

Distributive Laws and Decidable Properties of SOS Specifications

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SOS

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Stream systems: $\langle X, h : X \rightarrow A \times X \rangle$

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Stream SOS:

$$\frac{x \xrightarrow{a} x' \quad y \xrightarrow{b} y'}{\text{alt}(x, y) \xrightarrow{a} \text{alt}(y', x')}$$

$$\text{alt}(a_1 a_2 a_3 \cdots, b_1 b_2 b_3 \cdots) = a_1 b_2 a_3 b_4 \cdots$$

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Defines:

- operations on streams
- a stream system on terms

GSOS

Things can go **wrong**:

$$\frac{}{\mathbf{C} \xrightarrow{a} \mathbf{q}(\mathbf{C})} \quad \frac{x \xrightarrow{a} x' \xrightarrow{b} x''}{\mathbf{q}(x) \xrightarrow{b} \mathbf{q}(x'')} \quad (\text{for } a, b \in A)$$

this does not induce a system.

GSOS:

$$\frac{x_1 \xrightarrow{a_{1,1}} y_{1,1} \quad x_1 \xrightarrow{a_{1,2}} y_{1,2} \quad \cdots \quad x_i \xrightarrow{a_{i,j}} y_{i,j} \quad \cdots \quad x_i \xrightarrow{b_{i,j}} \cdots}{\mathbf{f}(x_1, \dots, x_k) \xrightarrow{b} \mathbf{t}}$$

Algebras and coalgebras

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$$\Sigma X \rightarrow X$$

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Rules may form a **distributive law** of Σ over F :

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Fact: such rules behave well.

Laws, laws, laws...

$$\lambda : \Sigma F \implies F \Sigma$$

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$$\frac{}{a.x \xrightarrow{a} x}$$

More laws...

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coGSOS: $\lambda : \Sigma F^\infty \Longrightarrow F(Id + \Sigma)$

$$\frac{x \xrightarrow{a} y \xrightarrow{b} z}{\mathbf{f}(x) \xrightarrow{c} \mathbf{g}(z)}$$

For $F X = A \times X$ **we have** $F^\infty X = (X \times A)^\omega$

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But what is it?

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Naive idea: allow both

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But perhaps this can be cleverly restricted?

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$$FX = A \times X \quad (\text{streams})$$

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Format = decidable property of specs.

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Def: λ extends ρ :

$$\begin{array}{ccc} \Sigma F^\infty & \xRightarrow{\rho} & F \Sigma^* \\ \downarrow \iota_{F^\infty} & & \uparrow \pi_{\Sigma^*} \\ \Sigma^* F^\infty & \xRightarrow{\lambda} & F^\infty \Sigma^* \end{array}$$

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Theorem:

It is undecidable whether a mixed-GSOS spec extends to a unique law

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Proof:

reduction from halting problem of queue machines.

Queue machines

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Fact: halting problem undecidable.

From a QM to SOS

Take a QM $\langle Q, A, q_0, \delta_0, \delta_1, \delta_2 \rangle$.

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(if $\delta_0(q) = (c, q')$)

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(if $\delta_1(q, a) = (c, q')$)

$$\frac{x \xrightarrow{a} y \xrightarrow{b} z}{q(x) \xrightarrow{c} q'(z)}$$

(if $\delta_2(q, a, b) = (c, q')$)

From a QM to SOS

Fact:

the QM halts

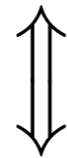


the SOS does **not** extend to a law.

From a QM to SOS

Fact:

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the SOS does **not** extend to a law.

Key step:

finite runs of the QM



initial segments of $\lambda_X(C)$

for each λ that extends the SOS.

The case of LTSs

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 - a beautiful notion, but
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 - existence of a stable model

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3. Also related to work on stream productivity.