Uniform Eilenberg Theorems: Syntactic Algebras For Free

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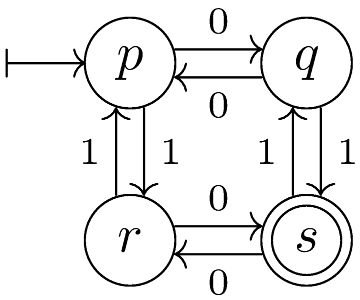
Behaviour of finite automata



regular languages $L \subseteq \Sigma^*$

regular Moore behaviours $\Sigma^* \to \mathcal{O}$

ω-regular languages

regular tree languages

Algebraic Acceptors

Rabin-Scott (1959)

$L \subseteq \Sigma^*$ regular $\iff \exists \quad \Sigma^* \overset{\varphi}{\twoheadrightarrow} M, \quad P \subseteq M, \quad L = \varphi^-(P)$

finite monoid

Eilenberg (1974)

regular sequential behaviours $\Sigma^* \to \mathcal{O}^*$ $\iff$ finite monoids

Wilke (1991)

ω-regular languages $\iff$ binoids (2-sorted)

Bojanczyk-Walukiewics (2006)

regular tree languages $\iff$ forest algebras (2-sorted)

Many Algebraic Acceptors

… For Regular Languages

Automata	Algebraic Acceptors	
deterministic	finite monoids M, $P : M \to \{0, 1\}$	Rabin-Scott (1959)
alternating	finite ordered monoids M, monotone $P : M \to \{0 \leq 1\}$	Pin (1986)
nondeterministic	finite idempotent semirings M, $\vee$-preserving $P : M \to \{0 \leq 1\}$	Polák (2001)
xor	finite $\mathbb{Z}_2$-algebra M, linear $P : M \to \mathbb{Z}_2$	Reutenauer (1978)

Canonical Algebraic Acceptors

- Syntactic monoid for regular languages
- Syntactic semigroup/monoid for regular Moore/Mealy behaviour
- Pin: syntactic ordered monoid
- Polák: syntactic idempotent semiring
- Reutenauer: syntactic $\mathbb{Z}_2$-algebra
- Wilke: syntactic binoid
- Bojanczyk/Walukiewics: syntactic forest algebra

Hallmark of Algebraic Automata Theory

Eilenberg‘s Theorem

form isomorphic lattices

varieties of regular languages $\cong$ pseudovarieties of monoids

$\mathbb{HSP}_f$-closed class of finite monoids

$\mathcal{V} = (V_\Sigma)_\Sigma$, $V_\Sigma \subseteq \mathsf{Reg}(\Sigma)$ and:

(1) $L \in V_\Sigma \implies f^{-1}(L) \in V_\Delta$ for all $f : \Delta^* \to \Sigma^*$

(2) each V_Σ closed under derivatives $w^{-1}L$ and Lw^{-1}

(3) each V_Σ closed under $\bot, \wedge, \neg$ (Boolean opns)

Other Eilenberg-type theorems

form isomorphic lattices

$\mathcal{V} = (V_\Sigma)_\Sigma$, $V_\Sigma \subseteq \mathrm{Reg}(\Sigma)$ such that (1), (2), and (3') each V_Σ closed under . . .		pseudovarieties of . . .
Eilenberg:	$\bot, \wedge, \neg$	monoids
Pin:	$\bot, \top, \wedge, \vee$	ordered monoids
Polák:	$\bot, \vee$	idempotent semirings
Reutenauer:	$\bot, \triangle$	$\mathbb{Z}_2$-algebras

Our goals …

- General results concerning duality of algebraic and coalgebraic recognition
- General version of Eilenberg‘s theorem uniformly

Rest of the Talk

- Technical details of our setting
- Statements of several versions of Generalized Eilenberg‘s Theorem

Recipe – Ingredients

Take: Σ signature of operations

$O_{\mathcal{D}}$ non-trivial finite Σ-algebra whose operations commute:

$|O_{\mathcal{D}}| \geq 2$

$$\sigma_{O_{\mathcal{D}}}(\tau_{O_{\mathcal{D}}}(x_{ij})_i)_j = \tau_{O_{\mathcal{D}}}(\sigma_{O_{\mathcal{D}}}(x_{ij})_j)_i \qquad \forall \sigma, \tau \in \Sigma$$

Ordered algebra if operations monotone.

Define: $\mathcal{D} = \mathbb{HSP}(O_{\mathcal{D}})$

Prove:

1. $\mathcal{D}$ is a commutative locally finite variety
2. $O_{\mathcal{D}}$ is an injective cogenerator
3. $\mathcal{D}$-epis are surjective

our assumptions

Examples of Ingredients

Σ	$O_{\mathcal{D}}$	$\mathcal{D} := \mathbb{HSP}(O_{\mathcal{D}})$
$\emptyset$	any set	Set
$\emptyset$	2-chain $0 \leq 1$	Poset
$\{0, \vee\}$	any $\vee$-0-semilattice	JSL_0
$\{0, +, (k.-)_{k \in \mathbb{F}}\}$	vector space $\mathbb{F}^n$	Vect$(\mathbb{F})$
$\{0\}$	any pointed set	Set_*
$\{m.- : m \in M\}$	$\exists O_{\mathcal{D}}$	M-sets
$\{0, +, (r.-)_{r \in R}\}$	$\exists O_{\mathcal{D}}$	Mod(R)

Recipe – Preparation

Shake $O_{\mathcal{D}}$ to define a new algebra $O_{\mathcal{C}}$:

$$\text{carrier} = \text{carrier of } O_{\mathcal{D}}$$

$$\text{operations} = \text{all homomorphisms } O_{\mathcal{D}}^n \to O_{\mathcal{D}} \quad (n \in \mathbb{N})$$

Cooking tip: only need operations that generate all others

Define: $\mathcal{C} = \mathbb{SP}(O_{\mathcal{C}})$

Post Preparation Situation

$O_{\mathcal{D}}$	$\mathcal{D}$	$O_{\mathcal{C}}$	$\mathcal{C} := \mathbb{SP}(B)$
set $2 = \{0, 1\}$	Set	boolean algebra 2	BA
2-chain $0 \leq 1$	Poset	lattice 2	DL$_{01}$
2-chain	JSL$_0$	2-chain	JSL$_0$
vector space $\mathbb{F}$	Vect$(\mathbb{F})$	vector space $\mathbb{F}$	Vect$(\mathbb{F})$
set n	Set	Post algebra of order n	Post$_n$
pointed set 2	Set$_*$	boolean ring 2 sans unit	BoolRing$_{nu}$
$\bullet \rightleftarrows \bullet \quad \bullet \quad \bullet$	$\mathbb{Z}_2$-sets	$p \rightleftarrows \neg p \quad \top \quad \bot$	$[\mathbb{Z}_2, \mathsf{BA}]$

Duality

Theorem. Clark-Davey (1998): Natural Duality

Finite $\mathcal{D}$-algebras are dual to finite $\mathcal{C}$-algebras:

Duality $\mathcal{D}_f^{op} \cong \mathcal{C}_f$ arises from

$$\mathcal{D}_f(-, O_{\mathcal{D}}) : \mathcal{D}_f^{op} \to \mathsf{Set} \qquad \mathcal{C}_f(-, O_{\mathcal{C}}) : \mathcal{C}_f^{op} \to \mathsf{Set}$$

by adding algebra structure to the hom-sets.

Recipe – The Secret Spice

Bimonoids

A $\mathcal{D}$-bimonoid consist of:

- an algebra $X \in \mathcal{D}$
- a monoid structure $|X| \times |X| \xrightarrow{\bullet} |X| \xleftarrow{e} 1$

bilinear $=$ $\mathcal{D}$-morphism is each argument

Examples of Bimonoids

$\mathcal{D}$	$\mathcal{D}$-bimonoids	$\mathcal{C}$
Set	monoids	BA
Poset	ordered monoids	DL_{01}
JSL_0	idempotent semirings with 0	JSL_0
$\mathsf{Vect}(\mathbb{F})$	associative algebras over $\mathbb{F}$	$\mathsf{Vect}(\mathbb{F})$
Set	monoids	Post_n
Set_*	monoids with zero	$\mathsf{BoolRing}_{nu}$
$\mathbb{Z}_2$-sets	monoids with compatible involution	$[\mathbb{Z}_2, \mathsf{BA}]$

Towards Eilenberg‘s Theorem

Recap: 1. Started with finite Σ-algebra $O_{\mathcal{D}}$.

2. Constructed $\mathcal{D}$, $O_{\mathcal{C}}$, $\mathcal{C}$ with $\mathcal{D}_f^{op} = \mathcal{C}_f$.

3. Defined $\mathcal{D}$-bimonoids.

Now consider:

Formal power series $f : \Sigma^* \to O_{\mathcal{C}}$

Define derivatives: $w^{-1}f = \lambda u.f(wu), \qquad fw^{-1} = \lambda u.f(uw).$

Theorem. The following are dual:

regular behaviours in $O_{\mathcal{C}}^{\Sigma^*}$

- The finite Σ-generated $\mathcal{D}$-bimonoids.
- The finite subalgebras $S \subseteq \mathrm{Reg}(\Sigma)$ closed under derivatives.

Proof. The duality restricts a algebra-coalgebra duality extending the natural duality.

Detour: Syntactic Algebras for Free

Recall characterizations of syntactic monoid Syn(L):

(1) Original: via monoid congruence $\sim \subseteq \Sigma^* \times \Sigma^*$

$$u \sim v \quad :\iff \quad \forall x, y \in \Sigma^*.(xuy \in L \iff xvy \in L)$$

(2) Construction: transition monoid of L's minimal dfa

(3) Universal property: smallest Σ-generated monoid recognizing L.

(4) Gehrke, Grigorieff and Pin showed:

Syn(L) dual to the boolean algebra generated
by the languages $u^{-1}Lv^{-1}$ where $u, v \in \Sigma^*$

i.e. atoms $=$ states; multiplication $=$ dual of quotients

Detour: Syntactic Algebras For Free

Theorem. The following are dual:

syntactic algebra for a rational Moore behaviour $\beta : \Sigma^* \to |\mathcal{O}_\mathcal{C}| = |\mathcal{O}_\mathcal{D}|$

least subalgebra $S \subseteq \mathcal{O}^{\Sigma^*}$ containing β and closed under derivatives

Proof. Follows directly by restriction duality for finite algebras.

(Definition of syntactic algebra for free.)

Example. Original case

$$\mathcal{O}_\mathcal{D} = 2 \in \mathcal{D} = \mathsf{Set} \qquad \mathcal{O}_\mathcal{C} = 2 \in \mathcal{C} = \mathsf{BA}$$

In other cases analogous characterizations as for syntactic monoid.

Local Eilenberg Theorem

Local $=$ fix alphabet Σ for varieties:

(1) A local variety of $\mathcal{C}$-behaviours is $V_\Sigma \subseteq \mathrm{Reg}(\Sigma)$ closed under

derivatives $w^{-1}f$ and fw^{-1}

$\mathcal{C}$-operations (e.g. Boolean opns $\bot, \neg, \wedge$)

(2) A local variety of $\mathcal{D}$-bimonoids is a collection of finite Σ-generated bimonoids closed under

subdirect products

homomorphic images

Theorem. The inclusion ordered lattices of (1) and (2) are isomorphic.

Proof. Take ideal completion of previous duality

(concerning finite Σ-generated $\mathcal{D}$-bimonoids)

Semi Local Eilenberg Theorem

Missing. preimage closure in $\mathcal{C}$-varieties

(1) A semi local variety of $\mathcal{C}$-behaviours is $V_\Sigma \subseteq \mathrm{Reg}(\Sigma)$ closed under

- derivatives $w^{-1}f$ and fw^{-1}
- $\mathcal{C}$-operations (e.g. Boolean opns $\bot, \neg, \wedge$)
- preimages of $\mathcal{D}$-bimonoid morphisms $f : \mathbf{F}\Sigma \to \mathbf{F}\Sigma$

(2) A semi local variety of $\mathcal{D}$-bimonoids is a collection of finite Σ-generated fully invariant bimonoids closed under

- subdirect products
- homomorphic images

$$\begin{tikzcd}
\mathbf{F}\Sigma \arrow[r, "\forall f"] \arrow[d, two heads] & \mathbf{F}\Sigma \arrow[d, two heads] \\
A \arrow[r, dashed, "\exists u_f"'] & A
\end{tikzcd}$$

Theorem. The inclusion ordered lattices of (1) and (2) are isomorphic.

Proof. Take ideal completion of restriction of previous duality.

Global Eilenberg Theorem

Still missing. varying alphabets

(1) A variety of $\mathcal{C}$-behaviours is $\mathcal{V} = (V_\Sigma)_\Sigma$, $V_\Sigma \subseteq \mathsf{Reg}(\Sigma)$ closed under

derivatives $w^{-1}f$ and fw^{-1}

$\mathcal{C}$-operations (e.g. Boolean opns $\bot, \neg, \wedge$)

preimages of $\mathcal{D}$-bimonoid morphisms $f : \mathbf{F}\Delta \to \mathbf{F}\Sigma$

(2) A pseudovariety of $\mathcal{D}$-bimonoids is an $\mathbb{HSP}_f$-closed collection of $\mathcal{D}$-bimonoids.

Theorem. The pointwise inclusion ordered lattices of (1) and (2) are isomorphic.

Proof. Take ideal completion of result concerning

finitely generated varieties of $\mathcal{D}$-bimonoids

finitely generated varieties of rational $\mathcal{C}$-behaviours

Conclusions

- Uniform category-theoretic approach to Eilenberg-type theorems
- local, semi-local and global theorem specializing to new and existing results (Gehrke/Grigorieff/Pin, Pin, Eilenberg, Polák, …)
- Lots of syntactic algebras for free (parametric in well-behaved finite algebras whose opns commute)

Future Work

- Profinite equations and Reitermann's theorem
- 2-sorted cases (e.g. forest algebras of Bonjanczyk/Walukiewics)
- Understand why automata-theoretic notions have algebraic counterpart → uniform decidability results