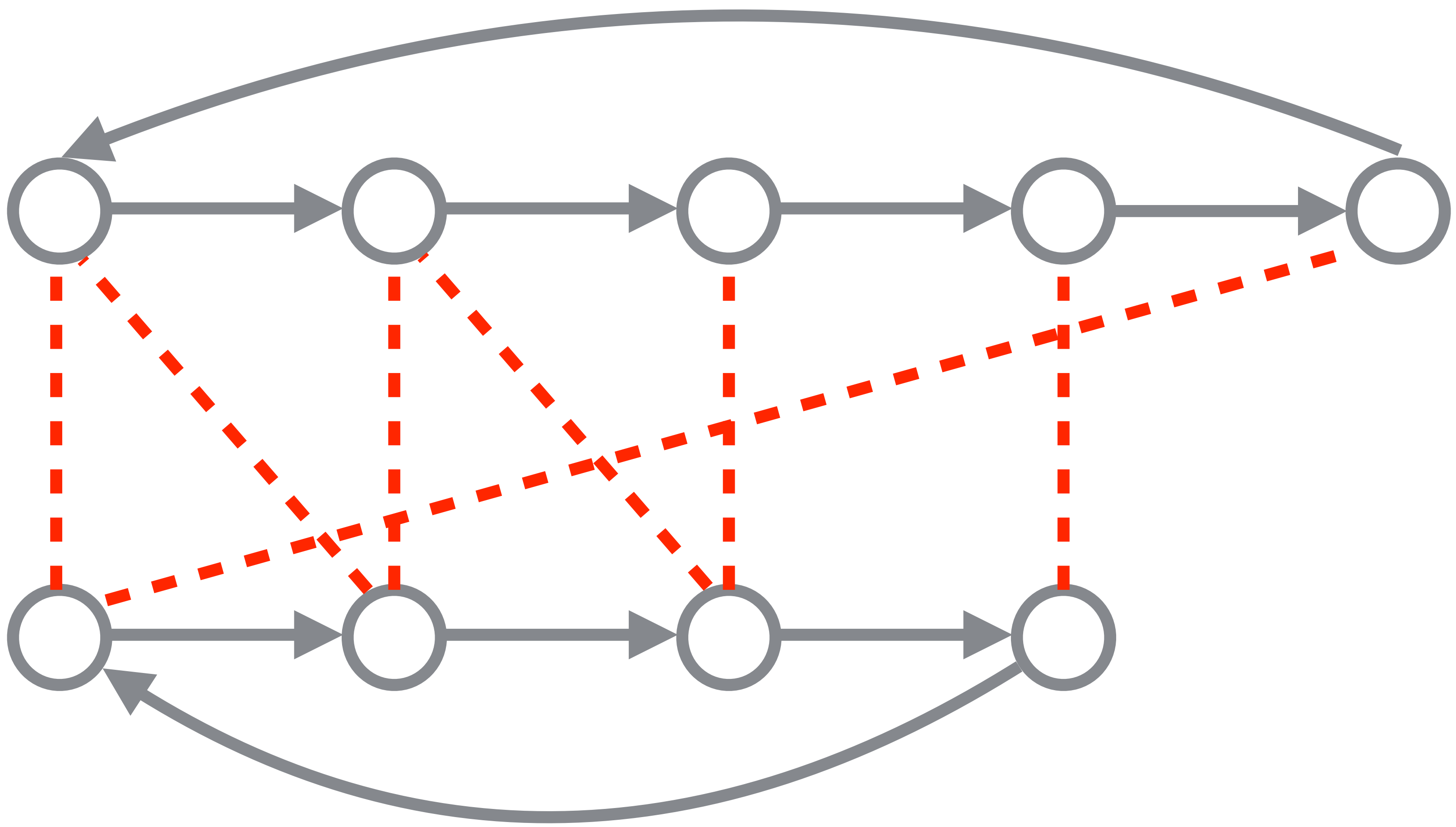

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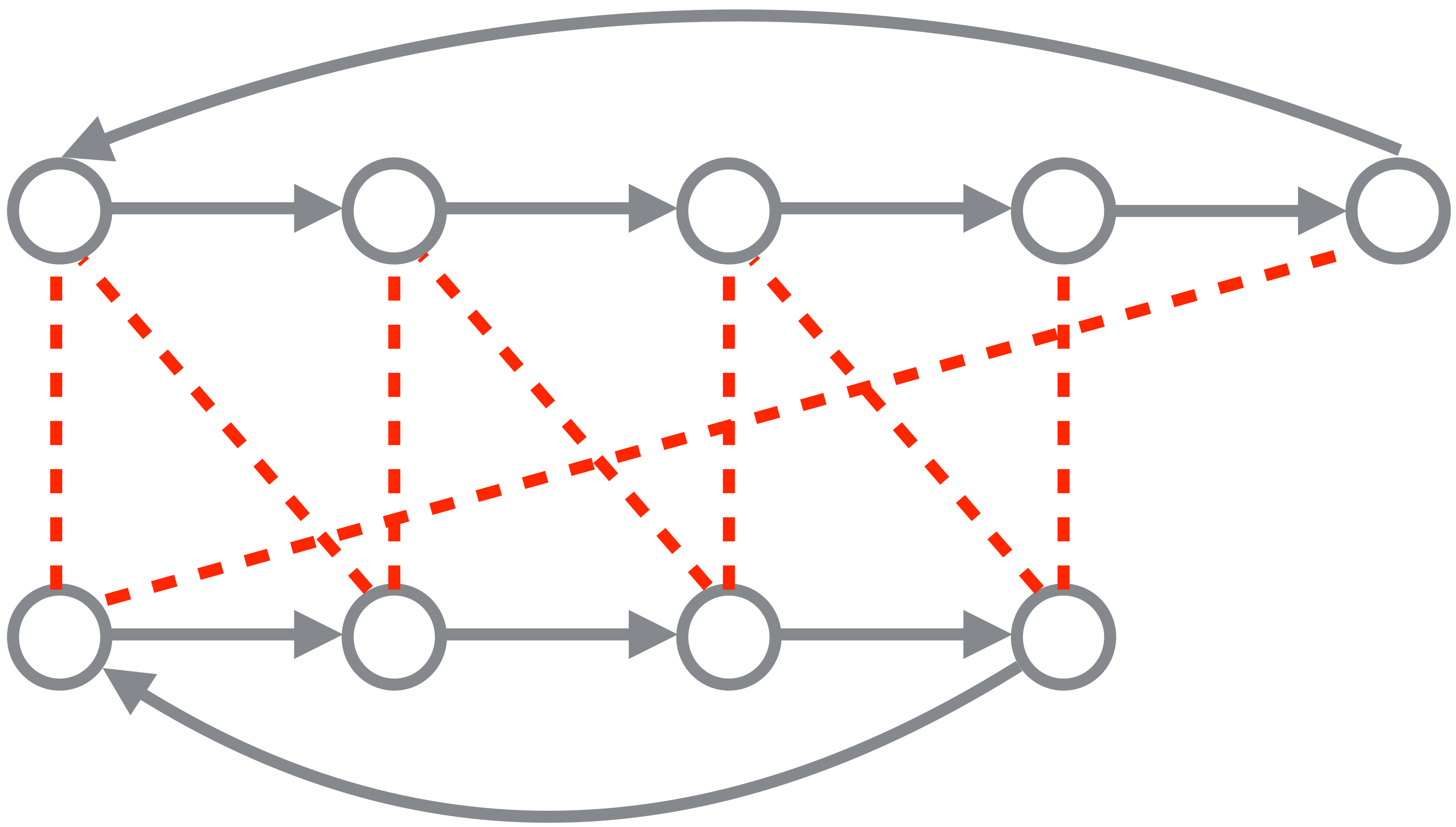

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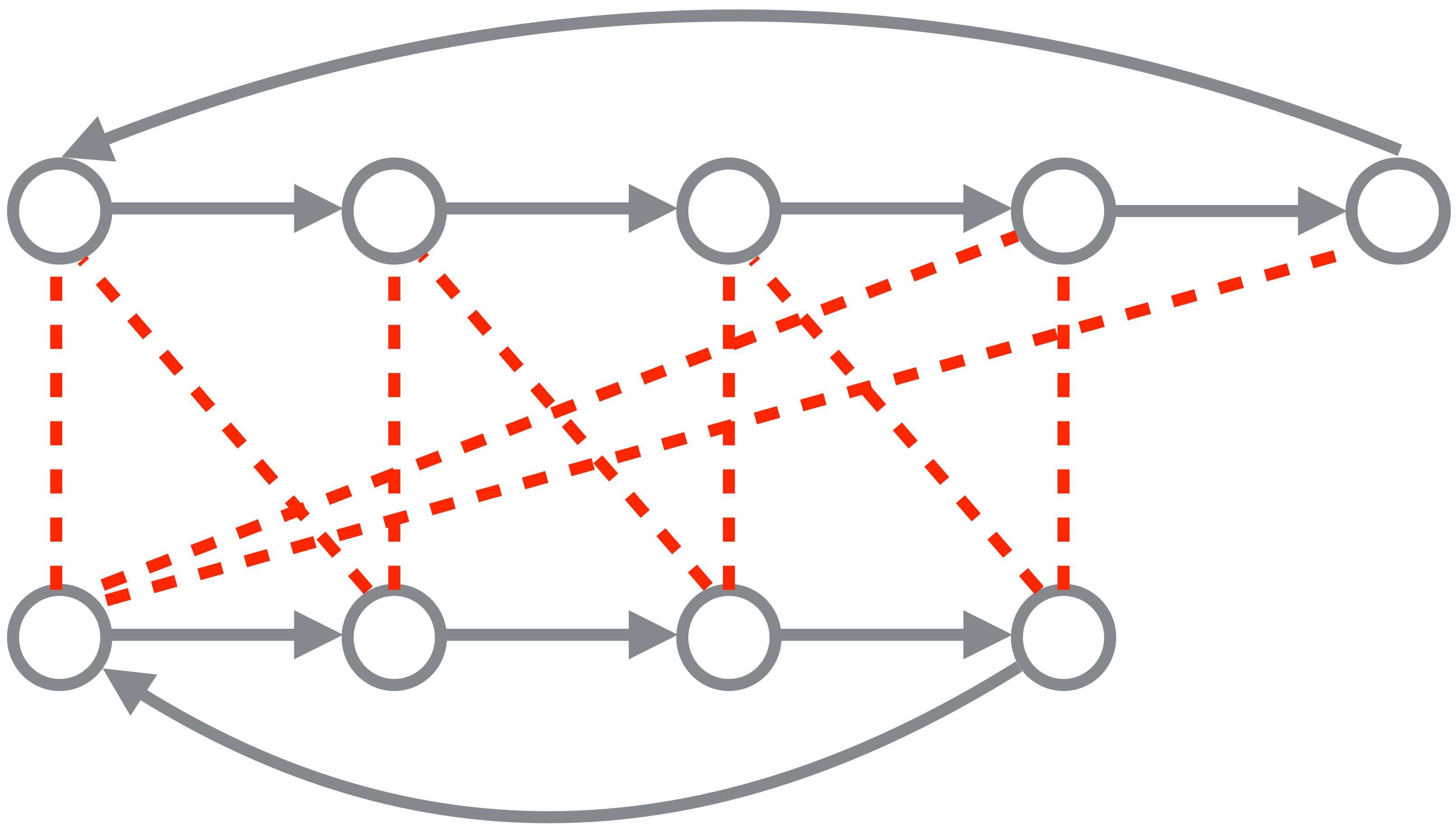

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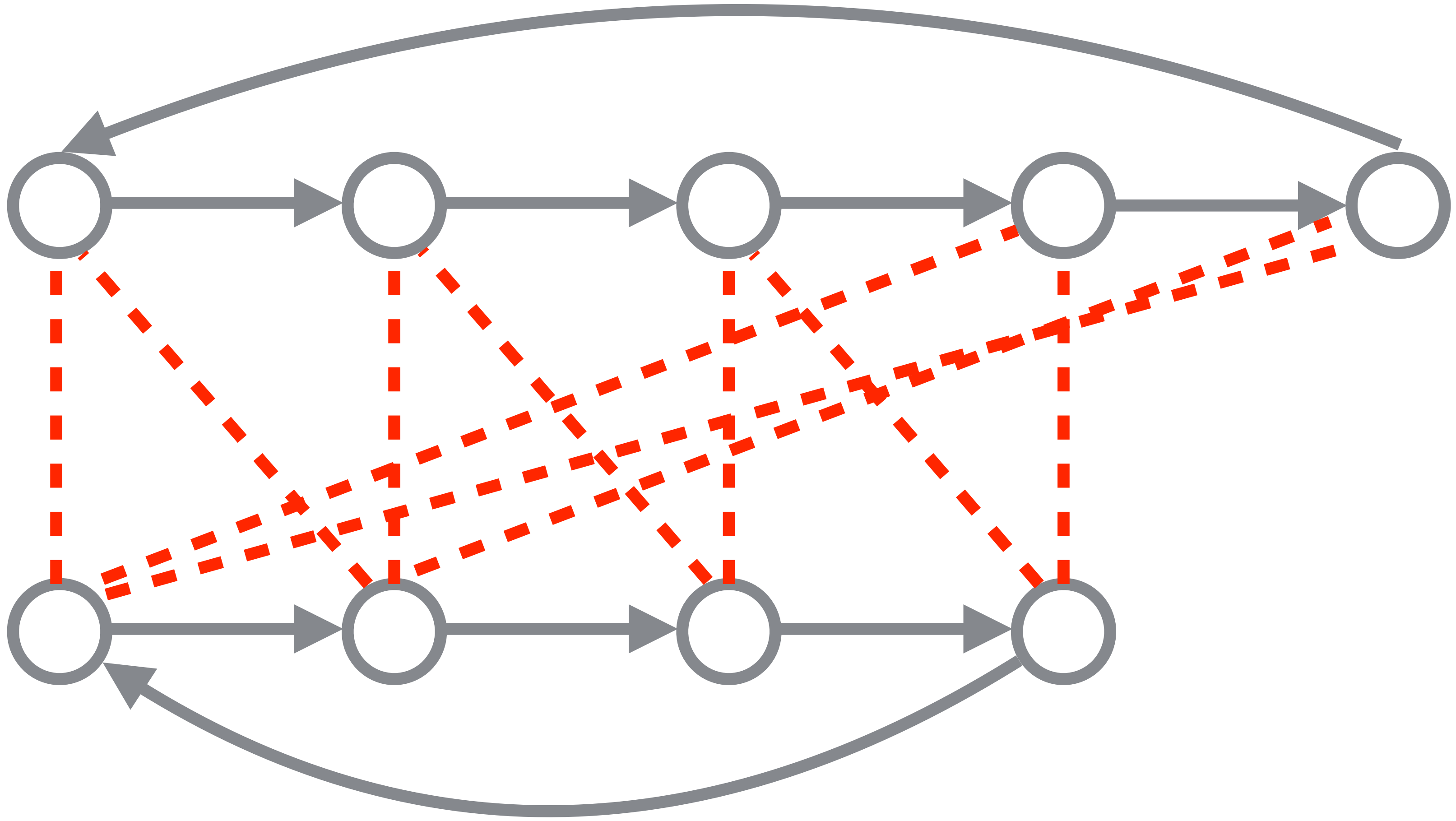

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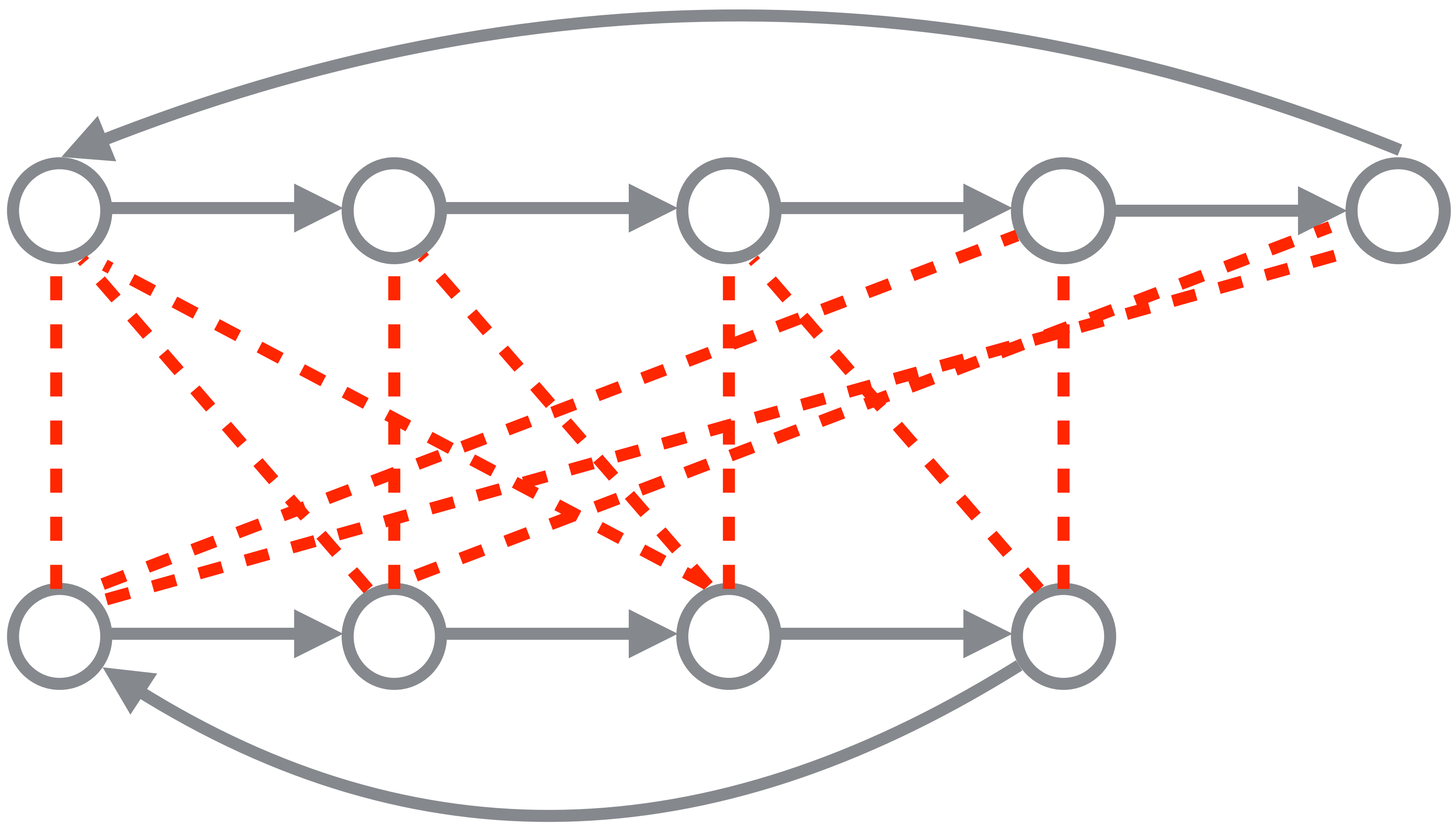

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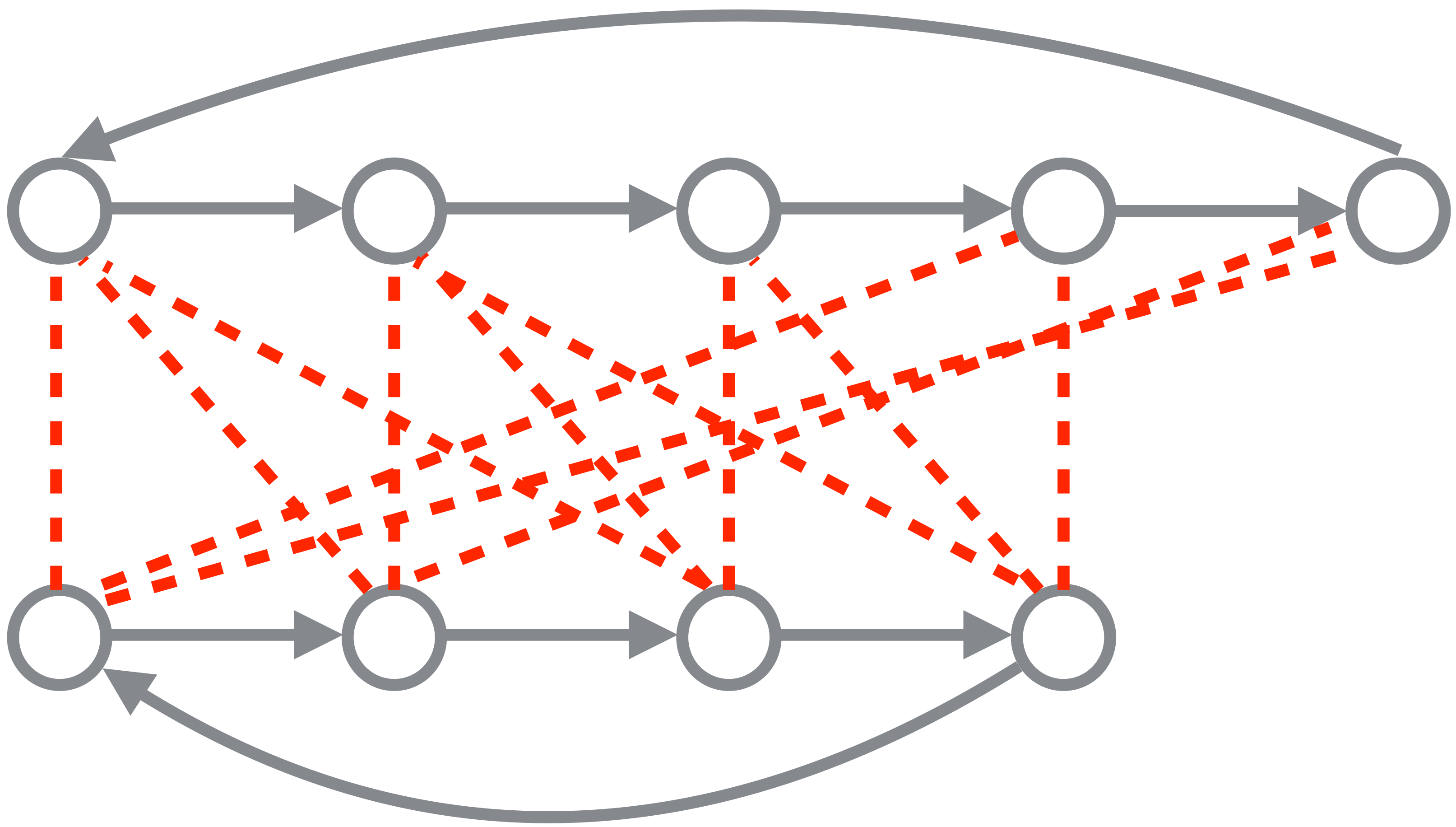

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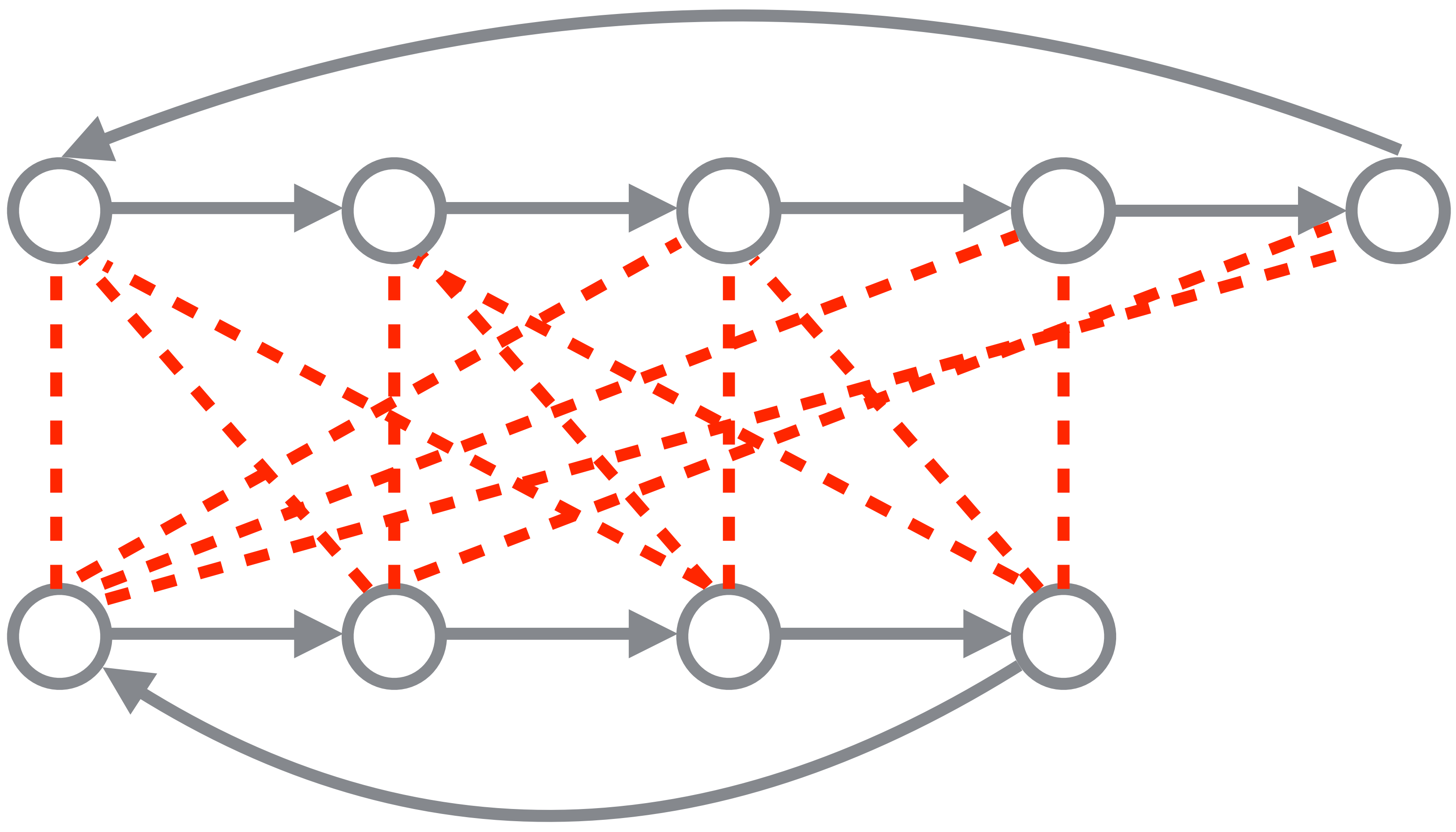

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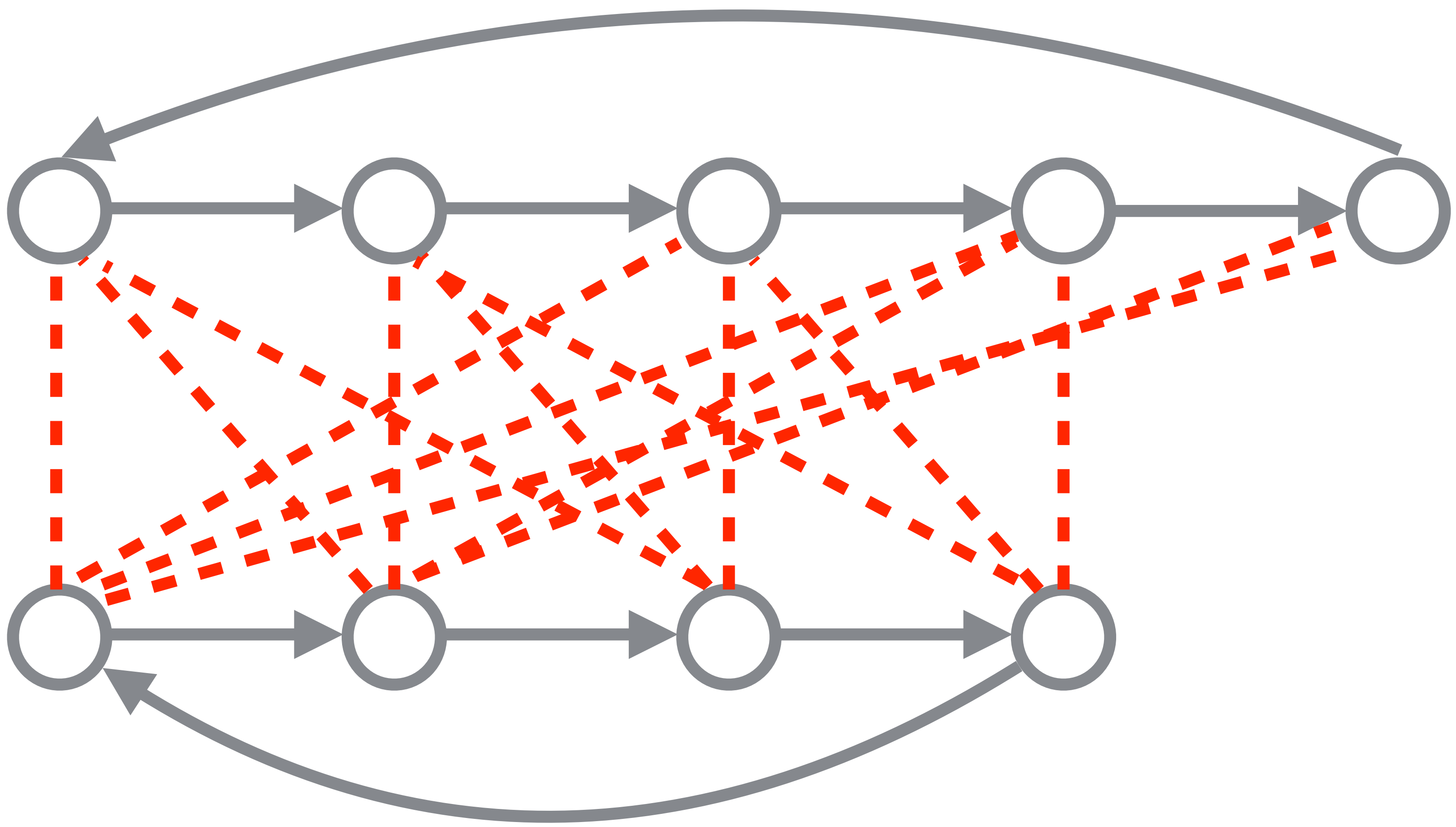

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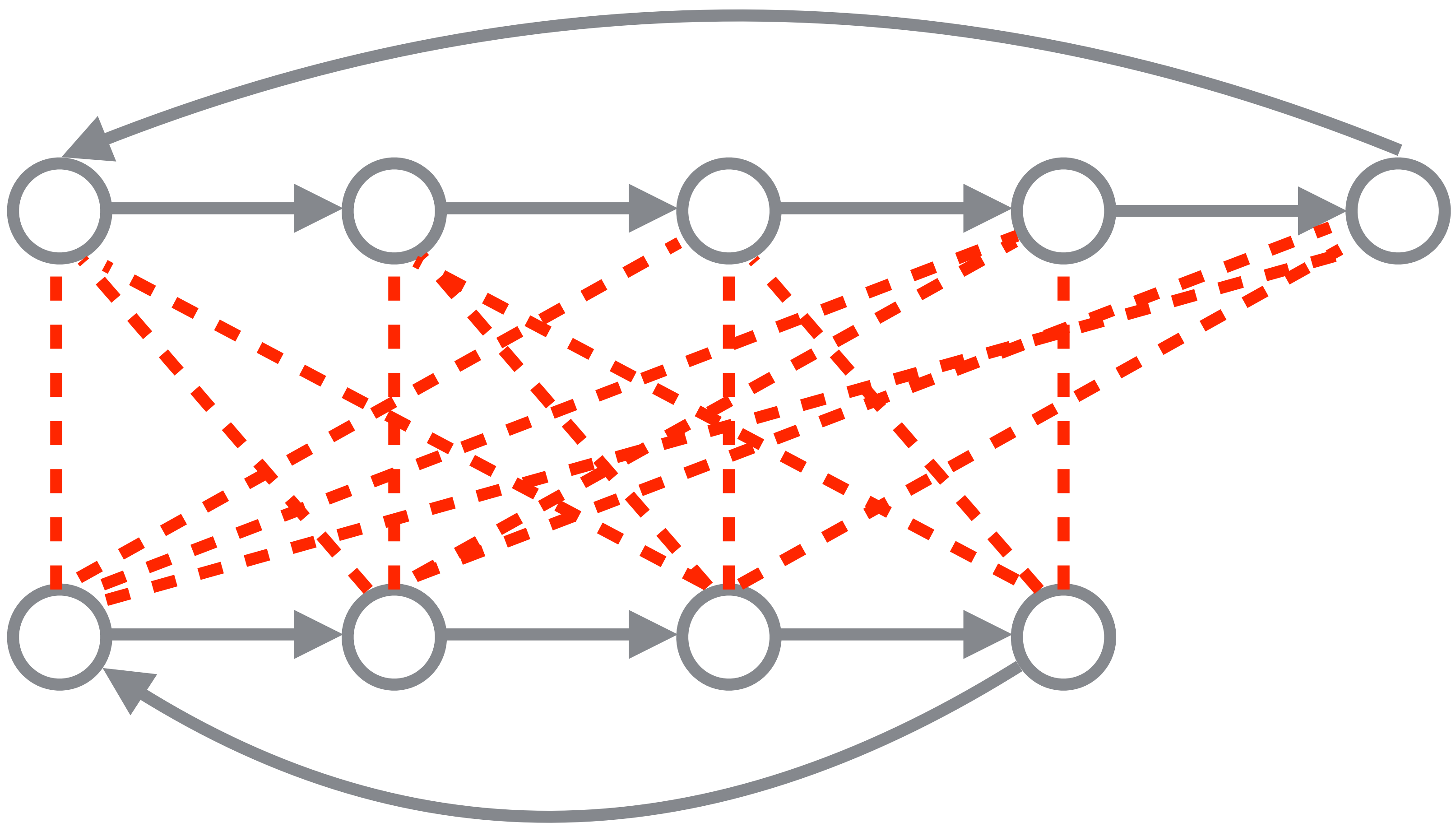

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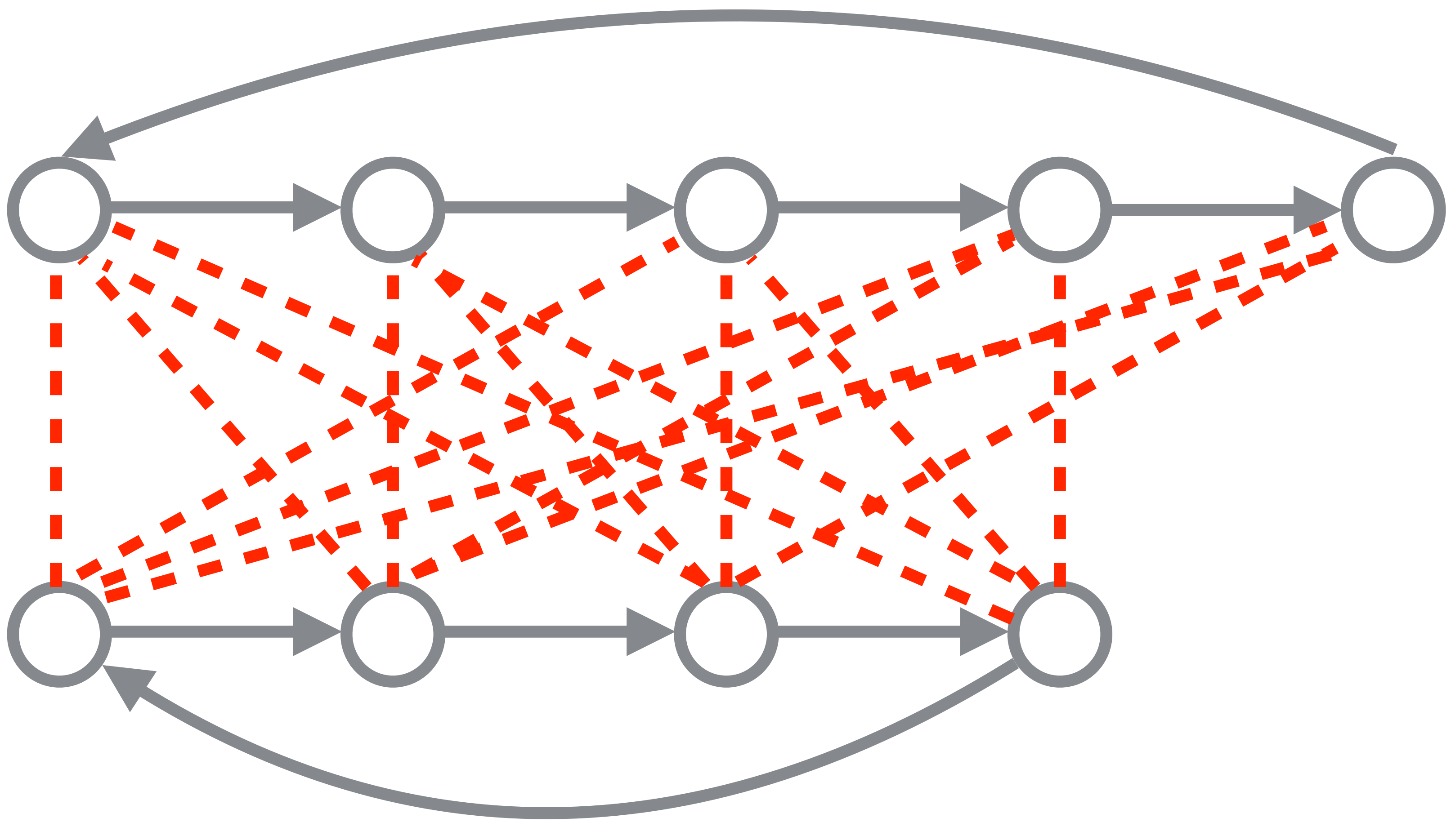

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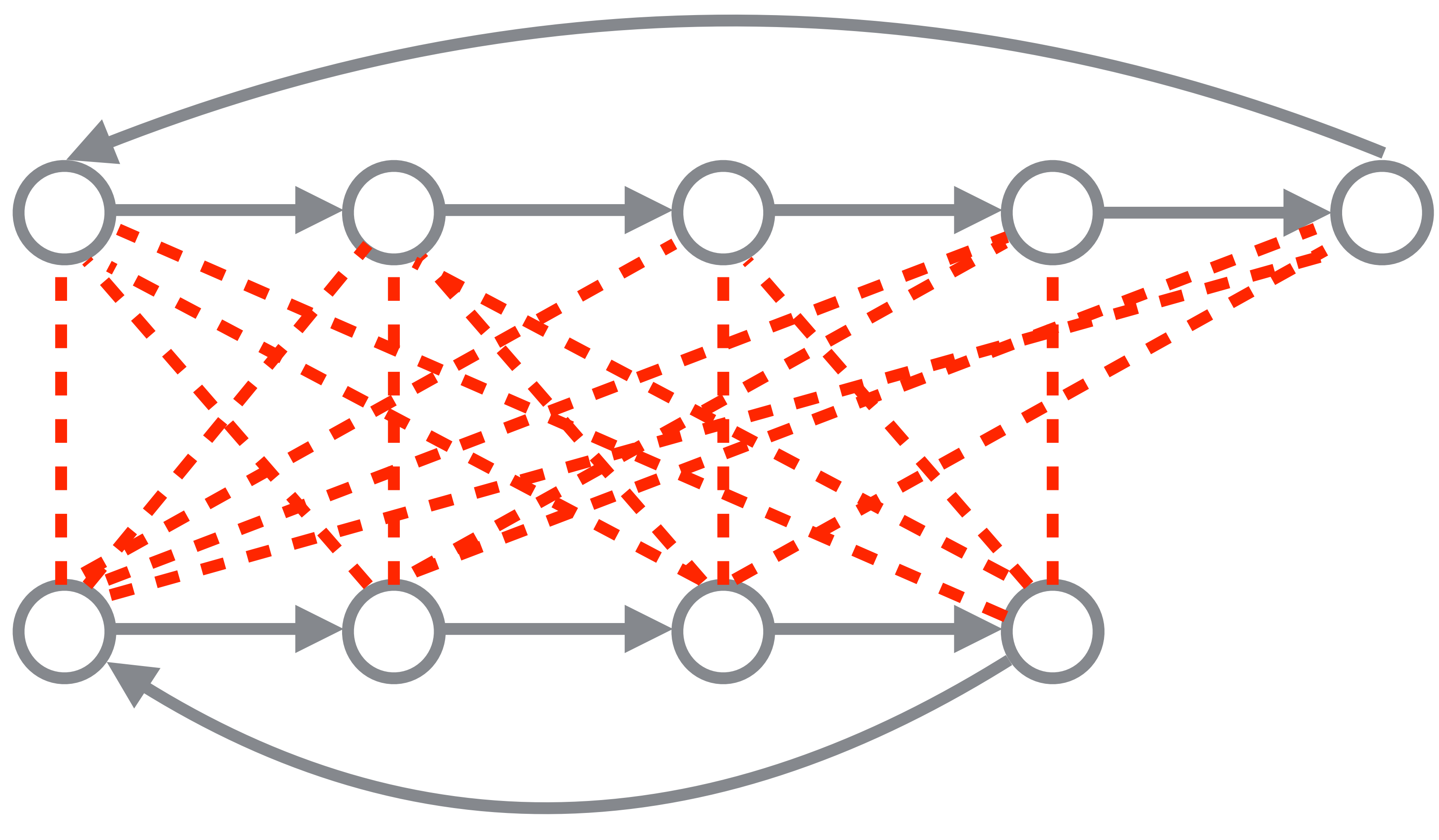

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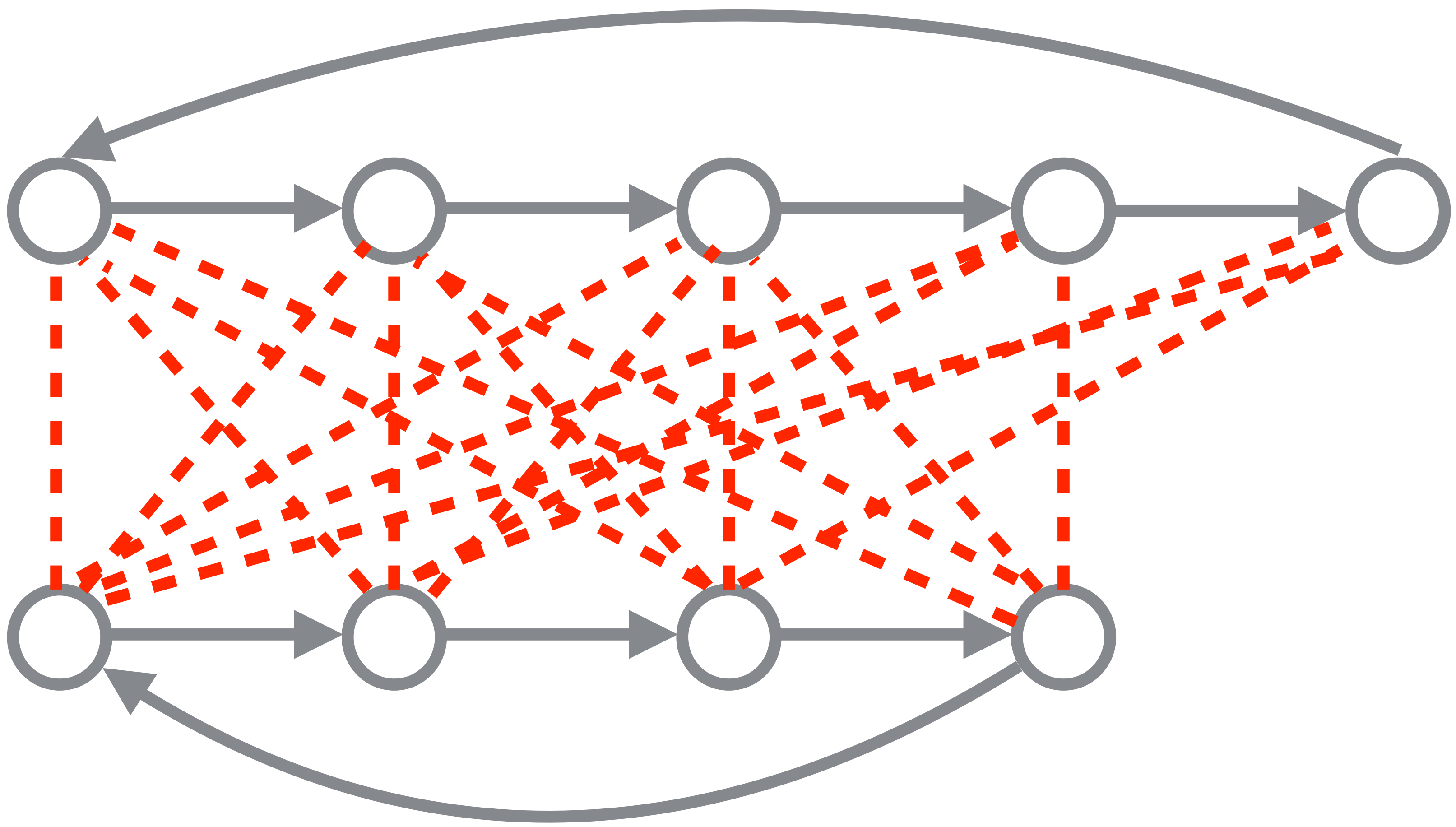

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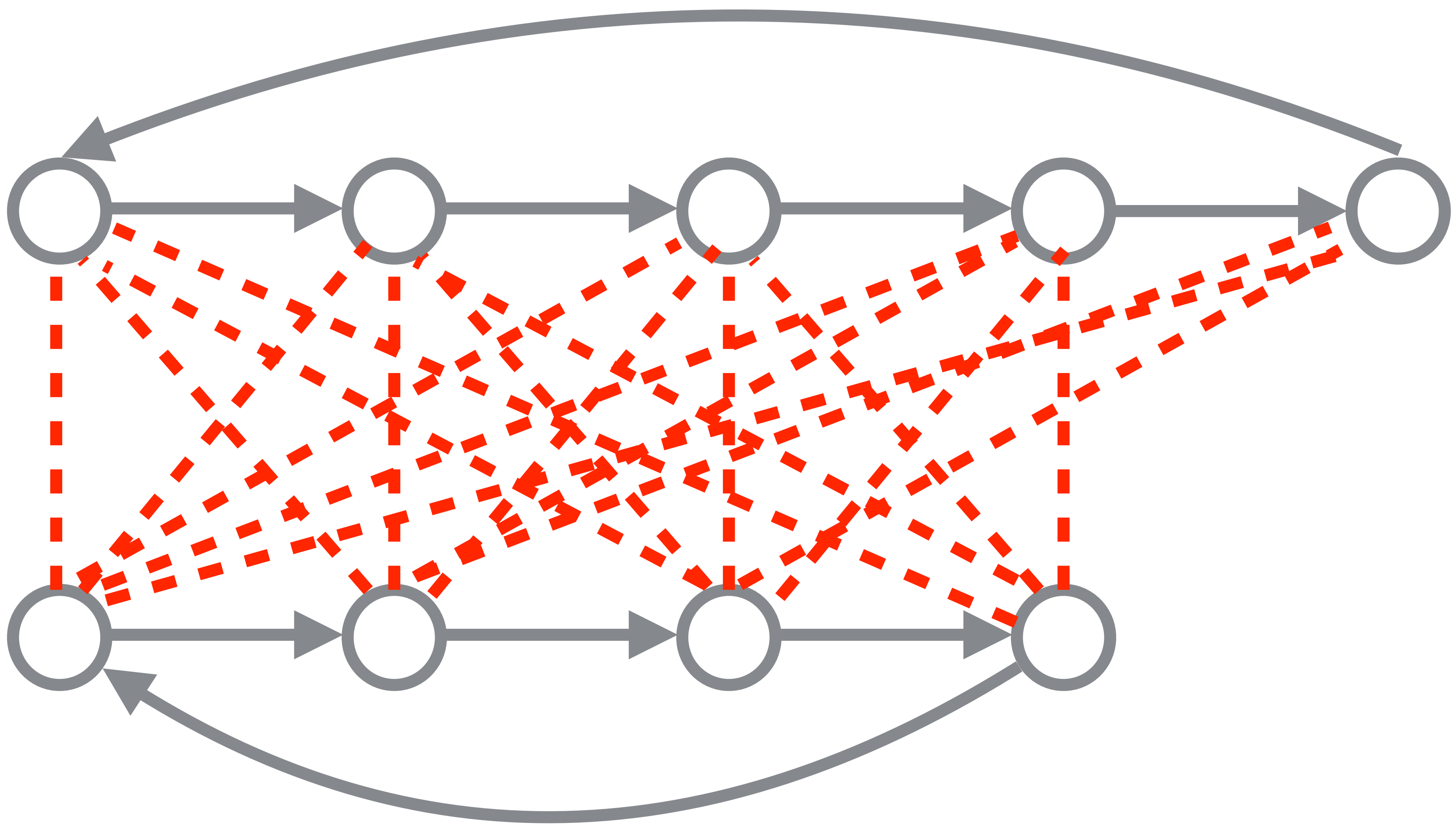

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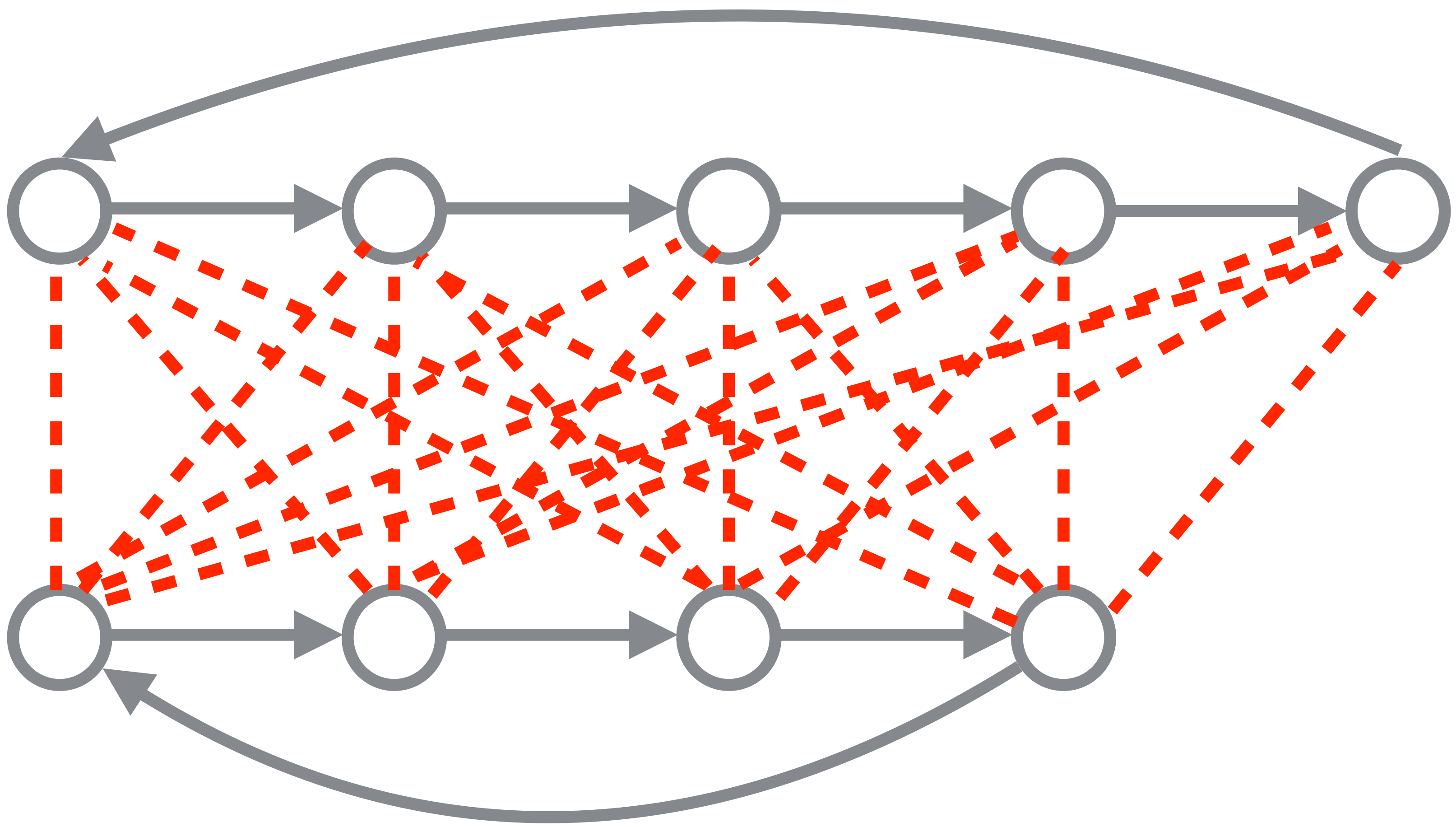

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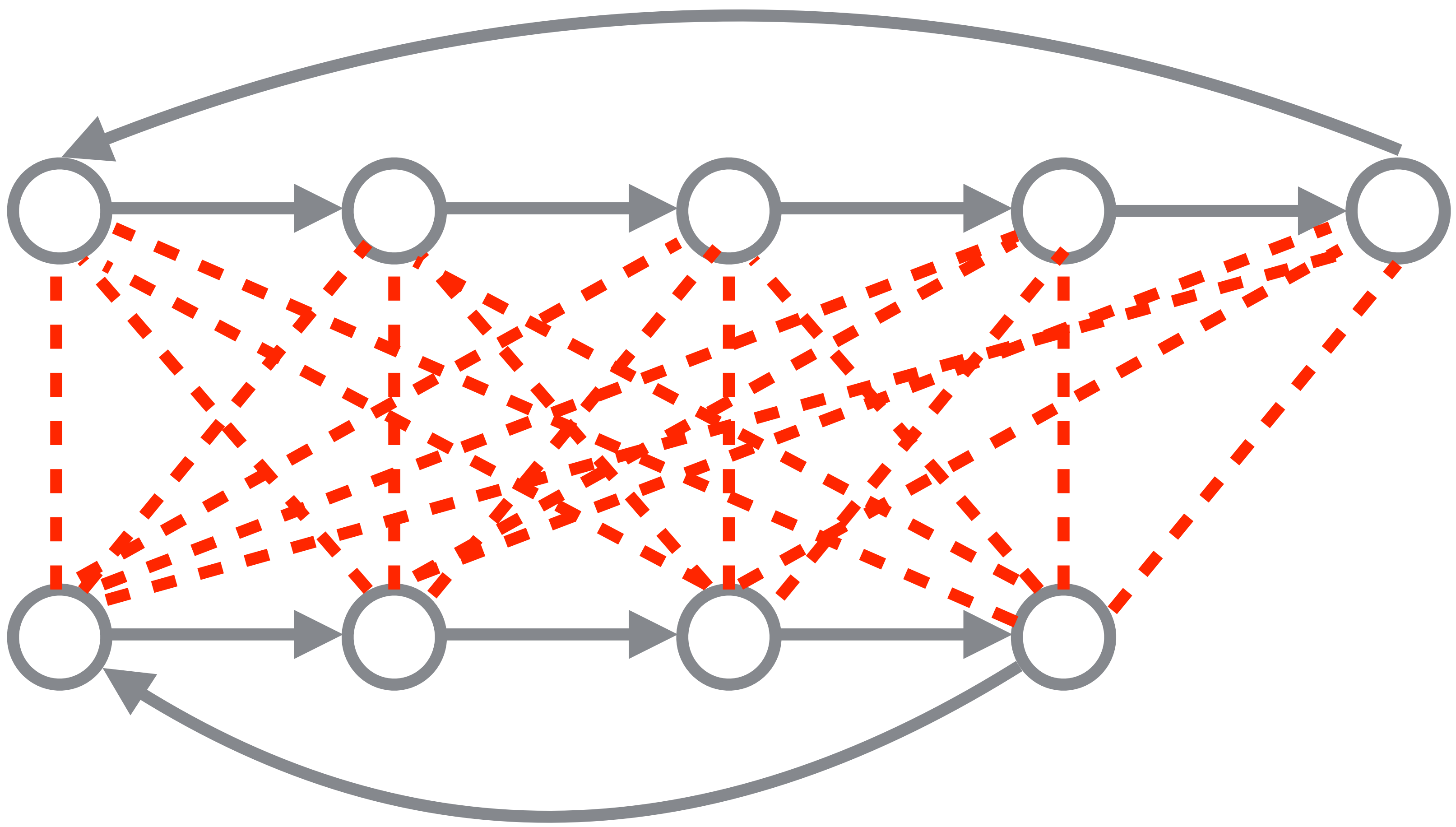

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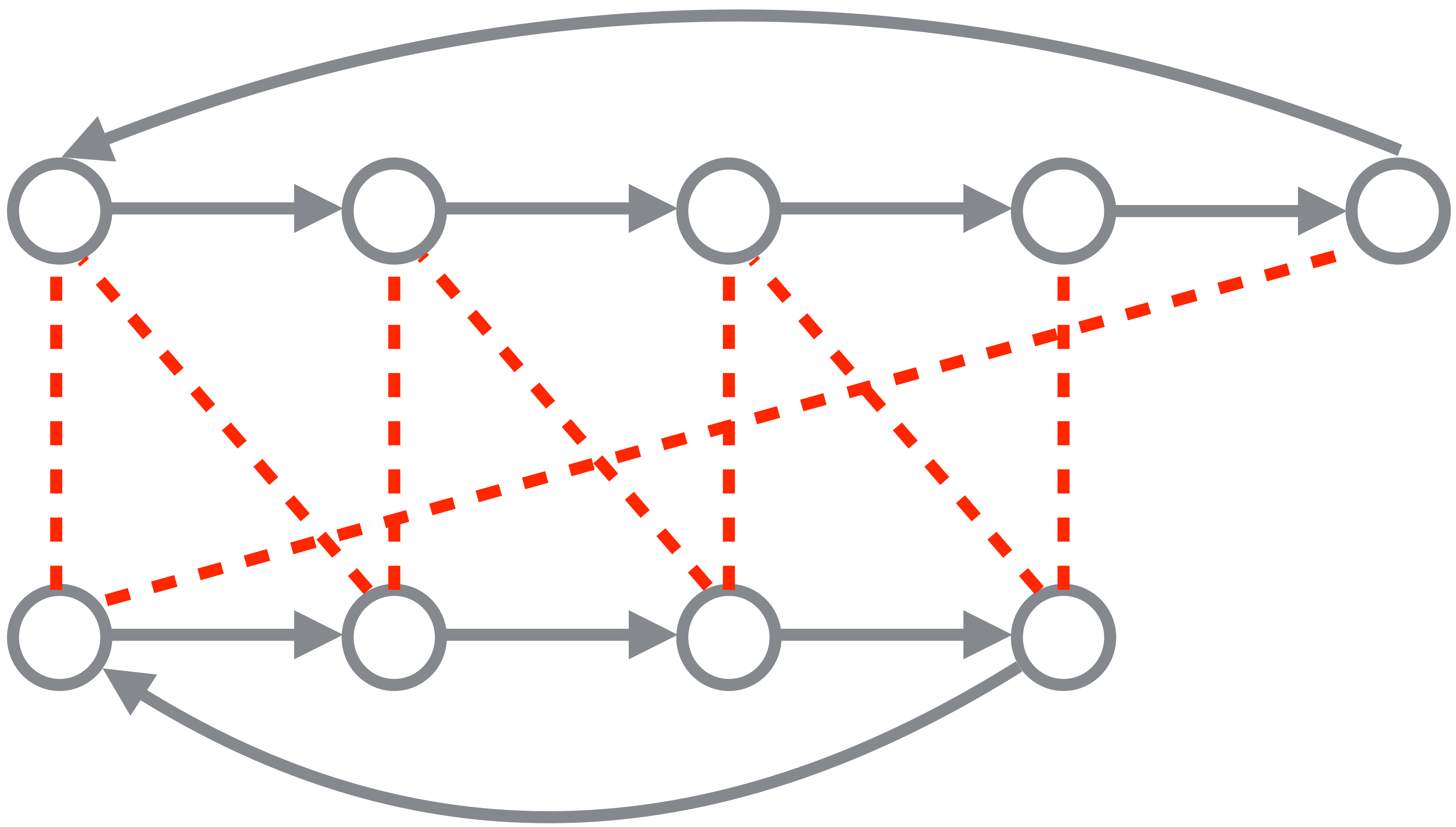

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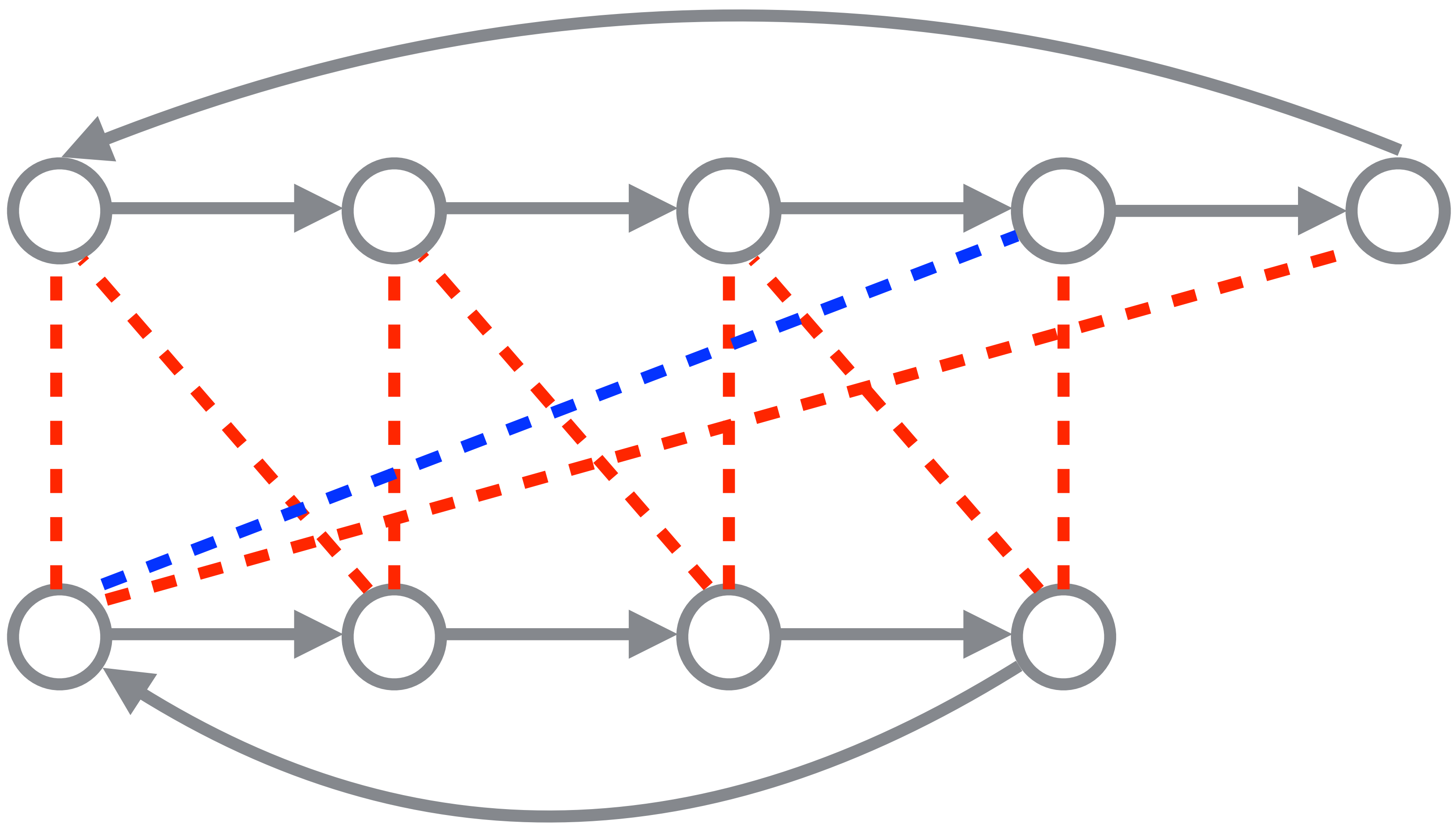

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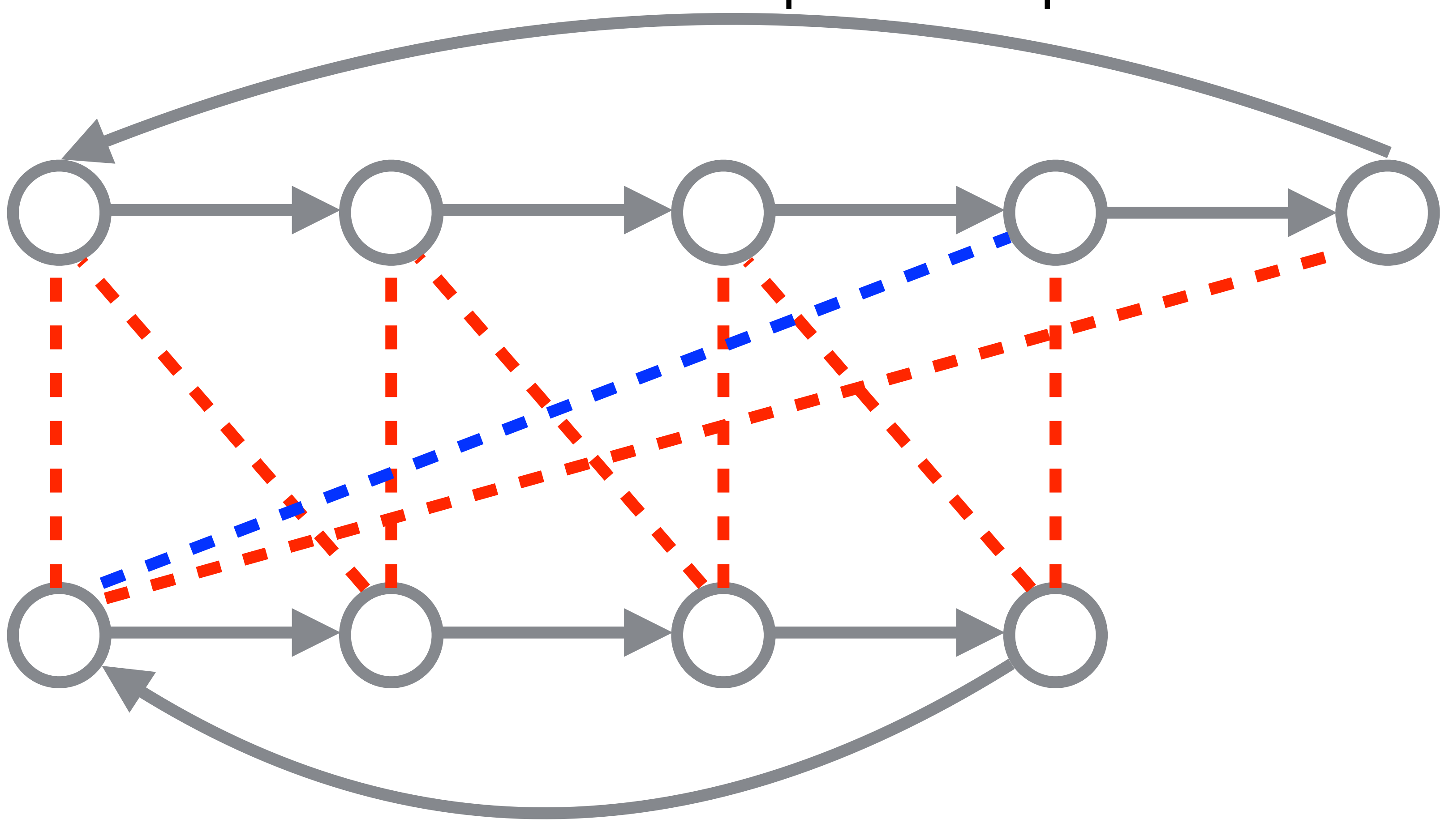

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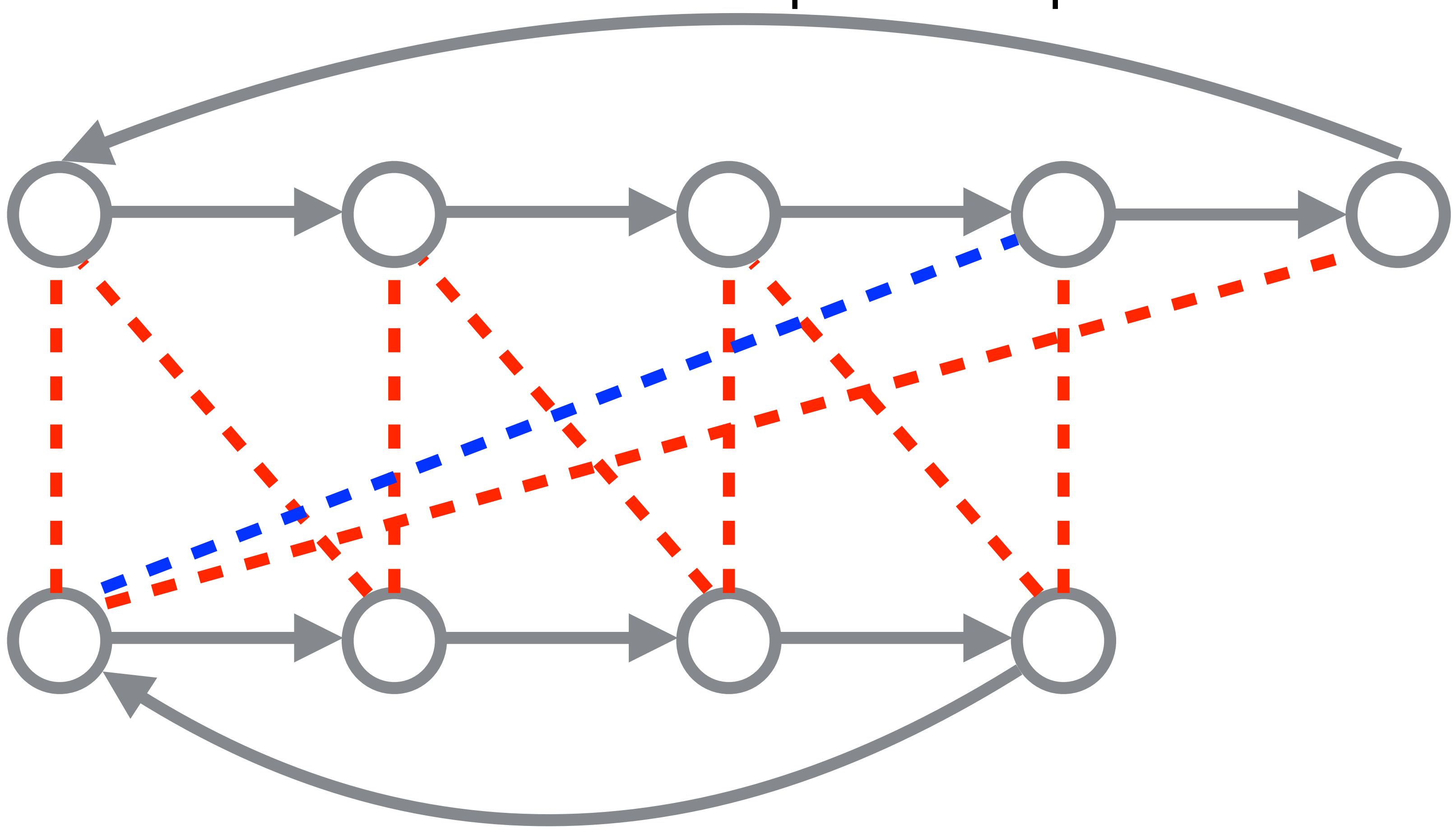

Eqv: **Rel$_X$**-->**Rel$_X$**

R $\subseteq$B(Eqv(R))

Hopcroft and Karp algorithm

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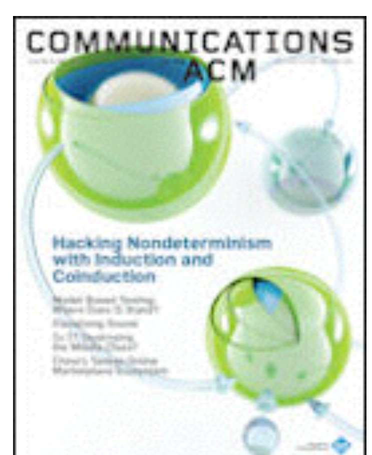

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$$\frac{}{0 \xrightarrow{a} 0} \qquad \frac{}{1 \xrightarrow{a} 0} \qquad \frac{}{a \xrightarrow{a} 1} \qquad \frac{b \neq a}{b \xrightarrow{a} 1} \qquad \frac{e \xrightarrow{a} e' \quad f \xrightarrow{a} f'}{e+f \xrightarrow{a} e'+f'}$$

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We can prove the soundness
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in a similar way, we can prove
that (RE,+,0) is a monoid

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Ctx: $\mathbf{Rel}_{\mathbf{RE}} \dashrightarrow \mathbf{Rel}_{\mathbf{RE}}$

$$\frac{e \; R \; f}{e \; Ctx(R) \; f} \qquad \frac{}{0 \; Ctx(R) \; 0} \qquad \frac{}{1 \; Ctx(R) \; 1} \qquad \frac{}{a \; Ctx(R) \; a}$$

$$\frac{e \; Ctx(R) \; e' \quad f \; Ctx(R) \; f'}{e+f \; Ctx(R) \; e'+f'} \qquad \frac{e \; Ctx(R) \; e' \quad f \; Ctx(R) \; f'}{ef \; Ctx(R) \; e'f'}$$

$$\frac{e \; Ctx(R) \; f}{e^* \; Ctx(R) \; f^*}$$

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$$

$$e'(f+g)+o(e)(f'+g') \; Ctx(R) \; (e'f+e'g)+(o(e)f'+o(e)g') \sim (e'f+o(e)f')+(e'g+o(e)g')$$

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$R \subseteq B(Bhv(Ctx(R)))$

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Arden's rule

Given two regular expressions k and m, the equation

$$e \sim ke+m$$

has solution $e=k^*m$, i.e., $k^*m \sim kk^*m + m$

Moreover:

1. $k\not\downarrow \Rightarrow k^*m$ is the *unique* solution, i.e., $f \sim kf+m \Rightarrow f \sim k^*m$
2. k^*m is the *smallest* solution, i.e., $f \sim kf+m \Rightarrow k^*m \lesssim f$

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2. k^*m is the *smallest* solution, i.e., $f \sim kf+m \Rightarrow k^*m \lesssim f$

language inclusion ($\lesssim$) is $\nu B'$

B':**Rel$_X$**-->**Rel$_X$** is defined as

$B'(R)=\{(x,y) \mid o(x) \leq o(y)$ and for all $a \in A$ $t(x)(a)\ R\ t(y)(a)\}$

Arden's rule

To show $f \sim kf+m \Rightarrow k^*m \lesssim f$

We prove that
$S = \{ (k^*m,f) \mid f \sim kf+m \}$
is a simulation up-to

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1. $k^*m\downarrow \Rightarrow m\downarrow \Rightarrow kf+m\downarrow \Rightarrow f\downarrow$
2. $k^*m \xrightarrow{a}$ ... $f \xrightarrow{a}$

$(k'k^*)m + 1m' \sim k'(k^*m) + m' \; Ctx(S) \; k'f+m' \lesssim (k'f + o(k)f') + m' \sim f'$

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$S \subseteq B'(Slf(Ctx(S)))$

Slf: $\mathbf{Rel}_{\mathbf{RE}} \text{-->} \mathbf{Rel}_{\mathbf{RE}}$

$Slf(S) = \{ (e,f) \mid e \lesssim e' \; S \; f' \lesssim f \}$

Proving Soundness

We need to prove that these techniques are sound
(they do NOT follow from Knaster-Tarski)

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Proving soundness is rather complicated and error prone

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Proving soundness is rather complicated and error prone

In Milner's book there are two mistakes:

Weak Bisimulation up to weak bisimilarity

Weak Bisimulation up to equivalence

Desiderata

We would like to be able to prove soundness for

- Different sort of up-to techniques (like Eqv, Bhv, Ctx, Slf)
- Different sort of coinductive predicates (like $\sim$ or $\lesssim$)
- Different sort of systems (like DA or LTS)

Moreover, we would like to prove the soundness of these techniques in a modular way:

$$\text{Ctx and Bhv are sound} \Rightarrow \text{Bhv}\circ\text{Ctx is sound}$$

The Double role of Coalgebra

	Coalgebras as Systems	**Coalgebras as Proofs**
Functor F	F: **Set**→**Set** Type of the systems	F: $\mathbf{Rel}_X \to \mathbf{Rel}_X$ Type of the Proof
F-coalgebra	System X→FX	Invariants X⊆FX
Final F-coalgebra	Universe of Behaviours	Coinductive Predicate νF

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An up-to technique is a functor G: $\mathbf{Rel}_X \to \mathbf{Rel}_X$

An F-invariant up-to G is a coalgebra X⊆FGX

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The Double role of Coalgebra

G is *sound* if there exists a functor H:**Coalg(FG)**→**Coalg(F)**
and a natural transformation κ:U⇒UH

$$
\begin{array}{ccc}
\mathbf{Coalg(FG)} & \xrightarrow{\quad H \quad} & \mathbf{Coalg(F)} \\
U \downarrow & \Nearrow & \downarrow U \\
\mathbf{Rel_X} & \xrightarrow{\quad Id \quad} & \mathbf{Rel_X}
\end{array}
$$

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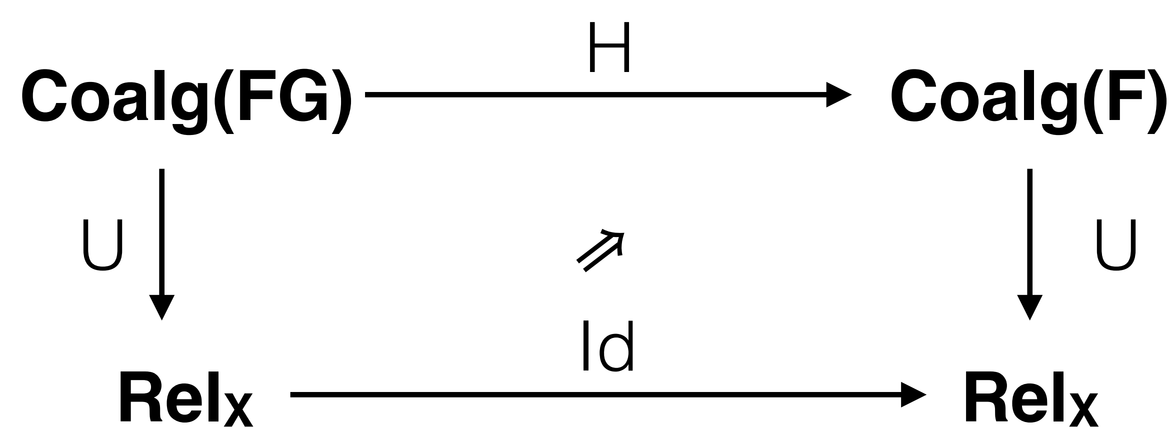

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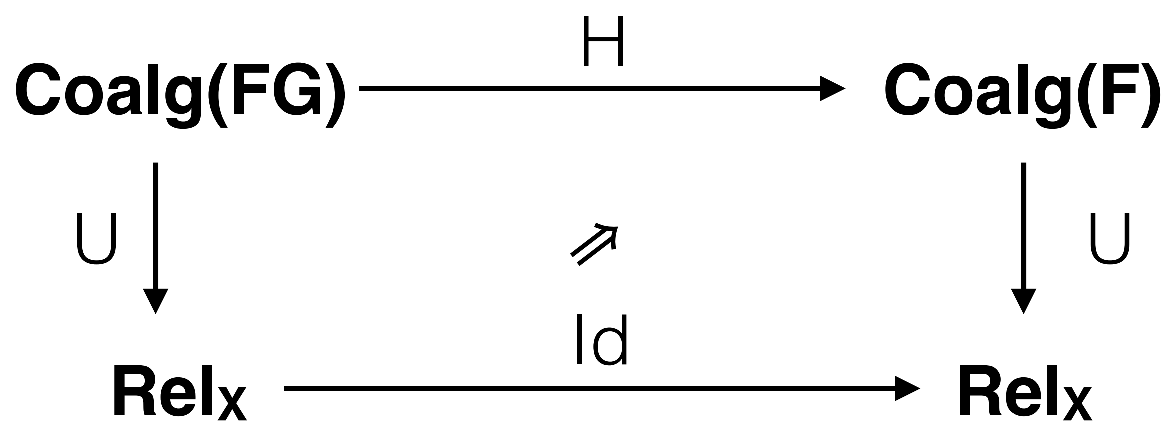

intuitively κ ensures that X⊆Y

∃Z, Z ⊆ X ⊆ FGX ⇒ ∃Z, Z ⊆ Y ⊆ FY ⇒ Z ⊆ νF

Generalised Powerset Construction

Silva, Bonchi, Bonsangue, Rutten - FSTTCS 2010

$$X \xrightarrow{\alpha} FTX$$

a functor F, a monad (T,η,μ) and a distributive law λ:TF ⇒FT

Generalised Powerset Construction

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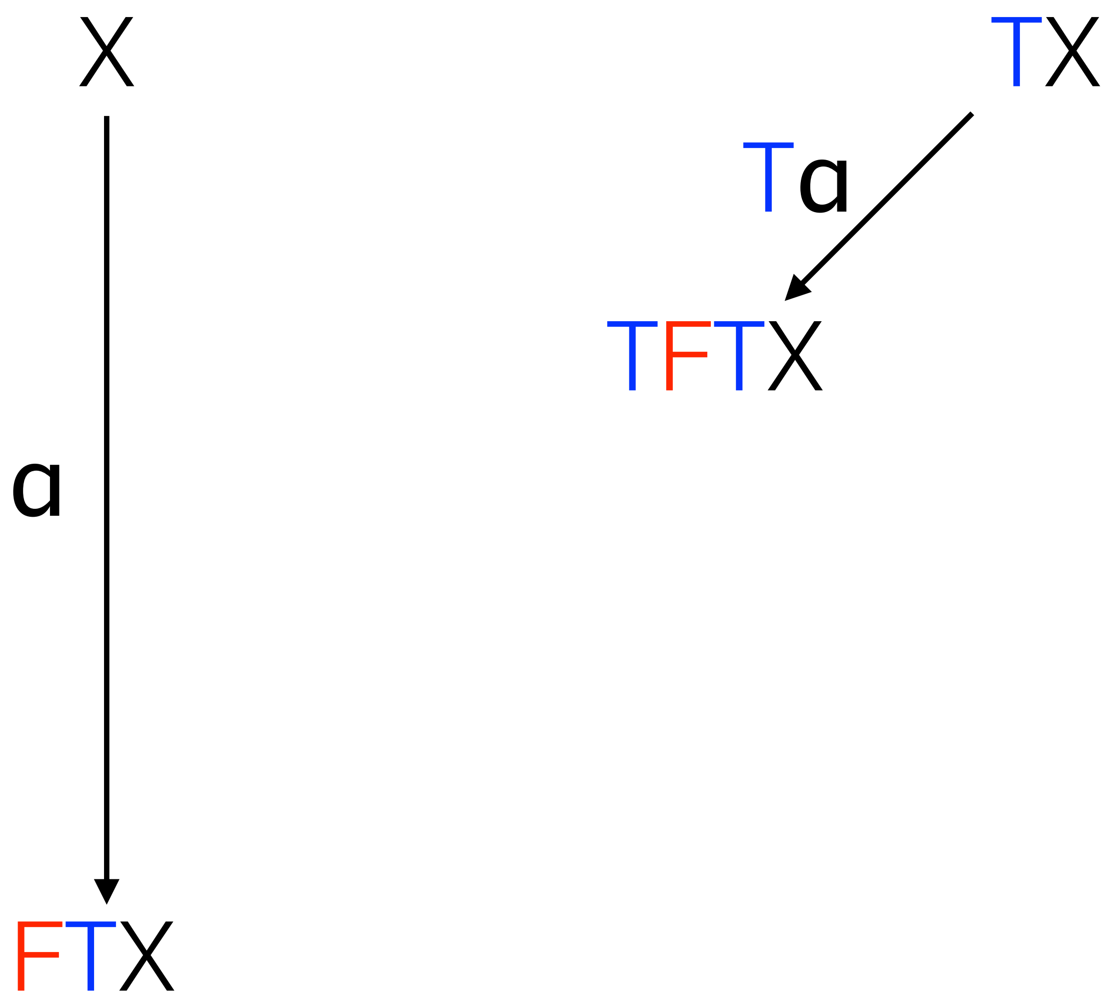

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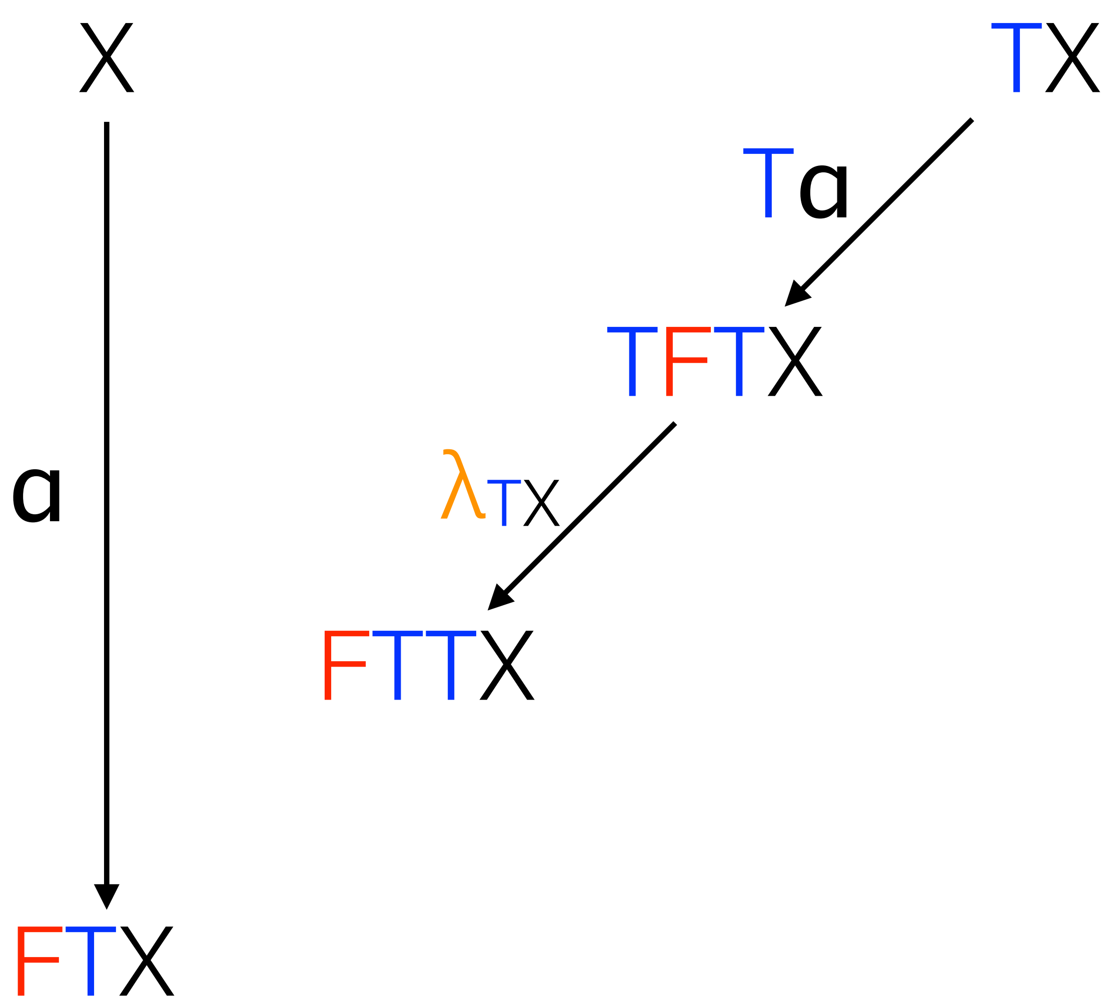

a functor F, a monad (T,η,μ) and a distributive law $\lambda : TF \Rightarrow FT$

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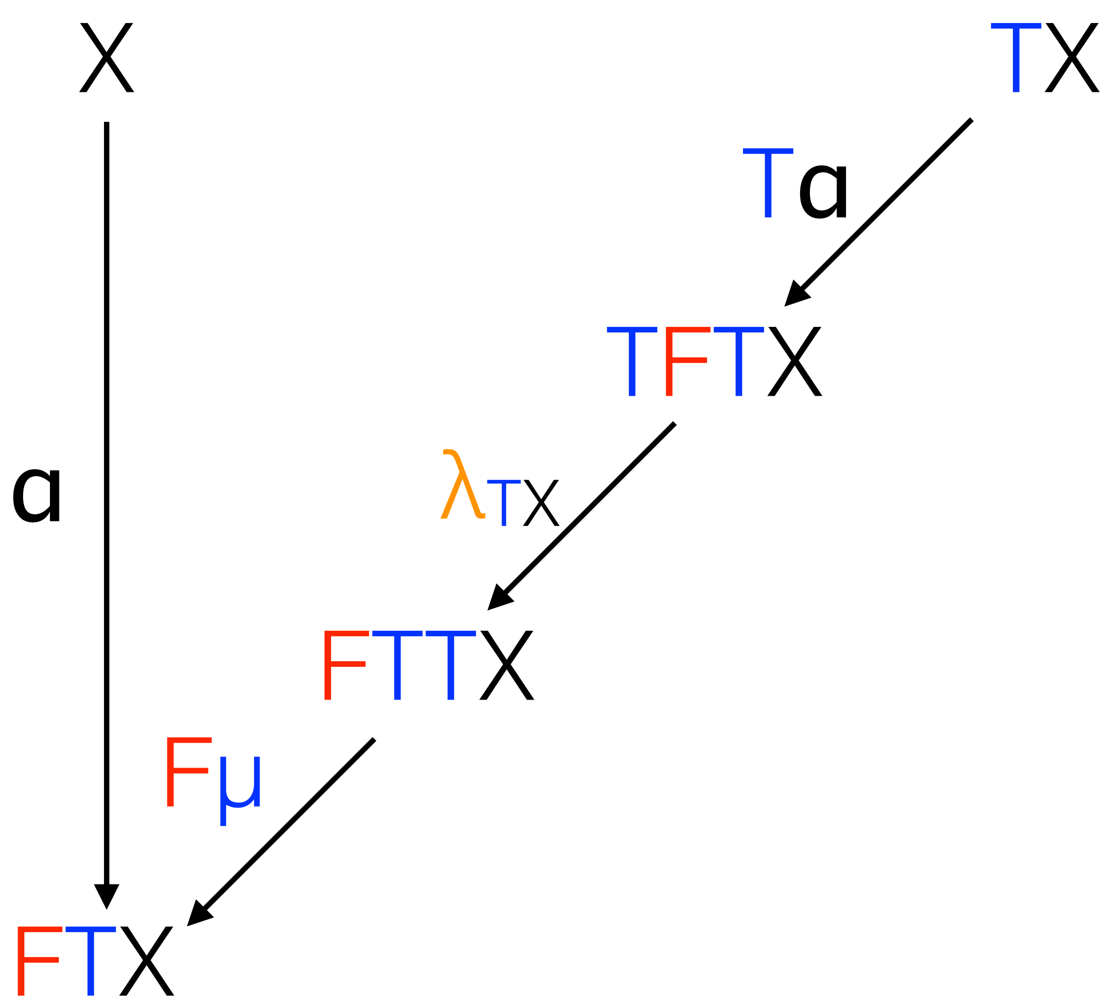

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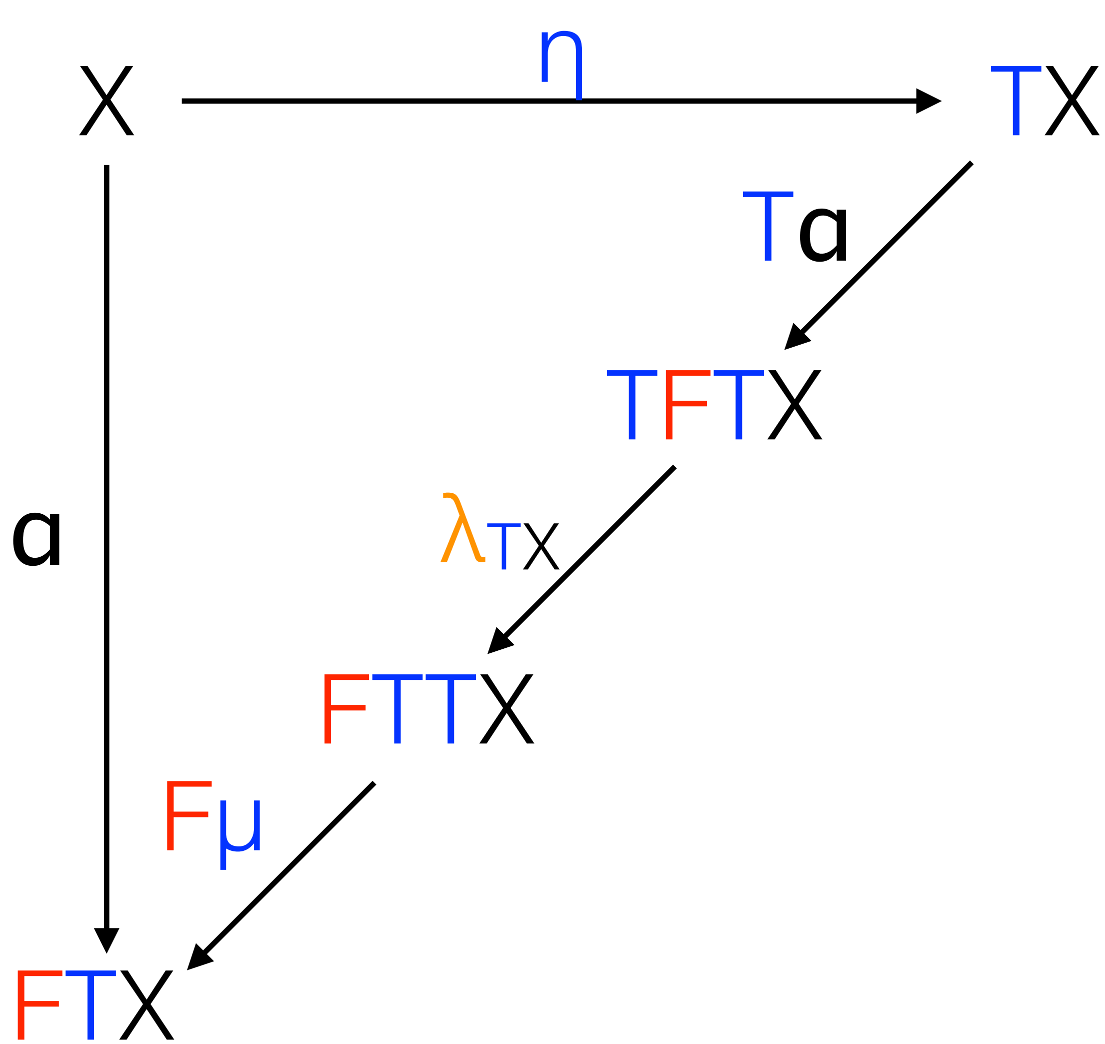

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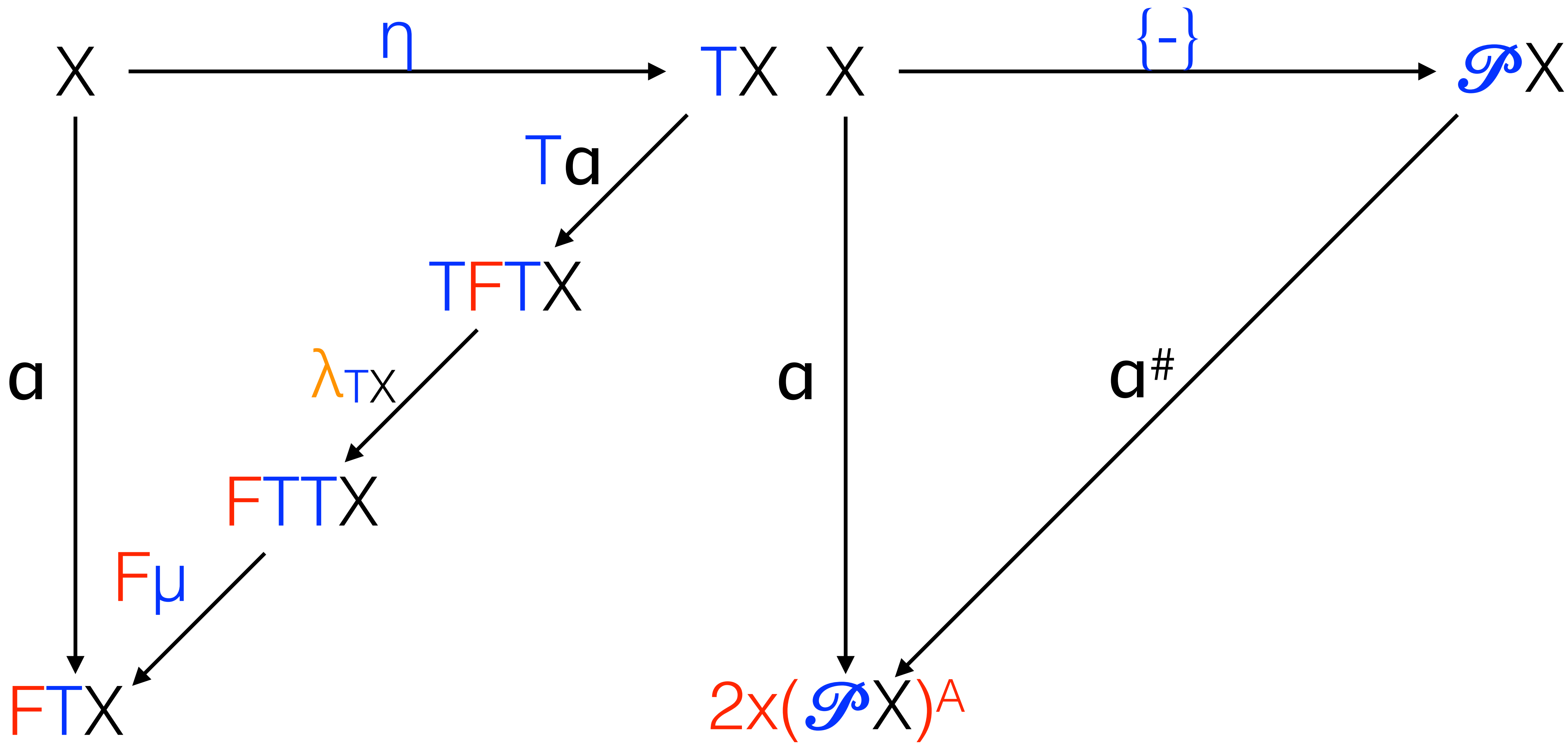

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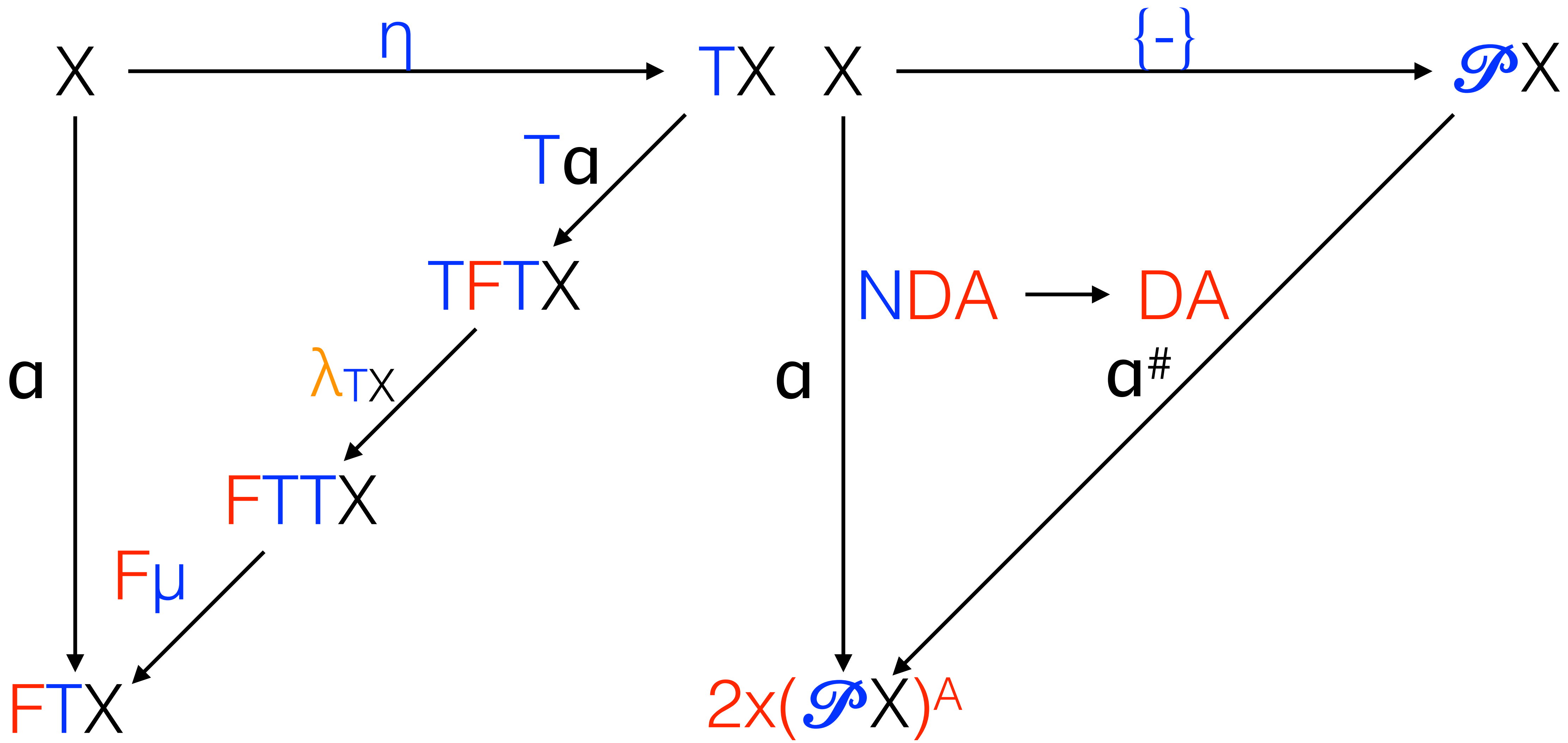

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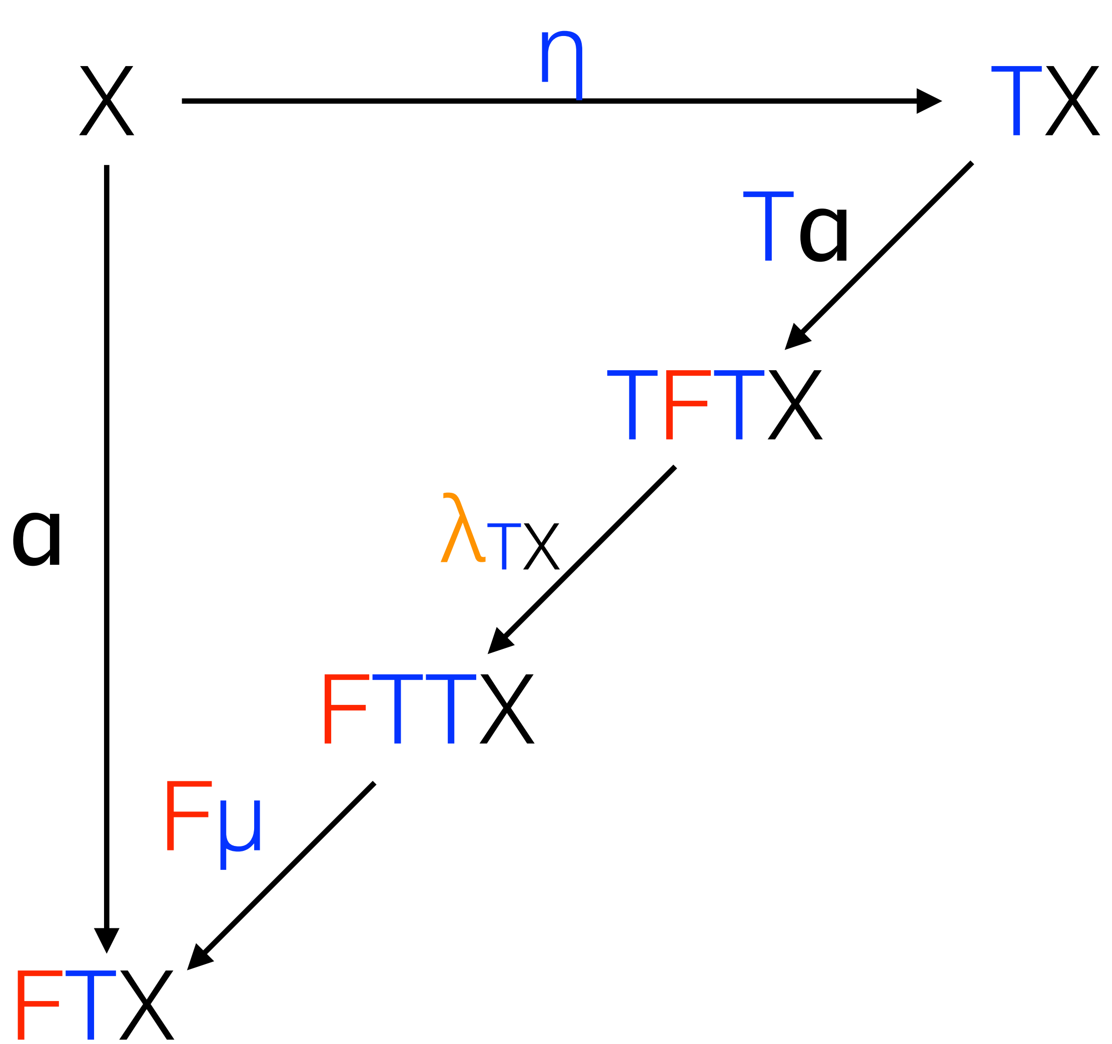

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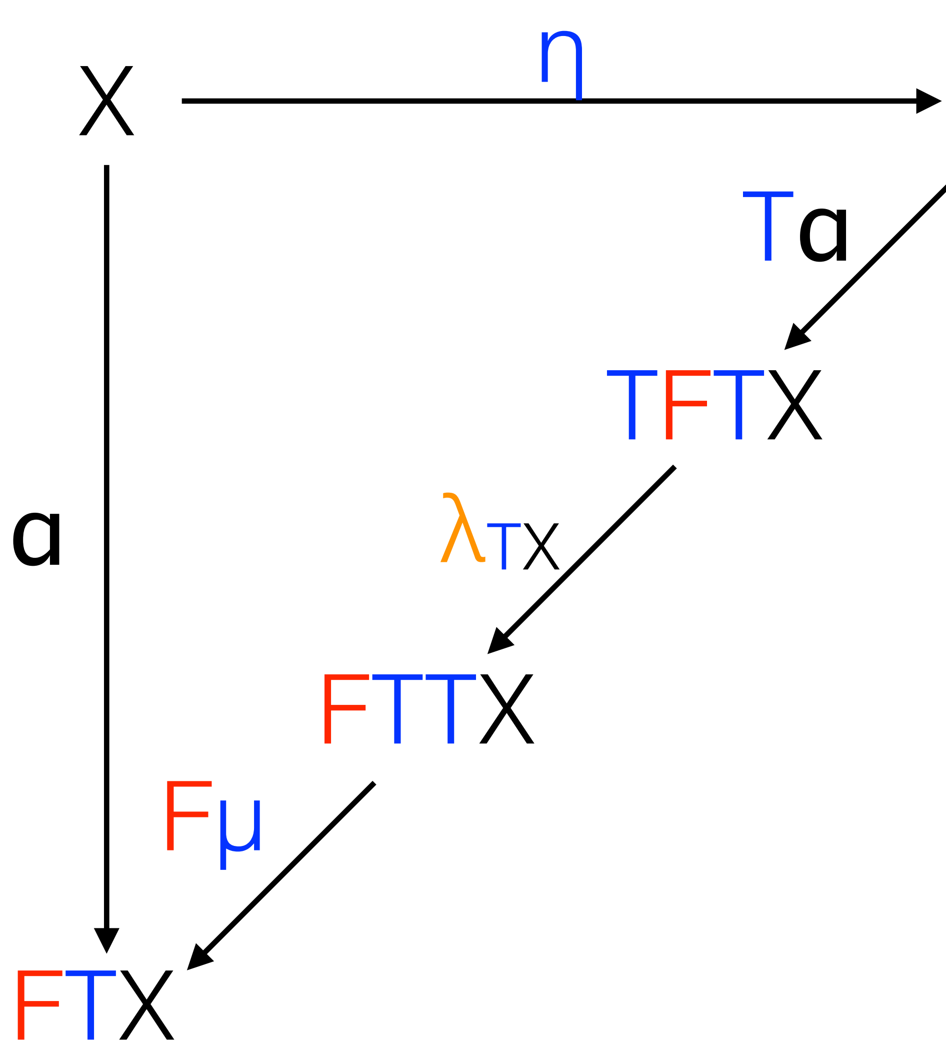
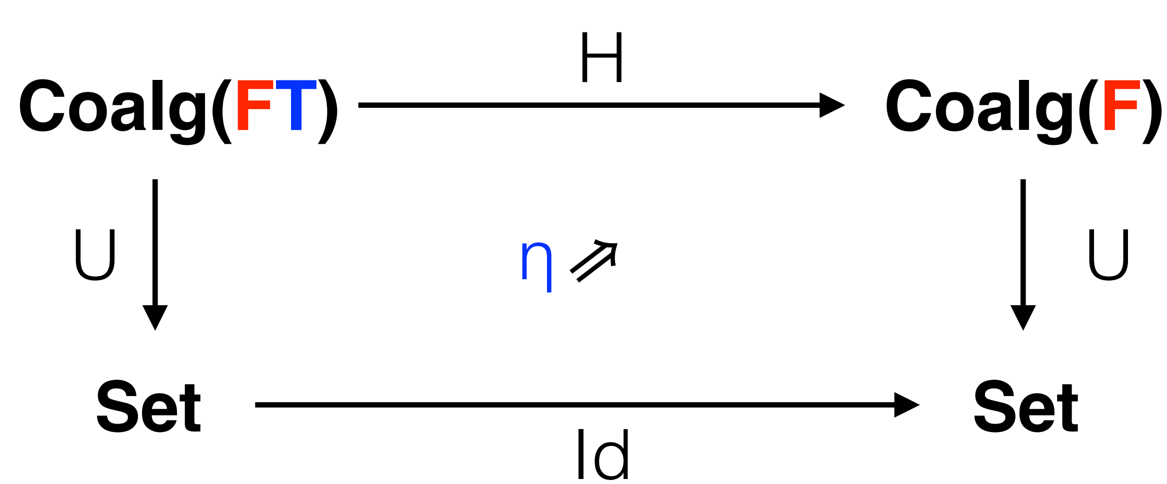


Theorem



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The Double role of Coalgebra

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FT-coalgebra	F-sytem with branching T	F-Invariants up-to T

Compatible Functors

Actually, we need much less than a monad T...

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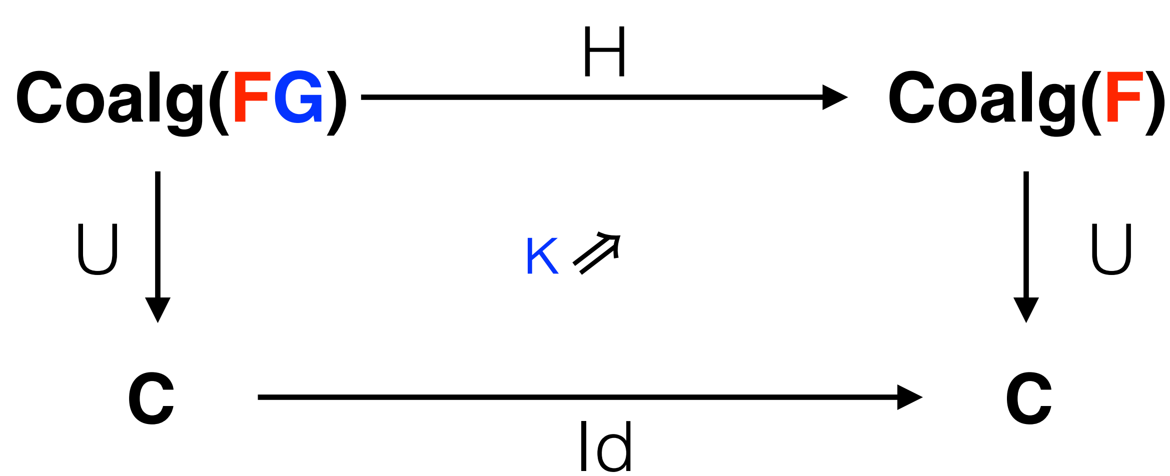
<u>Theorem</u>: a category **C** with countable coproducts F,G:**C**→**C** and λ:GF ⇒FG. Then



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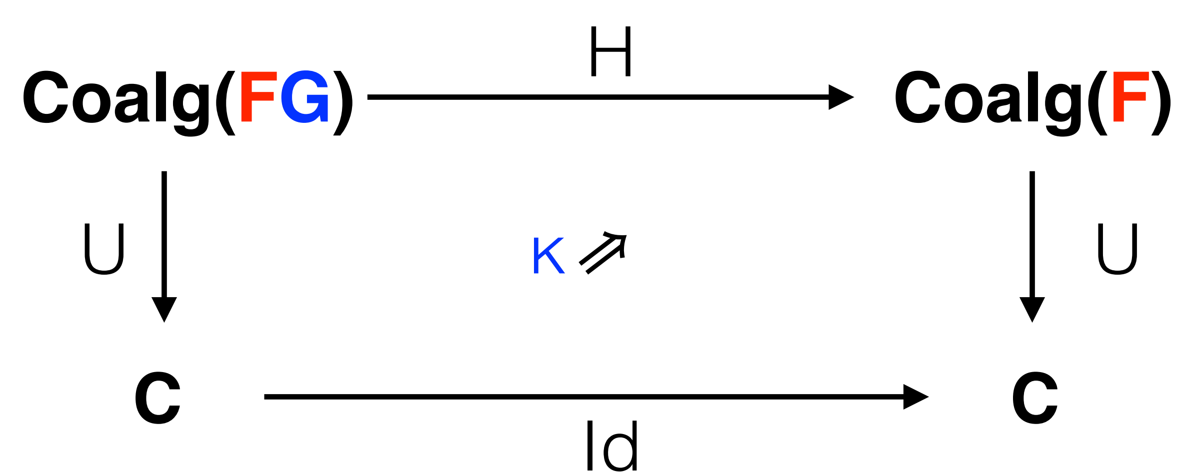

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<u>Theorem</u>: in a category **C** with countable coproducts, F-compatibility implies F-soundness

<u>Compositionality Theorem</u>
If G_1 and G_2 are compatibile with F,
then $G_1 \circ G_2$ is compatible with F

G is said to be *compatible* with F iff there is $\lambda : GF \Rightarrow FG$

Coinductive Predicates

Hermida and Jacobs - Information and Computation 1998

Category **Rel**

objects: $R \subseteq X \times X$

arrows $R \subseteq X \times X \to S \subseteq Y \times Y$: $f: X \to Y$ such that $f(R) \subseteq f(S)$

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$$\mathbf{Rel} \xrightarrow{\;p\;} \mathbf{Set}$$

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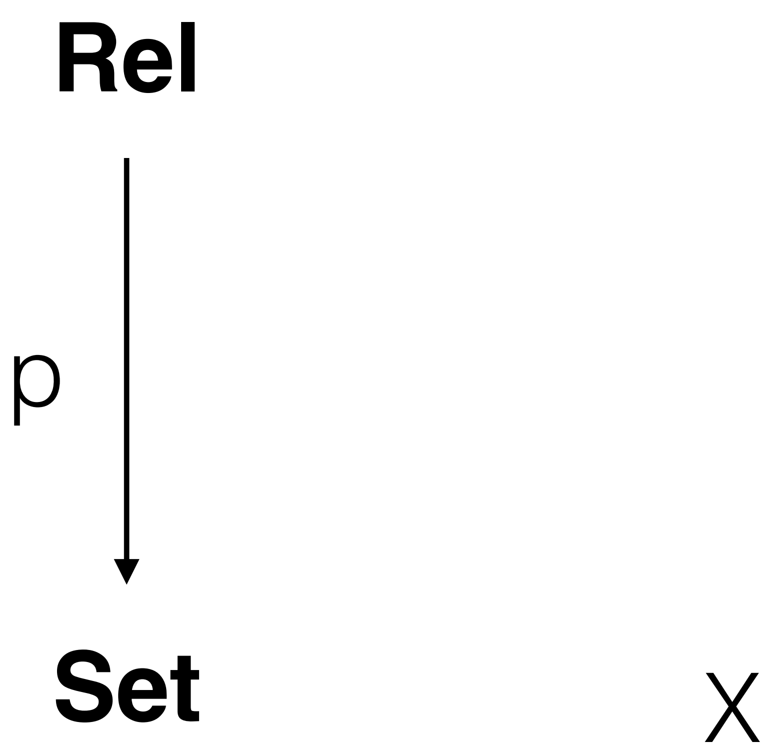

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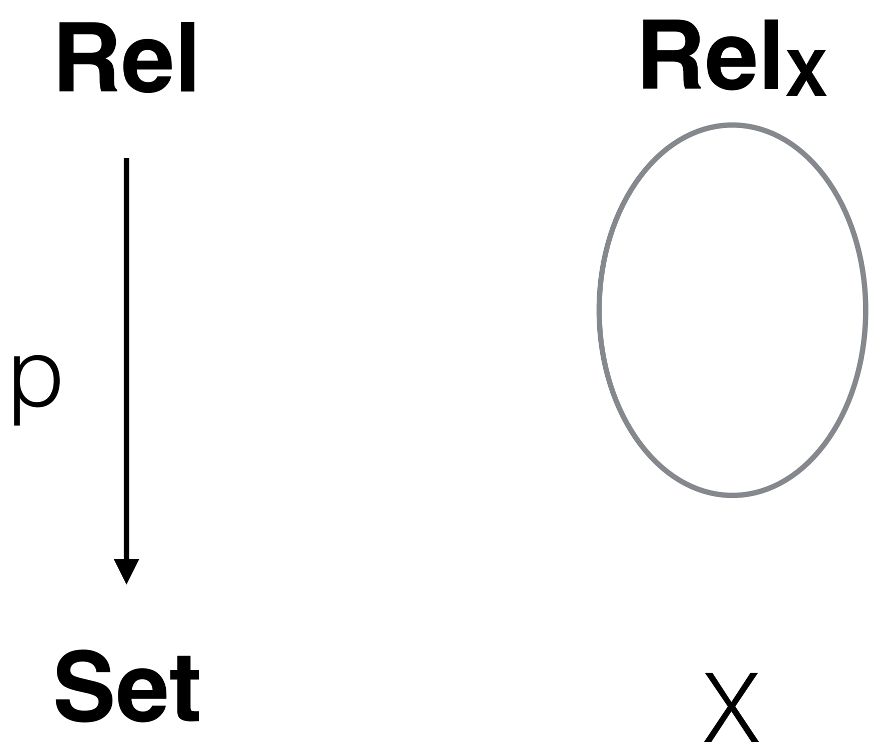

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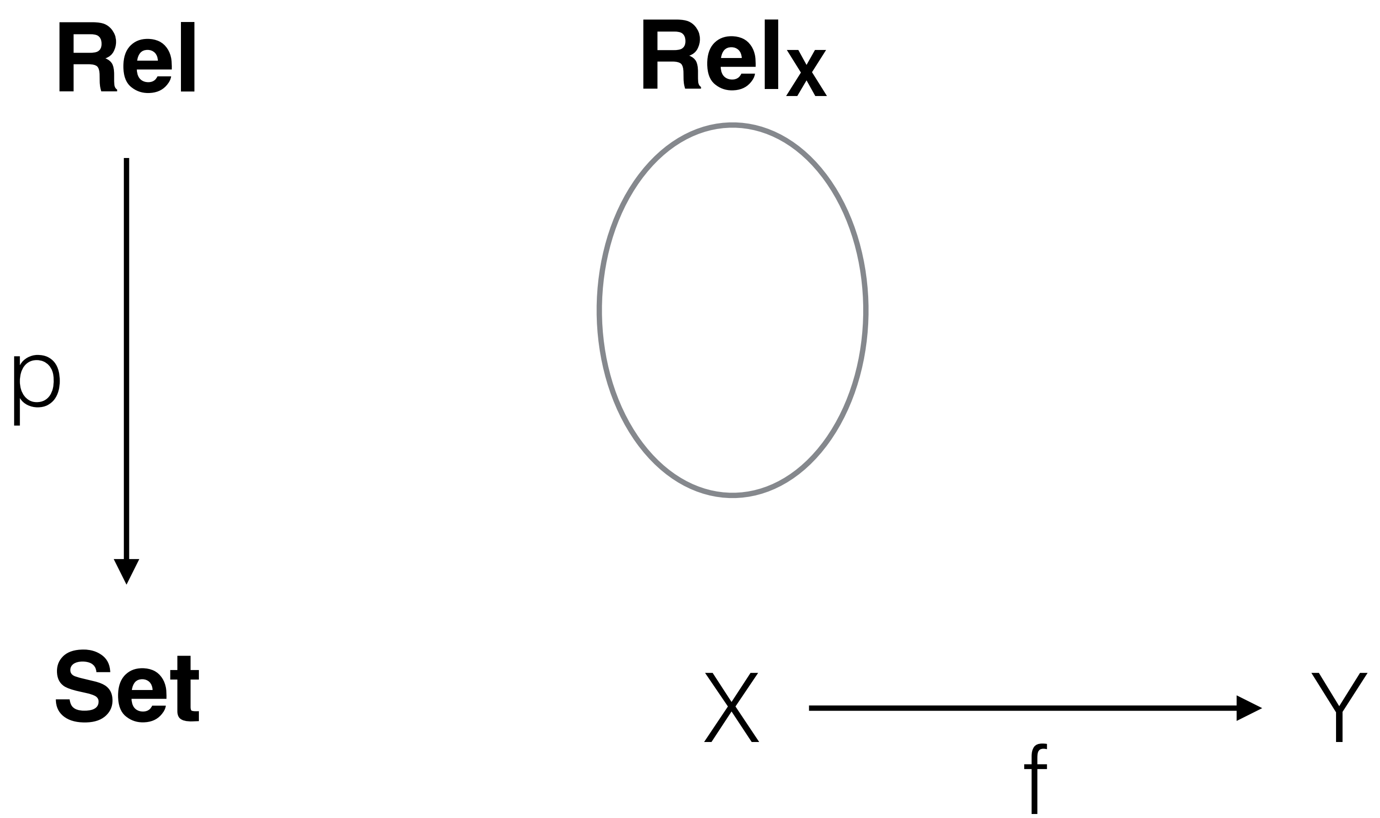

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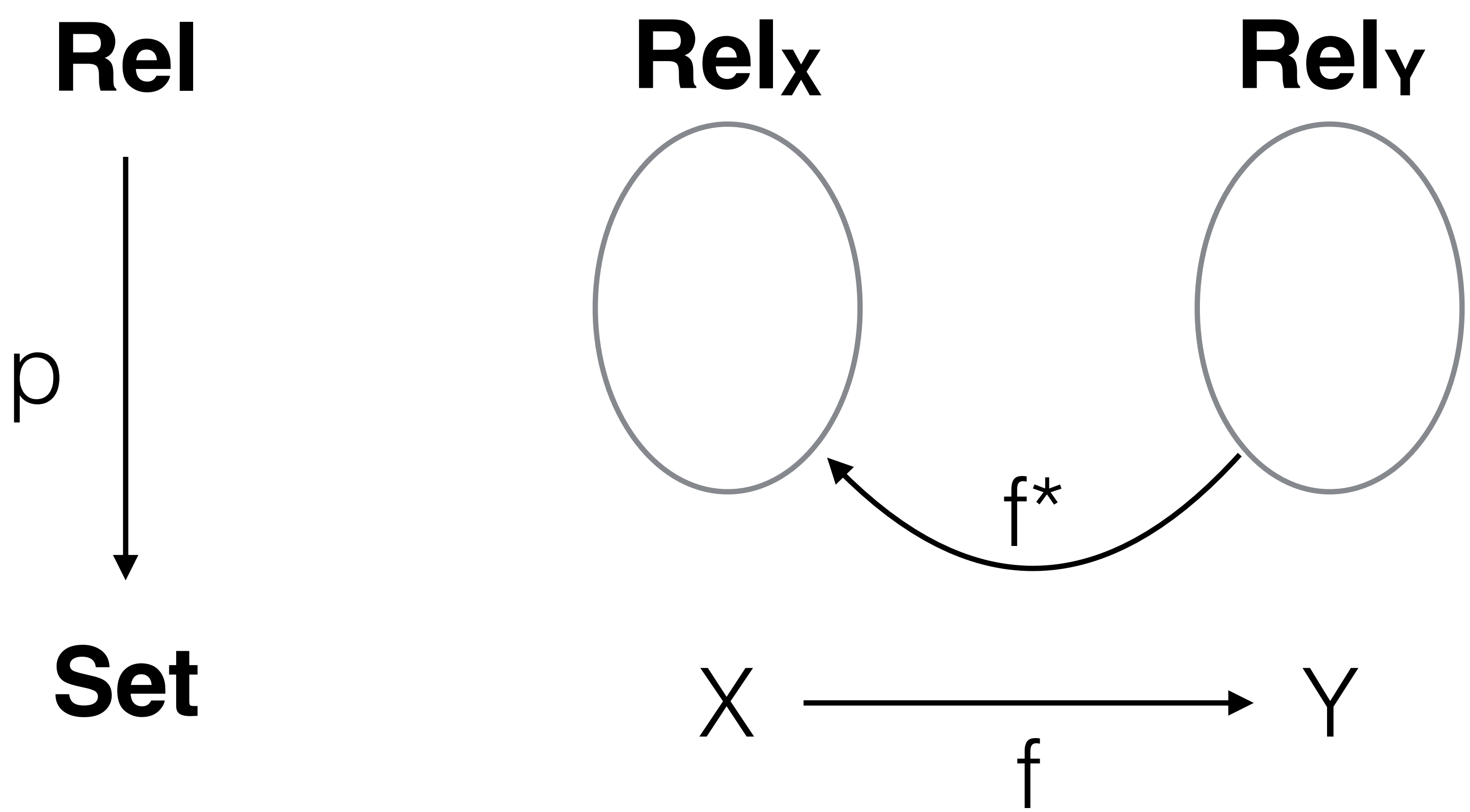

$$f^*(S) = \{ (x,y) \mid f(x)\ S\ f(y) \}$$

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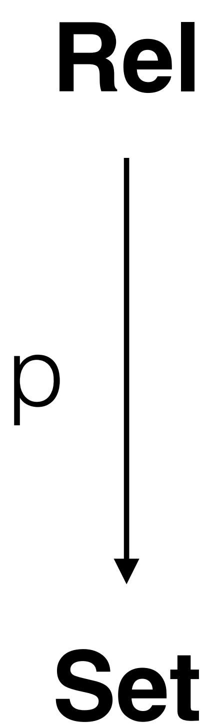

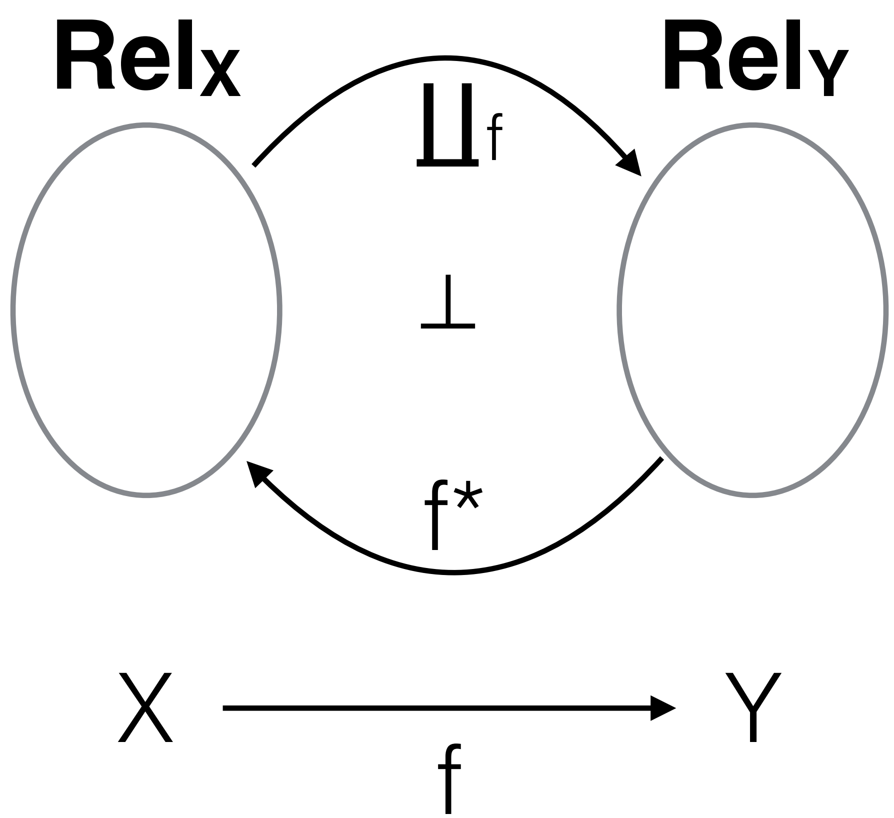

$\coprod_f(R)$
$=$
$\{ (f(x),f(y)) \mid x \; R \; y\}$

$f^*(S)$
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$\{ (x,y) \mid f(x) \; S \; f(y)\}$

Coinductive Predicates

Hermida and Jacobs - Information and Computation 1998

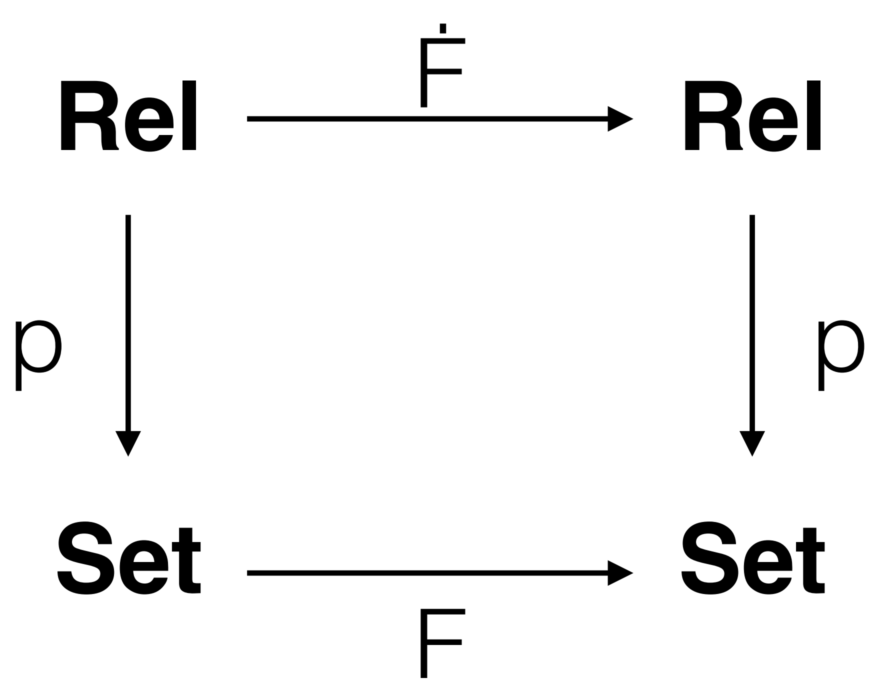
$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\dot{F}} & \mathbf{Rel} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{Set} & \xrightarrow[F]{} & \mathbf{Set} \end{array}$$

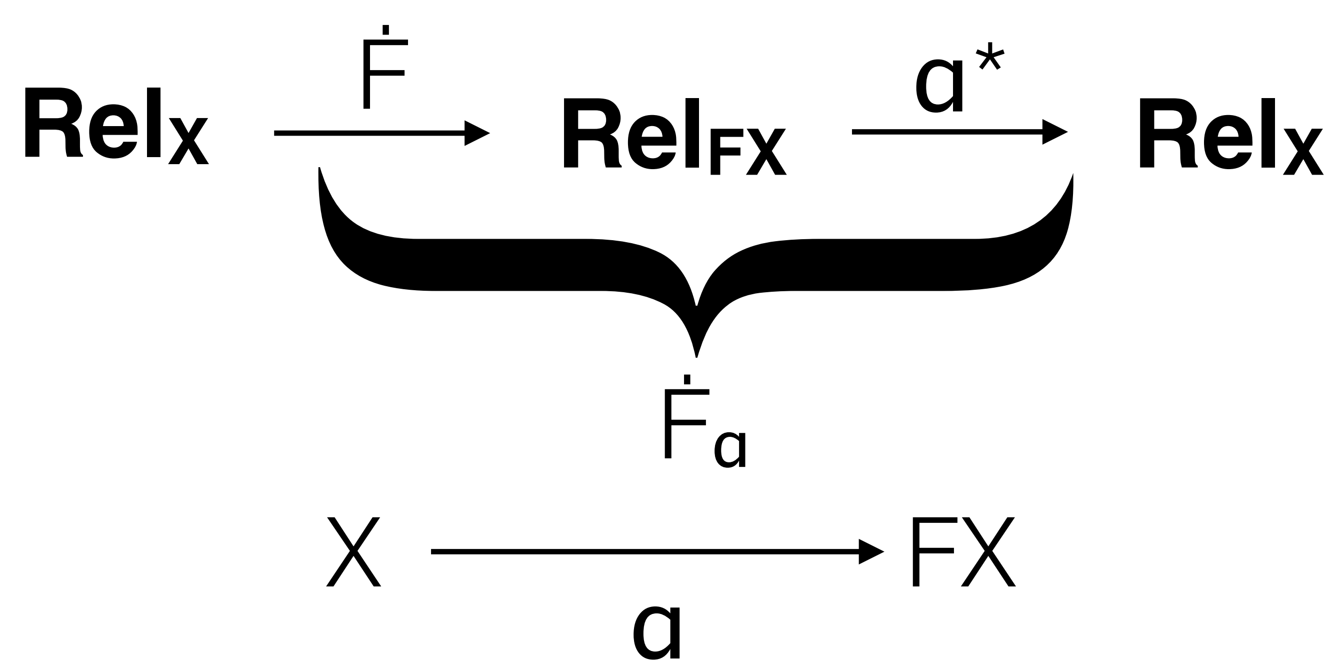
$$\mathbf{Rel}_X \xrightarrow{\dot{F}} \mathbf{Rel}_{FX} \xrightarrow{\alpha^*} \mathbf{Rel}_X$$

$$X \xrightarrow[\alpha]{} FX$$

Coinductive Predicates

Hermida and Jacobs - Information and Computation 1998





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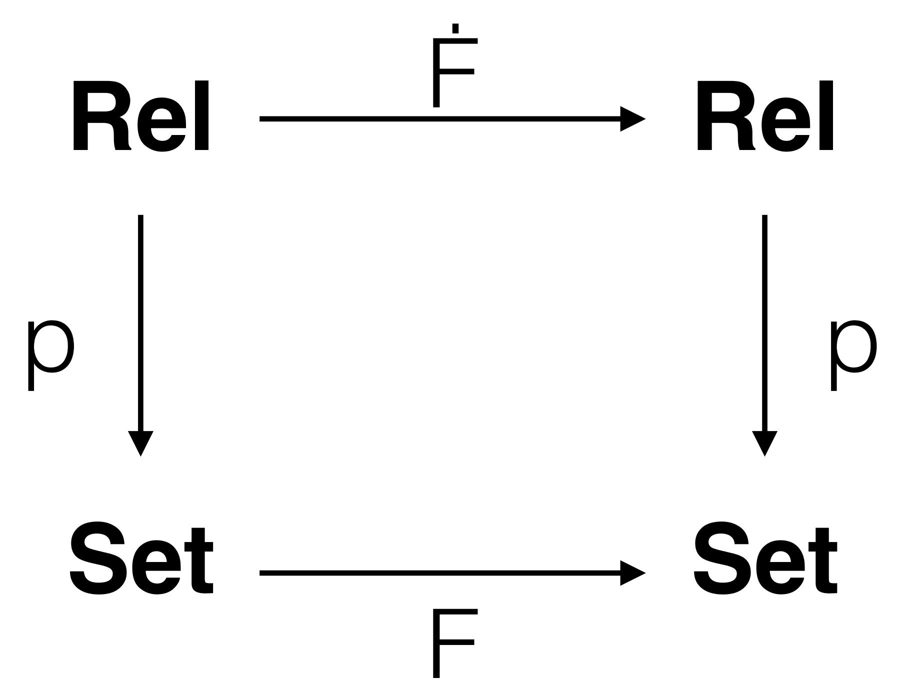
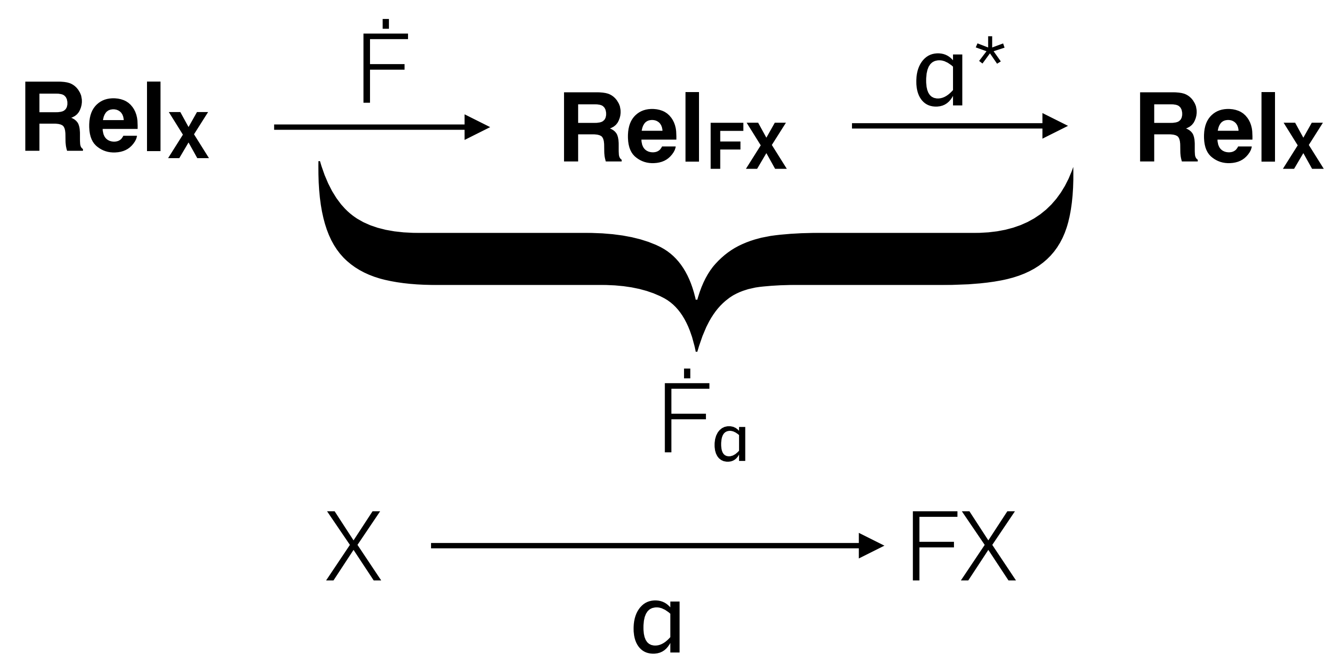
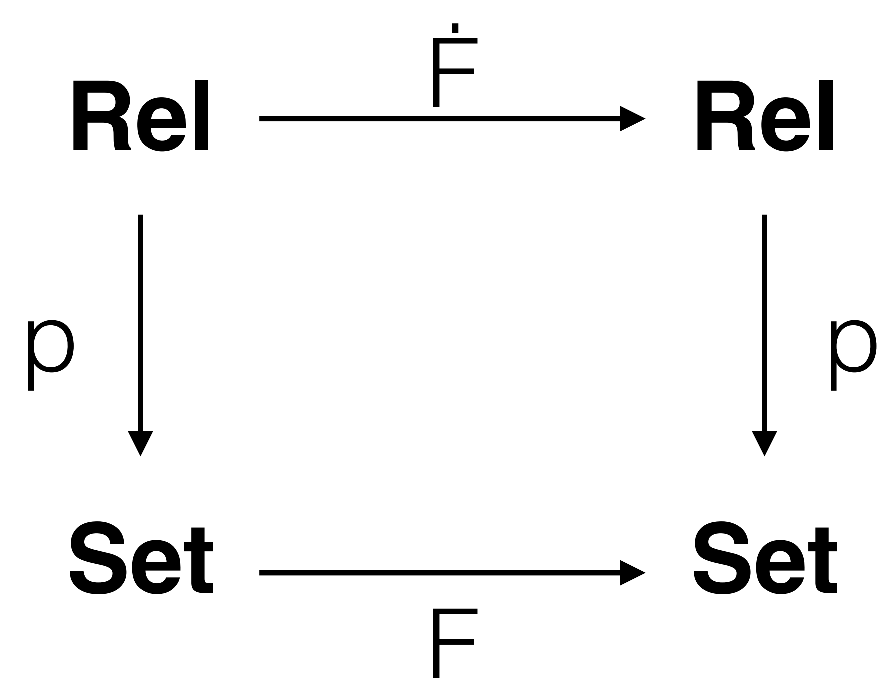


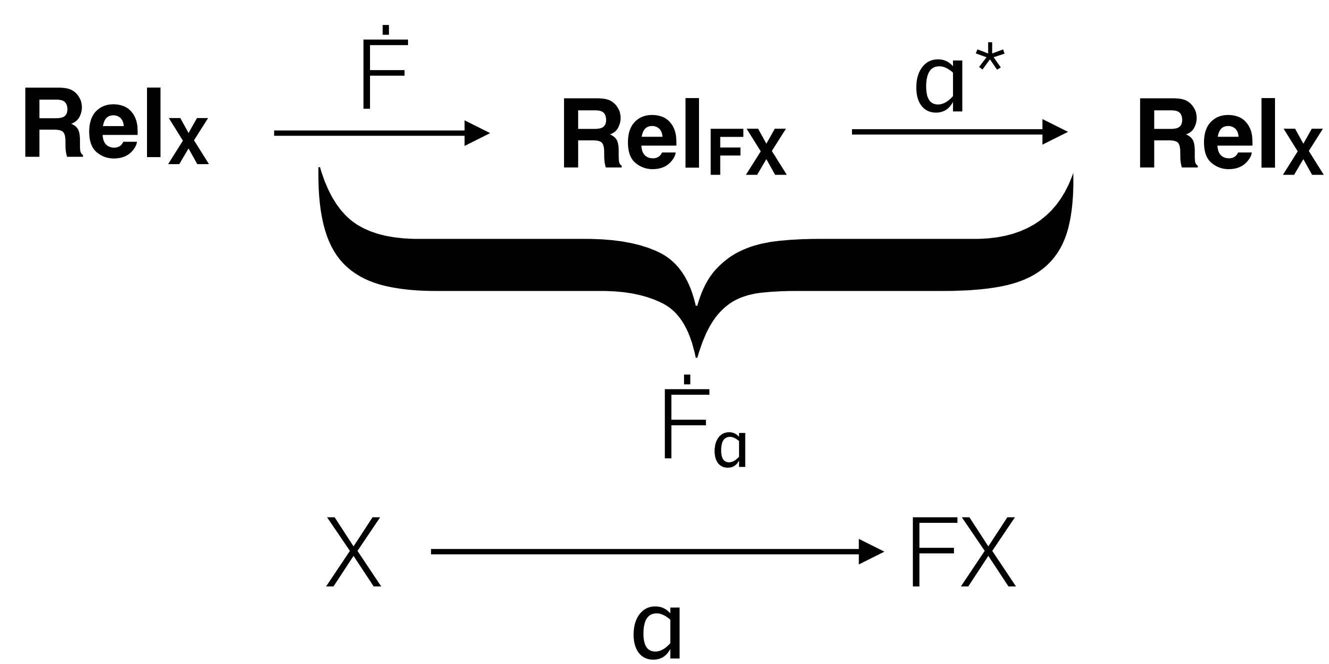


Coinductive Predicates

Hermida and Jacobs - Information and Computation 1998

$F = 2 \times Id^A$





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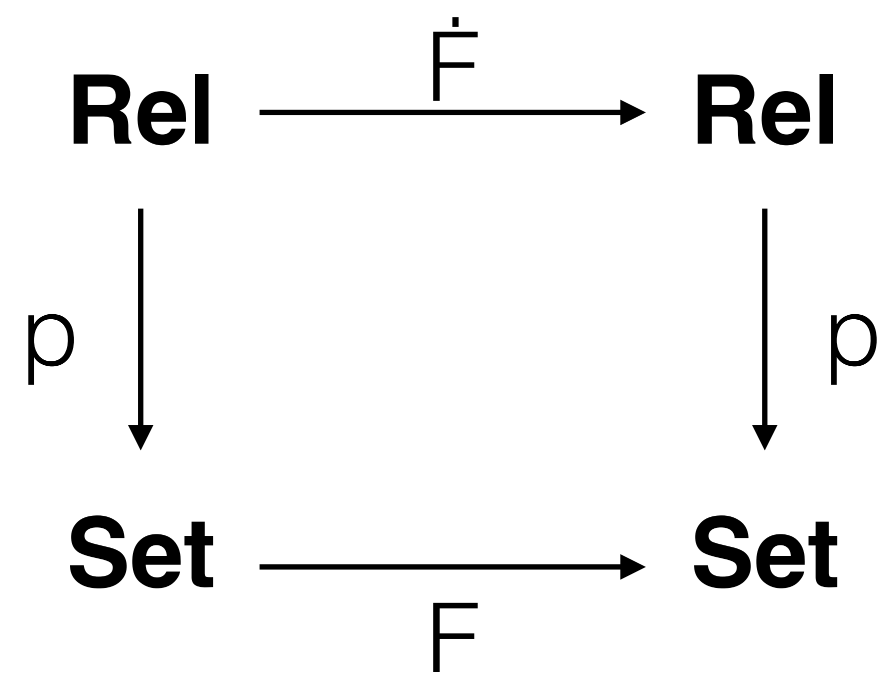

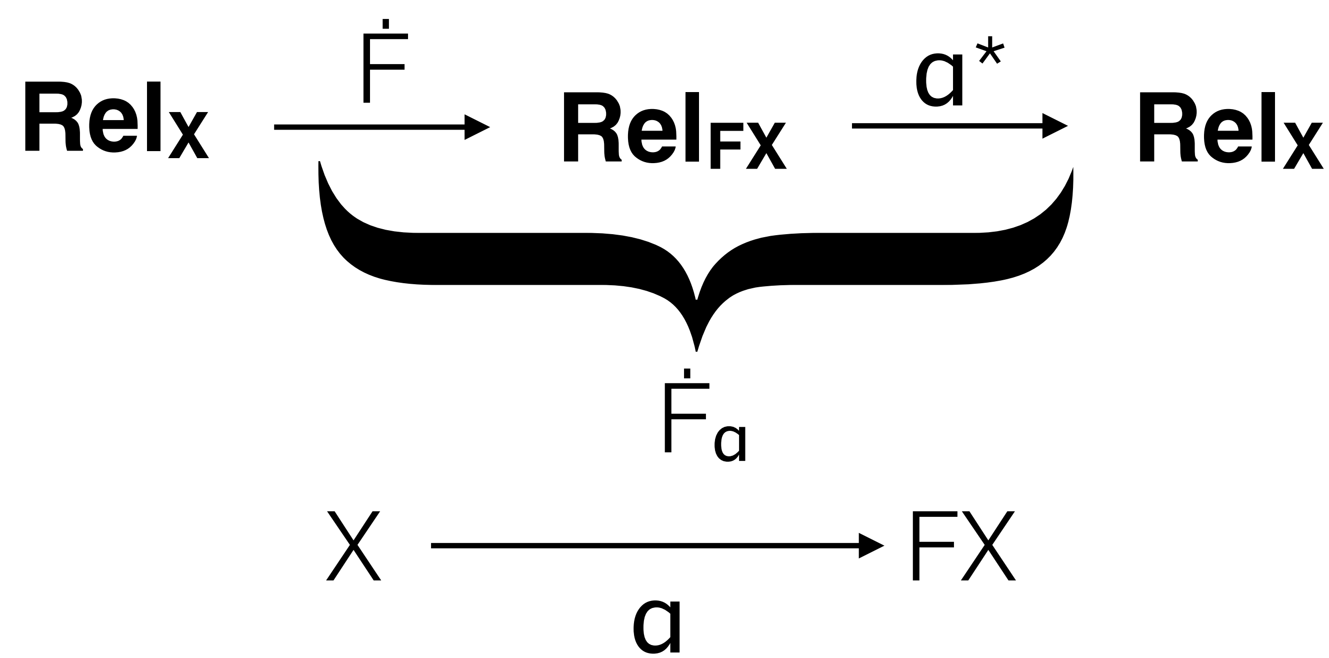

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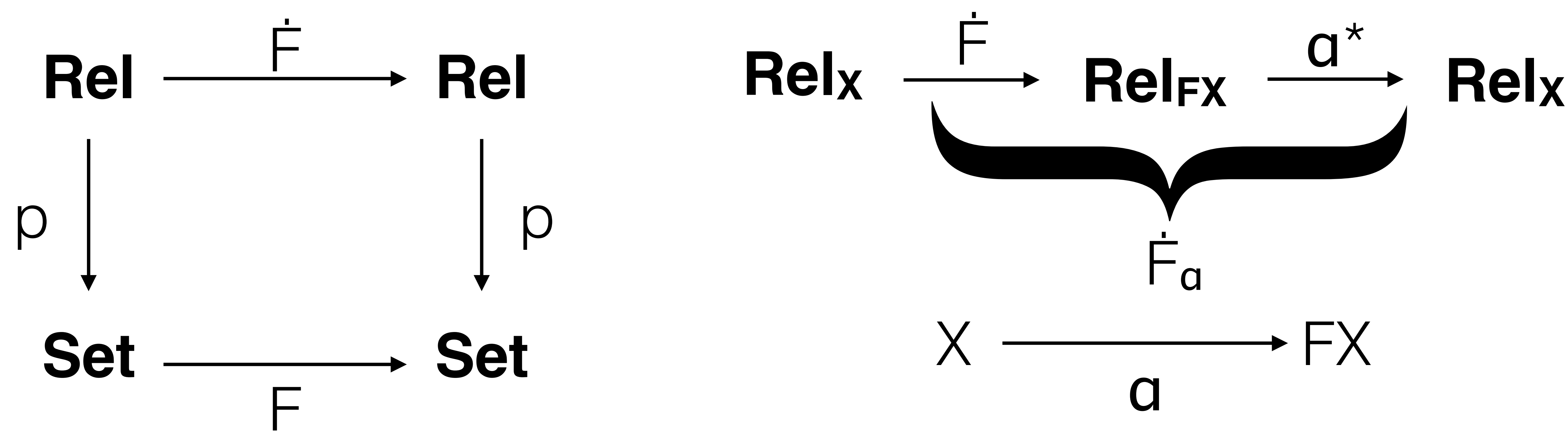

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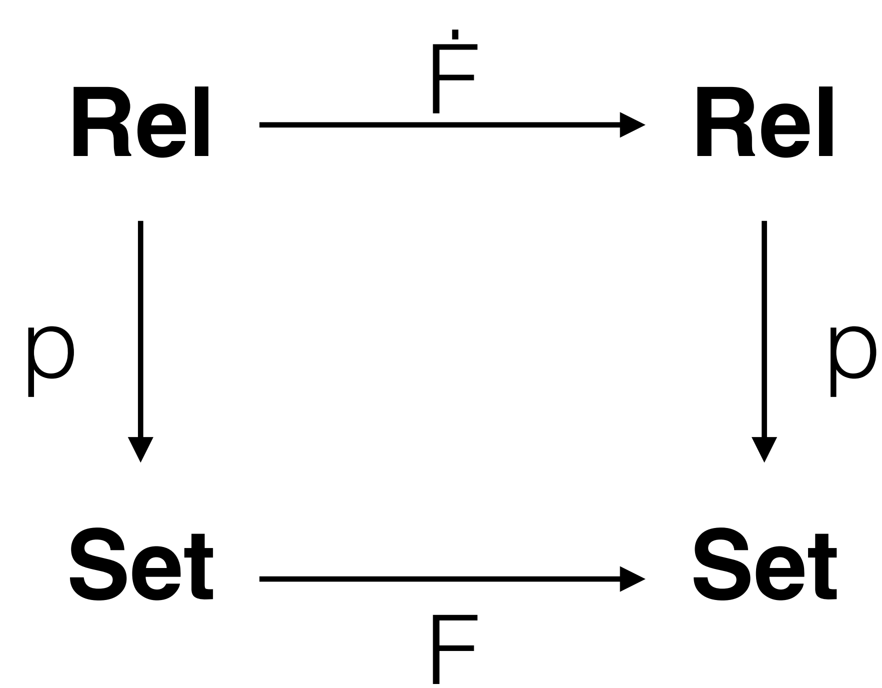
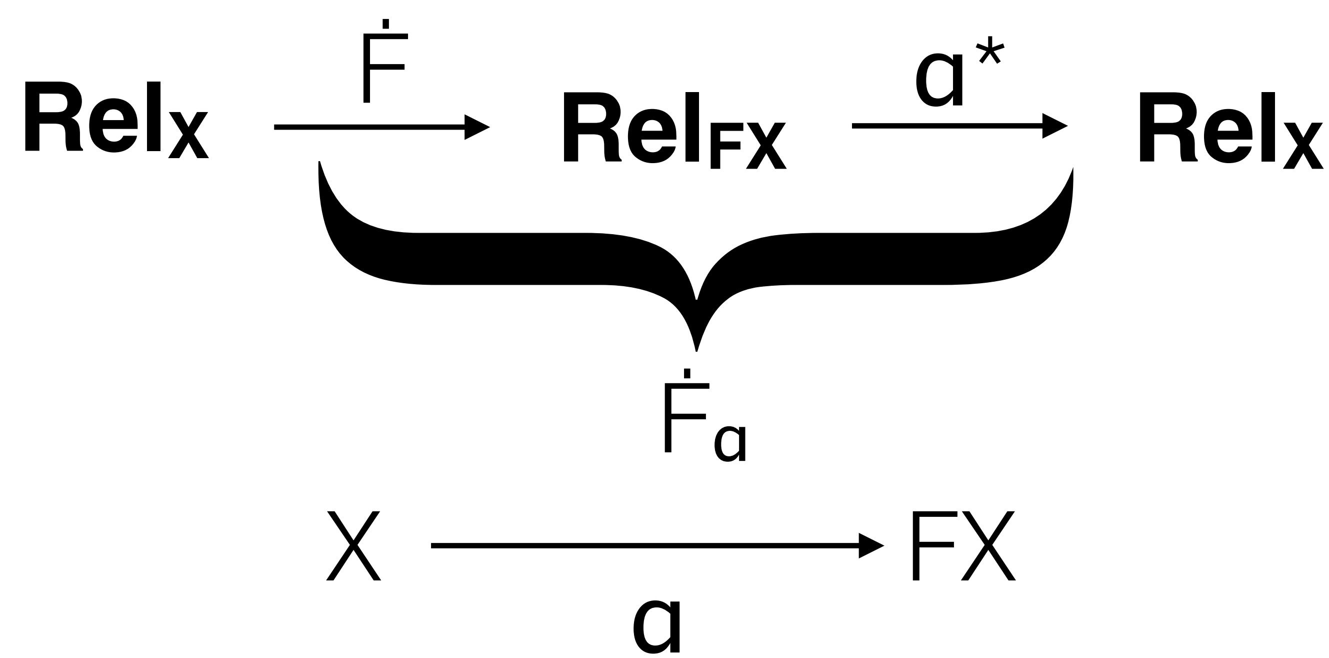
$$\dot{F}_{\langle o,t\rangle}(R)=B(R)$$





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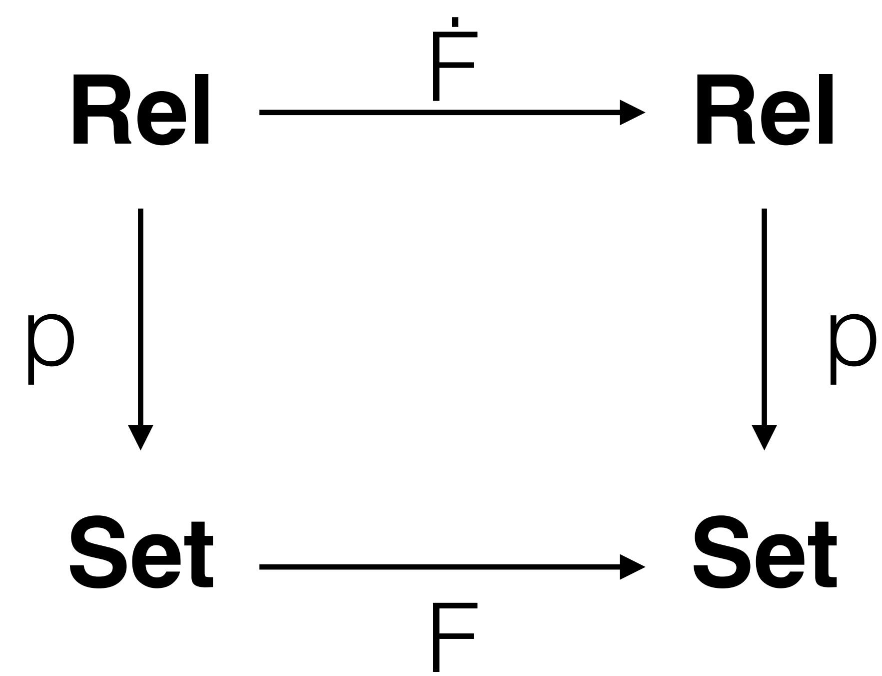

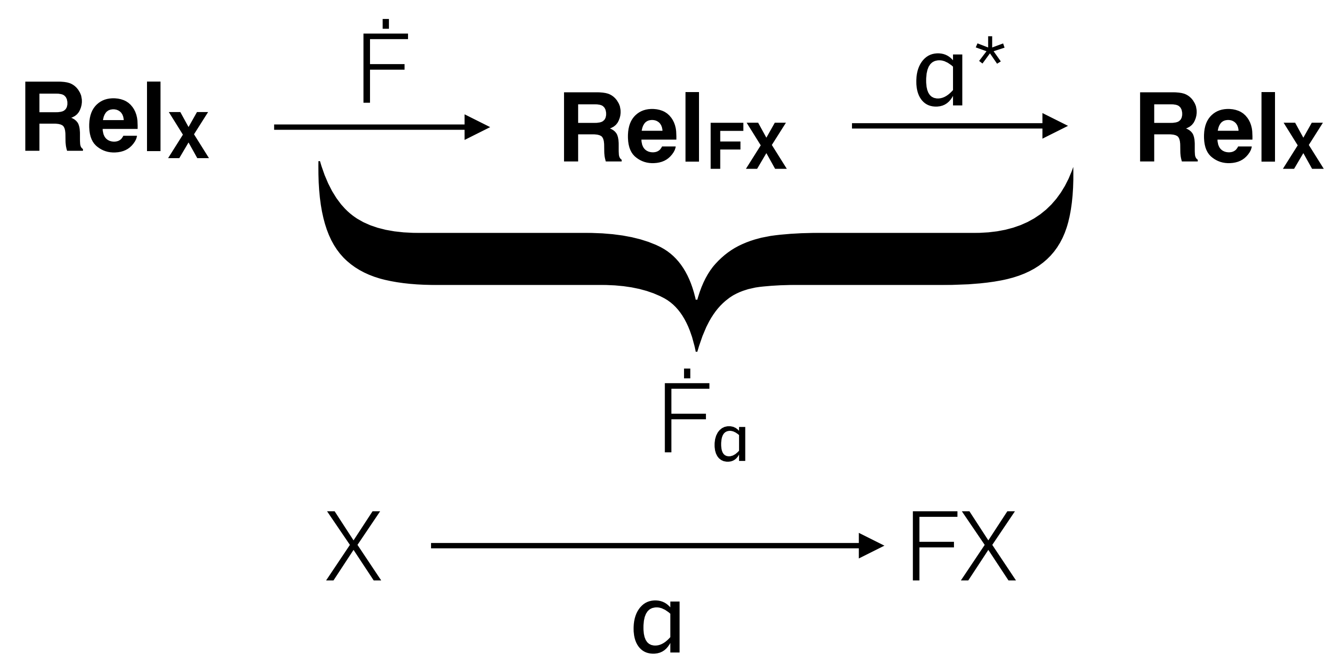

Actually, this can be defined for any bifibration

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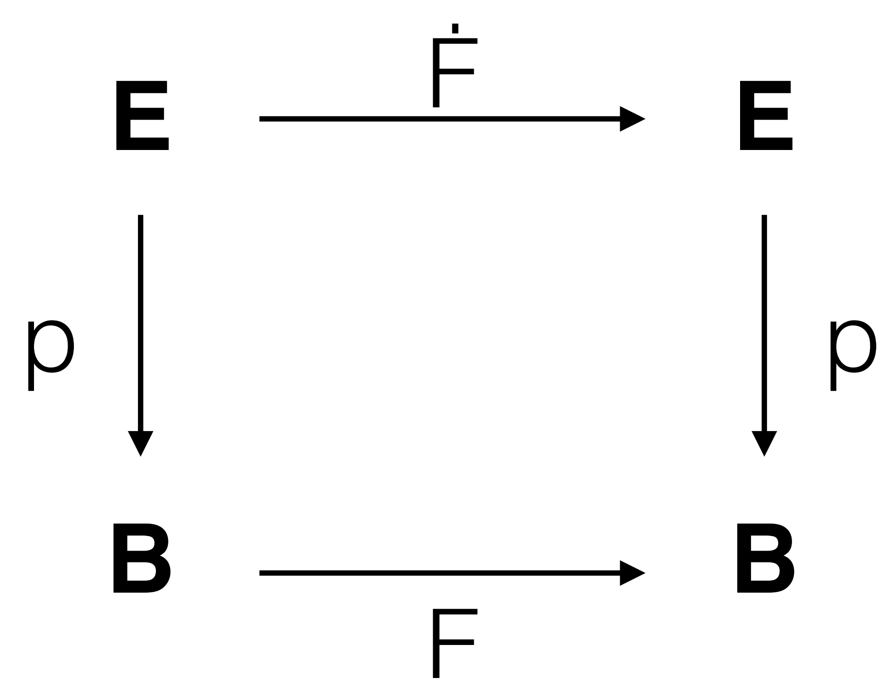

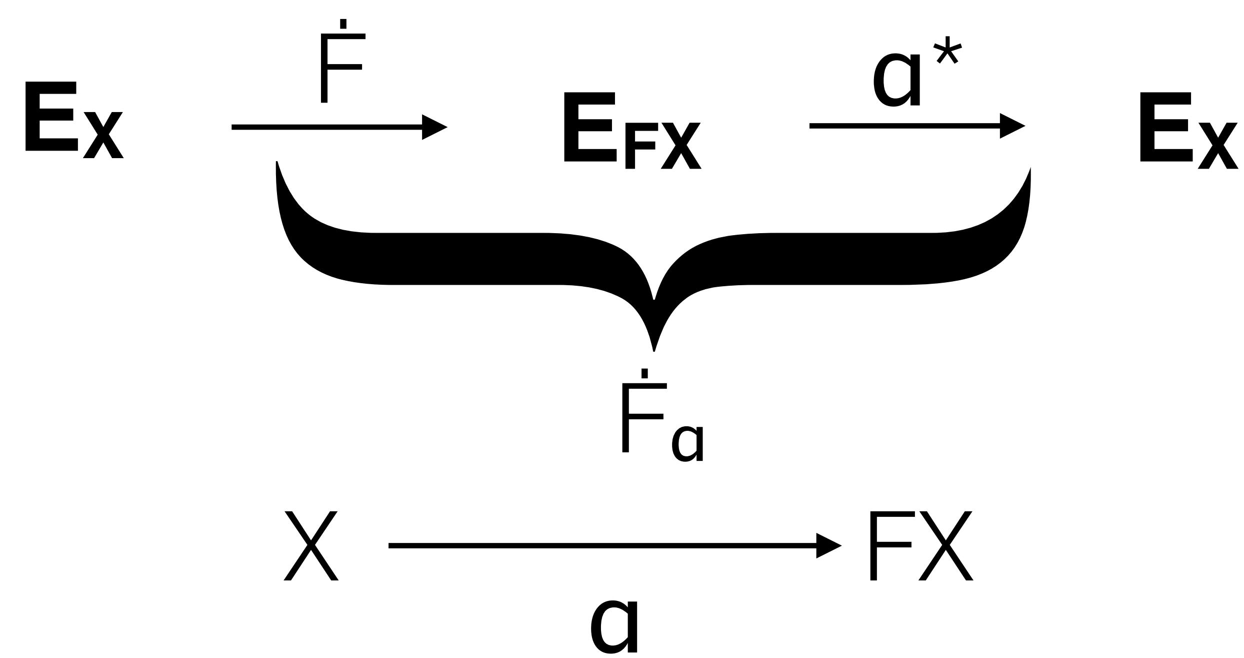

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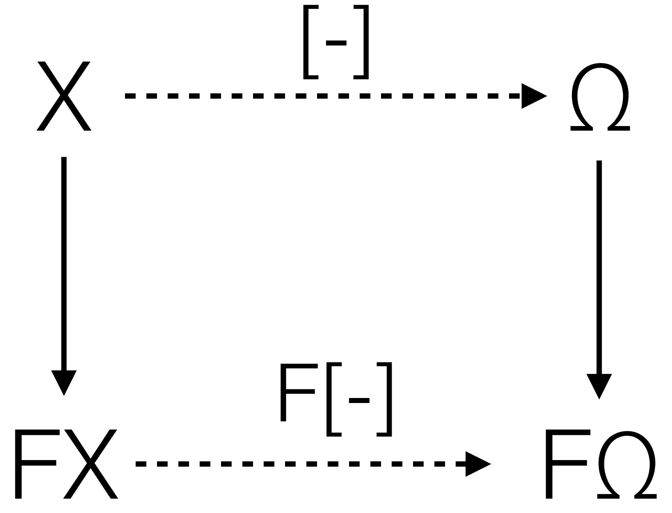
Compatibility of Bhv

Bhv: $\mathbf{Rel_X} \dashrightarrow \mathbf{Rel_X}$

$Bhv(R) = \{ (x,y) \mid x \sim x' \; R \; y' \sim y \}$

Compatibility of Bhv

Bhv: **Rel$_X$**-->**Rel$_X$**
Bhv(R)= { (x,y) | x∼x' R y'∼y }



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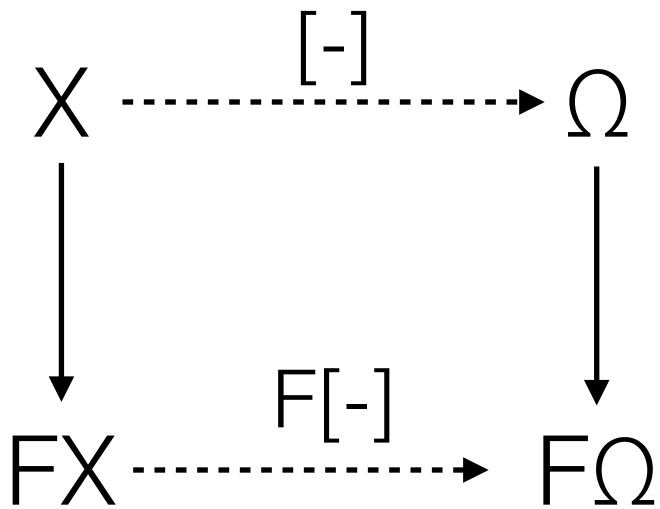

$x \sim y \overset{\text{def}}{\iff} [x]=[y]$

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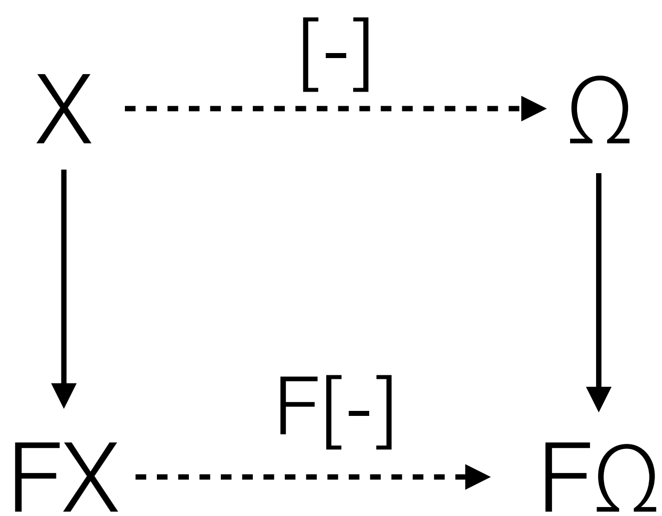

$$\mathbf{Rel}_X \xrightarrow{\coprod_{[-]}} \mathbf{Rel}_\Omega \xrightarrow{[-]^*} \mathbf{Rel}_X$$

$$x \sim y \overset{\text{def}}{\iff} [x] = [y]$$

Compatibility of Bhv

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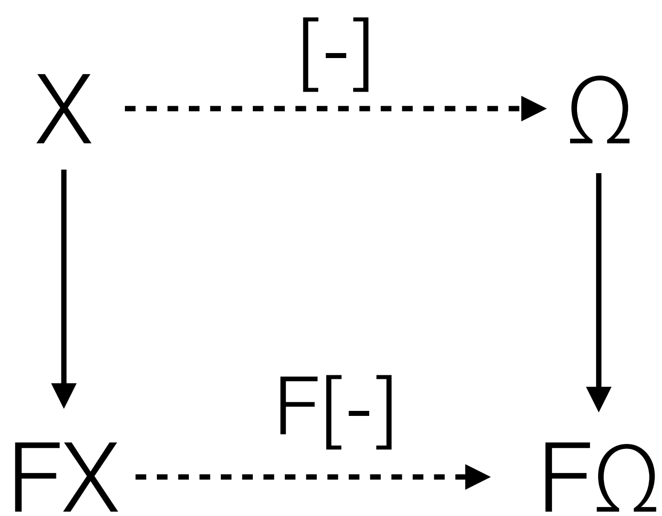

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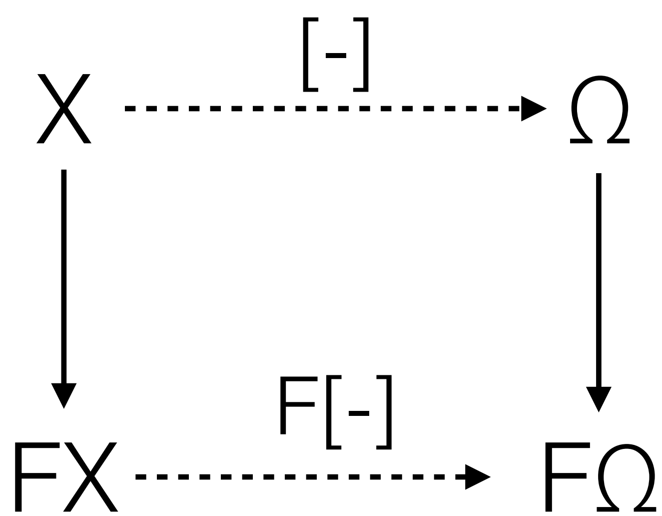

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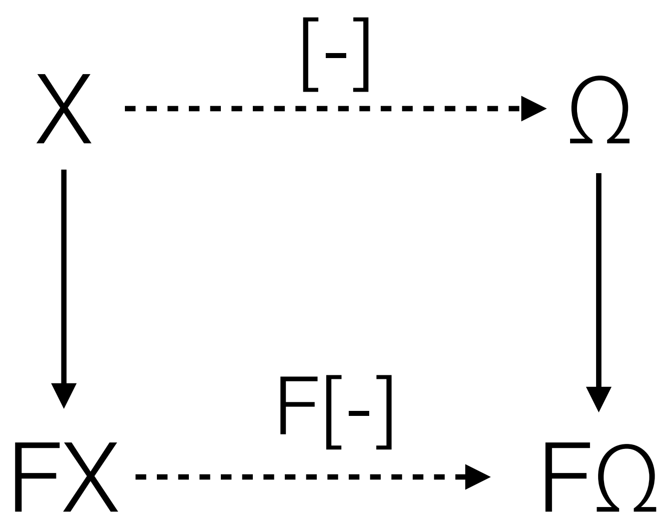

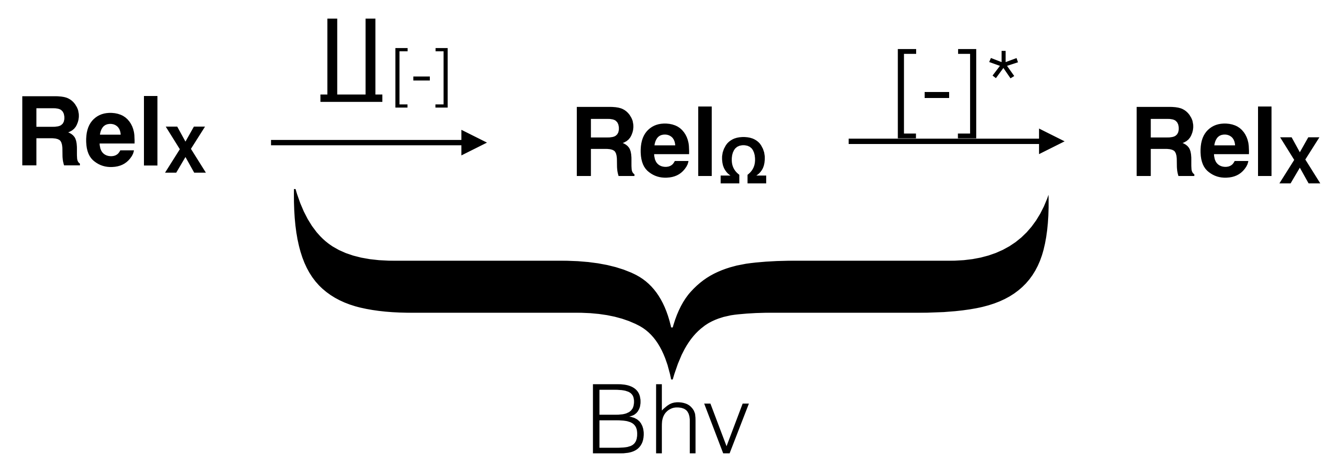

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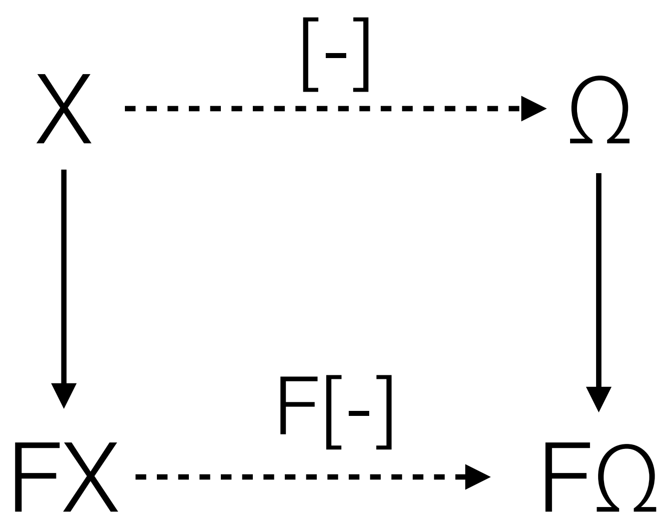

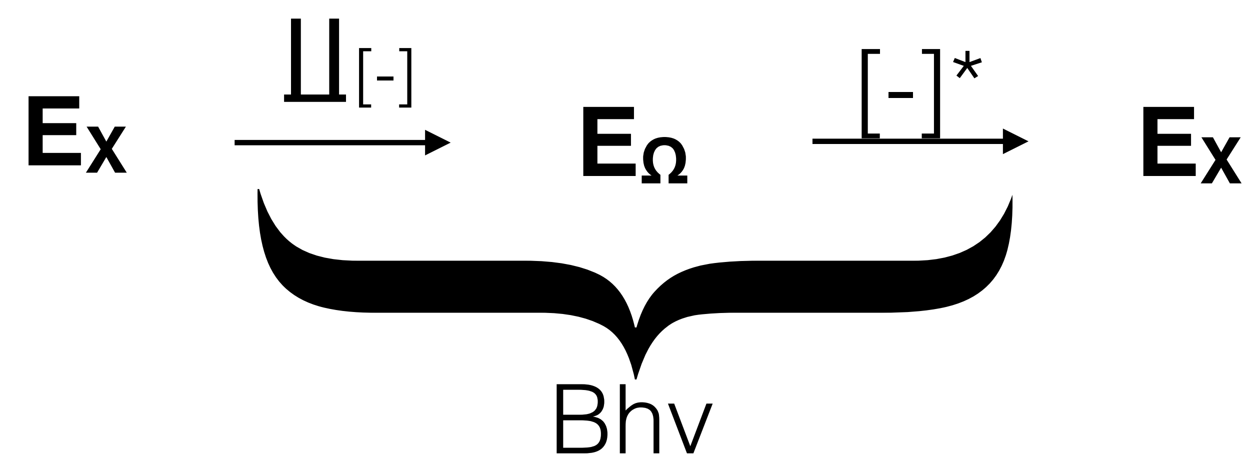

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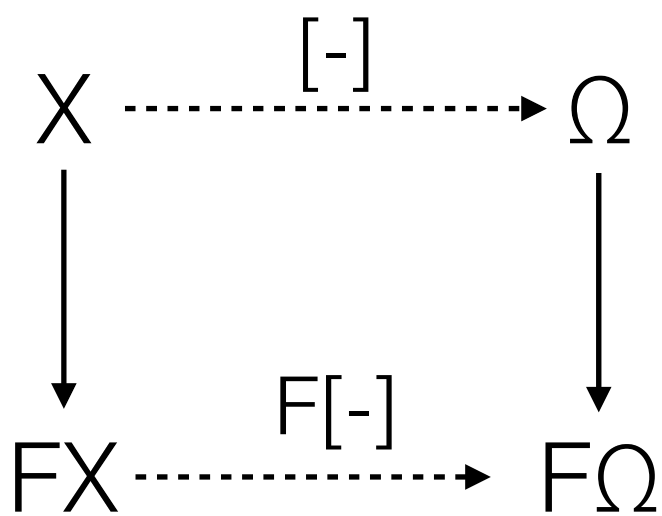

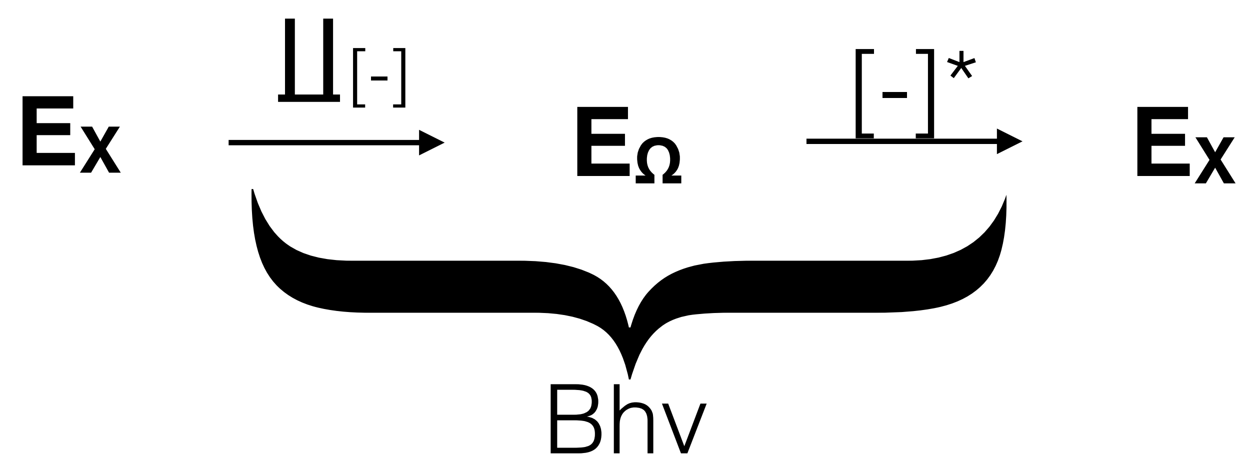

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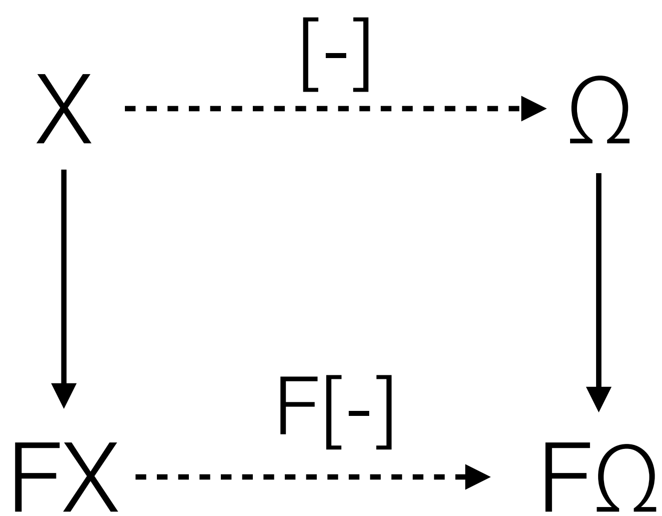

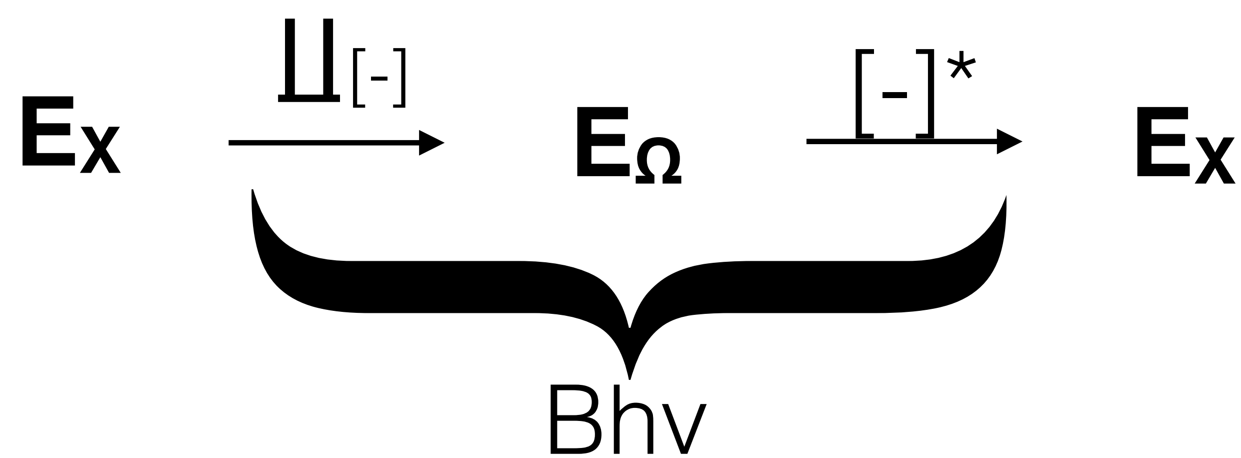

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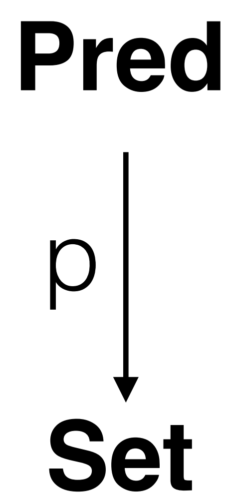

$Bhv(P \subseteq X) = \{x \mid x \sim x' \in P\}$

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Corollary:
For the monotone predicate lifting (in Coalgebraic modal logic) up-to Beh is compatible

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Corollary:
up-to language equivalence (at the beginning of this talk) and up-to bisimilarity (Milner) are compatible

References

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