Coalgebraic Traces Revisited

Corina Cîrstea
University of Southampton

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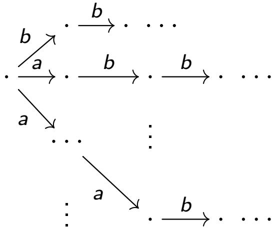
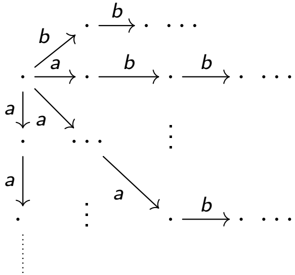
Motivation

- well understood theory of finite traces [HJS 2007]
  - coinduction in Kleisli category
- some work on infinite traces [Jacobs 2004], [Cîrstea 2011] ...
  - applies to non-deterministic and probabilistic systems
- but problems still remain!





This Talk

1. a unified account of finite and maximal (possibly infinite) traces
2. a fibrational account of maximal traces
    - based on [Hasuo et al 2013]

The General Setting

Similarly to [HJS 2007], we work with $\mathsf{T} \circ F$-coalgebras, where:

- commutative monad $\mathsf{T} : \mathsf{C} \to \mathsf{C}$ describes the branching type

  e.g. $\mathcal{P}$ (powerset), $\mathcal{S}$ (sub-probability distributions), $\mathcal{M}$ (finite multisets)

- $F : \mathsf{C} \to \mathsf{C}$ describes the transition type

  e.g. Id, $A \times \mathsf{Id}$, $1 + A \times \mathsf{Id}$

- distributive law $\lambda : F \circ \mathsf{T} \Rightarrow \mathsf{T} \circ F$ (compatible with monad structure)

This covers:

- non-deterministic systems: $\mathcal{P}(1 + A \times \mathsf{Id})$
- probabilistic systems: $\mathcal{S}(A \times \mathsf{Id})$
- weighted transition systems: $\mathcal{M}(A \times \mathsf{Id})$
- ...

Finite vs Maximal Traces: Our Expectations

Fix $\mathsf{T} \circ F$-coalgebra $\gamma : X \to \mathsf{T}FX$.

Finite traces:

- initial F-algebra (I, α) gives the possible finite traces
- finite trace map $\mathrm{ftr}_\gamma : X \to \mathsf{T}(I)$

Maximal traces:

- final F-coalgebra (Z, ζ) gives the possible infinite traces
- maximal trace map $\mathrm{mtr}_\gamma : X \to \mathsf{T}(Z)$

Both cases:

- $\mathsf{T} = \mathcal{P} \Rightarrow$ sets of traces
- $\mathsf{T} = \mathcal{S} \Rightarrow$ sub-probability *distributions* over traces
- trace map must "commute" with coalgebra structure ...

Maximal Traces: Expectations Made Formal

- working in Kl(T) abstracts away branching structure [HJS 2007] !
- $F : \mathsf{Set} \to \mathsf{Set}$ lifts to $\overline{F} : \mathsf{Kl(T)} \to \mathsf{Kl(T)}$ (using λ)

$$\begin{tikzcd} \mathsf{Kl(T)} \arrow[r, "\overline{F}"] & \mathsf{Kl(T)} \\ \mathsf{Set} \arrow[u] \arrow[r, "F"'] & \mathsf{Set} \arrow[u] \end{tikzcd}$$

- $\gamma : X \to \mathsf{T}FX$ yields $\overline{\gamma} : X \to \overline{F}X$
- $\zeta : Z \to FZ$ lifts to $\overline{\zeta} : Z \to \overline{F}Z$
- trace maps must define $\overline{F}$-coalgebra morphism:

$$\begin{tikzcd} X \arrow[r, "\overline{\gamma}"] \arrow[d, "\mathsf{mtr}_\gamma"'] & \overline{F}X \arrow[d, "\overline{F}\mathsf{mtr}_\gamma"] \\ Z \arrow[r, "\overline{\zeta}"'] & \overline{F}Z \end{tikzcd}$$

From Comm. Monads to Comm. Monoids [Coumans & Jacobs 2011]

For $\mathrm{T} : \mathrm{C} \to \mathrm{C}$ a commutative monad:

- commutative monoid $(\mathrm{T}1, \eta_1(*), \bullet)$
  - unit from $\eta_1 : 1 \to \mathrm{T}(1)$
  - multiplication:

$$T(1) \times T(1) \xrightarrow{st} T(1 \times T(1)) \xrightarrow{T\pi_2} T^2(1) \xrightarrow{\mu_1} T(1)$$

- e.g. $\mathrm{T} = \mathcal{P}$: $\{0, 1\}$ with $\wedge$
- e.g. $\mathrm{T} = \mathcal{S}$: $[0, 1]$ with $*$
- e.g. $\mathrm{T} = \mathcal{M}$: $\mathbb{N}$ with $*$

From Additive Comm. Monads to Comm. Monoids [Coumans & Jacobs 2011]

For $\mathsf{T} : \mathsf{C} \to \mathsf{C}$ a commutative monad with $\mathsf{T}0 = 1$:

- 0 element in $\mathsf{T}X$: $$\begin{array}{ccc} T0 & \xrightarrow{T!_X} & TX \\ * & \longmapsto & 0 \end{array}$$
- zero map $0 : Y \to TX$: $Y \xrightarrow{!_Y} T0 \xrightarrow{T!_X} TX$
- $T(X+Y) \xrightarrow{\langle \mu \circ Tp_1, \mu \circ Tp_2 \rangle} T(X) \times T(Y)$

  where $X + Y \xrightarrow{p_1 = [\eta, 0]} TX$
- $\langle \mu \circ Tp_1, \mu \circ Tp_2 \rangle$ iso $\implies$ comm. monoid $(\mathsf{T}X, 0, +)$
  - e.g. $\mathsf{T} = \mathcal{P}$: $\{0, 1\}$ with $\vee$
  - e.g. $\mathsf{T} = \mathcal{S}$: not an iso
  - e.g. $\mathsf{T} = \mathcal{M}$: $\mathbb{N}$ with $+$

From Partially Additive Comm. Monads to Partial Comm. Monoids

- $\langle \mu \circ Tp_1, \mu \circ Tp_2 \rangle$ mono $\implies$ partial comm. monoid: $(\mathrm{T}(X), 0, +)$:

$$\begin{tikzcd} T(X) & T(X+X) \arrow[l, "{[1_X,1_X]}"'] \arrow[r, shift left, "{\langle \mu\circ Tp_1, \mu\circ Tp_2 \rangle}"] & T(X)\times T(X) \arrow[l, shift left, dashed, "partial"] \arrow[ll, bend left, "+"] \end{tikzcd}$$

  - e.g. $\mathrm{T} = \mathcal{S}$: $[0,1]$ with partial $+$

Some Partial Orders

- assume partial comm. monoid $(\mathsf{T}(X), 0, +)$
- for $x, y \in TX$:

$$x \sqsubseteq y \;\text{ iff }\; \exists z \in TX. x + z = y$$

- above yields partial order with $\bot = 0$ on $\mathsf{T}1$
  - instantiates to natural orders for $\mathsf{T} = \mathcal{P}$, $\mathsf{T} = \mathcal{S}$, $\mathsf{T} = \mathcal{M}$
- also yields (pointwise) partial order with $\bot$ on $\mathsf{Kl}(\mathsf{T})$-homsets, in particular on $(X \to \mathsf{T}Z)$
- finally, it yields (pointwise) partial order on $X \to (Z \to \mathsf{T}1)$

Note: in all our examples, $\mathsf{T}Z$ is (up to iso) a subset of $(Z \to \mathsf{T}1)$!

Summary of Examples

- $\mathsf{T} = \mathcal{P}$: $(\{0,1\}, 1, \wedge, 0, +, \leq)$, $\bot = 0$, $\top = 1$
- $\mathsf{T} = \mathcal{S}$: $([0,1], 1, *, 0, +, \leq)$, $\bot = 0$, $\top = 1$
- $\mathsf{T} = \mathcal{M}$: $(\mathbb{N}, 1, *, 0, +, \leq)$, $\bot = 0$, no $\top$

Summary of Finite Trace Semantics [HJS 2007]

- initial object in Set gives final object in Kl(T)
- initial F-algebra (I, α) in Set gives final $\overline{F}$-coalgebra $(I, \overline{\alpha})$ in Kl(T)
- finality in Coalg($\overline{F}$) yields finite trace semantics

$$\mathrm{ftr}_\gamma : (X, \overline{\gamma}) \to (I, \overline{\alpha})$$

in Kl(T) :

$$\begin{tikzcd}
X \arrow[rrr, dashed, "\mathrm{ftr}_\gamma"] \arrow[d] \arrow[dr] \arrow[drr] & & & I \arrow[dlll, dashed] \arrow[dll, dashed] \arrow[dl, dashed] \\
0 & \overline{F}0 \arrow[l] & \overline{F}^2 0 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

in Set :

$$0 \longrightarrow F0 \longrightarrow F^2 0 \longrightarrow \ldots$$

Example: Finite Traces for LTSs

$$\begin{tikzcd}
x_0 \arrow[r, "a"] \arrow[dr, "a"'] & y_0 & \\
& \cdots \arrow[dr, "a"'] & \vdots \\
& \vdots & .
\end{tikzcd}$$

step 0: $x_0 \mapsto \{\}, \quad y_0 \mapsto \{\}$

step 1: $x_0 \mapsto \{\}, \quad y_0 \mapsto \{*\}$

step 2: $x_0 \mapsto \{a*\}, \quad y_0 \mapsto \{*\}$

...

step n: $x_0 \mapsto \{a*, \ldots, a^n*\}$

...

ω: $x_0 \mapsto \{a^n* \mid n = 1, 2, \ldots\}$

Maximal Traces – Some Problems

- replace initial F-algebra by final F-coalgebra (Z, ζ)
- viewing $\mathsf{T} \circ F$-coalgebra as $\overline{F}$-coalgebra $(X, \overline{\gamma})$ yields

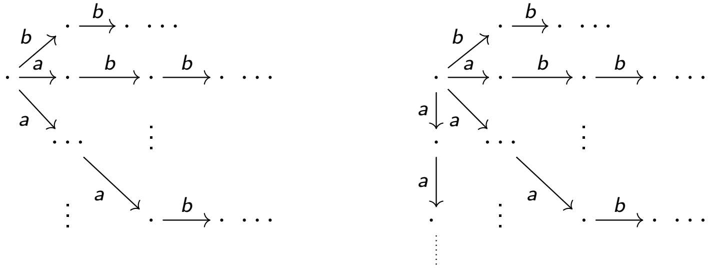
$$
\begin{tikzcd}
 & X \arrow[d, "\overline{\gamma}_0"'] \arrow[dr, "\overline{\gamma}_1"] \arrow[drr, "\overline{\gamma}_2"] & & & Z \arrow[dlll, dashed, "J\pi_0"'] \arrow[dll, dashed, "J\pi_1"'] \arrow[dl, dashed, "J\pi_2"] \\
\text{in } \mathsf{Kl}(\mathsf{T}) : & 1 & \overline{F}1 \arrow[l] & \overline{F}^2 1 \arrow[l] & \ldots \arrow[l] \\
\text{in } \mathsf{C} : & 1 & F1 \arrow[l] & F^2 1 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}
$$

Some problems:

- $(\overline{\gamma}_i)_{i \in \omega}$ only defines a cone when T affine ($\eta_1 : 1 \to \mathsf{T}1$ iso)
- even for affine T, $(Z, (J\pi_i)_{i \in \omega})$ not always limiting!

An Example

Two $\mathcal{P}(A \times \mathrm{Id})$-systems:



Cannot distinguish above systems based on the induced (lax) cones:

$$\begin{tikzcd}
b & bb \arrow[l, mapsto] & bbb \arrow[l, mapsto] & \cdots \\
a & ab \arrow[l, mapsto] & abb \arrow[l, mapsto] & \cdots \\
& aa \arrow[ul, mapsto] & aab \arrow[l, mapsto] & \cdots \\
& & aaa \arrow[ul, mapsto] & aaab \arrow[l, mapsto] & \cdots \\
& & & aaaa \arrow[ul, mapsto] & \cdots
\end{tikzcd}$$

Maximal Traces – Towards a Solution

- work with lax cones:

$$\begin{tikzcd}
X \arrow[rrr, dashed, "m"] \arrow[d, "\overline{\gamma}_0"'] \arrow[dr, "\overline{\gamma}_1"] \arrow[drr, "\overline{\gamma}_2"] \arrow[d, phantom, shift left=3, "\sqsupseteq"] & & & Z \arrow[dlll, dashed] \arrow[dll, dashed] \arrow[dl, dashed] \\
1 & \overline{F}1 \arrow[l] & \overline{F}^2 1 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

  - works if $\eta_1(*)$ is $\top$ for $\sqsubseteq$ on $\mathsf{T}(1)$

- trace map not uniquely determined by $(\overline{\gamma}_i)_{i \in \omega}$!
- many $m : X \to Z$ s.t. $m; J\pi_i \sqsubseteq \overline{\gamma}_i$ for $i \in \omega$
  - maximal one only defines oplax coalgebra morphism $m; \overline{\zeta} \sqsupseteq \overline{\gamma}; \overline{F}m$
  - ...and is not always the right choice!

Lesson: need to go beyond ω ...

Finite Traces Revisited

- assume $\mathcal{K}\ell(\mathsf{T})(X, Z)$ is a poset with $\perp$ and limits of increasing chains
  - true for the powerset, sub-distribution and multiset monads
- monotone operator on $\mathcal{K}\ell(\mathsf{T})(X, Z)$:

$$\mathcal{K}\ell(\mathsf{T}): \qquad X \xrightarrow{f} I \qquad \stackrel{O}{\longmapsto} \qquad X \xrightarrow{\overline{\gamma}} \overline{F}X \xrightarrow{\overline{F}f} \overline{F}I \xrightarrow{\overline{\alpha}} I$$

Definition

Let $\gamma : X \to \mathsf{T}FX$. The finite trace map $\mathrm{ftr}_\gamma : X \to \mathsf{T}I$ of γ is the least fixed point of the operator O.

Maximal Traces - a Solution

- no $\top$ in $\mathsf{Kl}(\mathsf{T})(X, Z) \ldots$
- so instead of maps $X \to Z$ in $\mathsf{Kl}(\mathsf{T})$ consider maps $X \to (\mathsf{T}1)^Z$ in Set
- assume T1-predicate lifting for F: $\tilde{\lambda}_X : F(T1)^X \to (T1)^{FX}$
  - think of elements of $(T1)^Y$ as predicates over Y
- assume T1 is a poset with $\top$ and limits of decreasing chains
  - true for the powerset and sub-distribution monads
  - extends to poset $(X \to (\mathsf{T}1)^Z)$
- monotone operator $\tilde{O}$ on $(X \to (\mathsf{T}1)^Z)$, mapping $X \xrightarrow{f} (\mathsf{T}1)^Z$ to

$$X \xrightarrow{\gamma} \mathsf{T}FX \xrightarrow{\mathsf{T}Ff} \mathsf{T}F(\mathsf{T}1)^Z \xrightarrow{\mathsf{T}\tilde{\lambda}_Z} \mathsf{T}(\mathsf{T}1)^{FZ} \xrightarrow{\alpha} (\mathsf{T}1)^{FZ} \xrightarrow{(\mathsf{T}1)^{\zeta}} (\mathsf{T}1)^Z$$

Definition

Let $\gamma : X \to \mathsf{T}FX$. The maximal trace map $\mathsf{mtr}_\gamma : X \to (\mathsf{T}1)^Z$ of γ is the greatest fixed point of the operator $\tilde{O}$.

Conjecture: mtr_γ is a map in $\mathsf{Kl}(\mathsf{T})$.

Example: Maximal Traces for LTSs

$$
\begin{tikzcd}
x_0 \arrow[r, "b"] \arrow[dr, "a"'] & y_0 & & & & \\
& x_1 \arrow[r, "b"] \arrow[dr, "a"'] & \cdot \arrow[r, "b"] & y_1 & & \\
& & x_2 \arrow[r, "b"] \arrow[dr, dotted, no head] & \cdot \arrow[r, "b"] & \cdot \arrow[r, "b"] & y_2 \\
& & & {} & &
\end{tikzcd}
$$

step 0: $x_i \mapsto \overline{A},\ y_i \mapsto \overline{A}$ $\qquad\qquad \overline{A} = A^* \cup A^\omega$

step 1: $x_i \mapsto b\overline{A} \cup a\overline{A},\ y_i \mapsto \{*\}$

step 2: $x_0 \mapsto \{b*\} \cup ab\overline{A} \cup aa\overline{A}$

$\qquad\qquad x_{i+1} \mapsto bb\overline{A} \cup ab\overline{A} \cup aa\overline{A}$

...

step ω: $x_i \mapsto \{a^j b \mid j \in \mathbb{N}\} \cup \{aa\ldots\}$

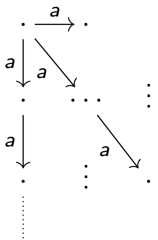
$\Longrightarrow$ all invalid traces (including infinite ones) rejected in finite time!

Is this always the case ?

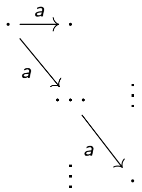
Example: Maximal Traces for LTSs

No!

(1)



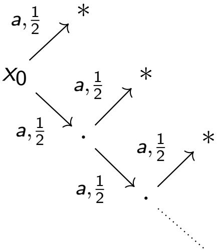
(2)



Infinite trace $aa\ldots$ not rejected at ω for system (2)!

Question: when can we stop at ω?

Example: Maximal Traces for PTSs



After ω steps: $x_0 \mapsto (a{*} \mapsto \frac{1}{2},\ aa{*} \mapsto \frac{1}{4},\ \ldots,\ a^\omega \mapsto 0)$

Fixpoint always reached at (or before) ω ! Reason: $\mathcal{S}$ preserves limits of ω^{op}-chains.

Coinductive Predicates [Hasuo et al 2013]

- poset fibration: $\begin{tikzcd} \mathbb{P} \arrow[d, "p"'] \\ \mathrm{Set} \end{tikzcd}$ (each fibre is a poset)
- predicate lifting: $\begin{tikzcd} \mathbb{P} \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{P} \arrow[d, "p"] \\ \mathrm{Set} \arrow[r, "F"'] & \mathrm{Set} \end{tikzcd}$ (φ preserves cartesian maps)
- φ-coinductive predicate for $\gamma : X \to FX$: final $\gamma^* \circ \varphi_X$-coalgebra

$$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{\gamma^*} \mathbb{P}_X$$

- φ-invariant for $\gamma : X \to FX$: $\gamma^* \circ \varphi_X$-coalgebra
- e.g. can be used to interpret basic fragment of modal μ-calculus

Maximal Traces from Coind. Predicates

- predicate over X: $\begin{tikzcd} X \arrow[d, "q"'] \\ (Z \to \mathsf{T}1) \end{tikzcd}$

- predicate lifting for F: $\begin{tikzcd} X \arrow[d, "q"'] \\ (Z \to \mathsf{T}1) \end{tikzcd}$ $\begin{tikzcd} FX \arrow[d, "Fq"] \\ F(Z \to \mathsf{T}1) = F\mathsf{T}Z \arrow[d, "\lambda_Z"] \\ (FZ \to \mathsf{T}1) = \mathsf{T}FZ \arrow[d, "\text{-} \circ \zeta"] \\ (Z \to \mathsf{T}1) \end{tikzcd}$

- predicate lifting for T: $\begin{tikzcd} Y \arrow[d, "q"'] \\ (Z \to \mathsf{T}1) \end{tikzcd}$ $\begin{tikzcd} \mathsf{T}Y \arrow[d, "\mathsf{T}q"] \\ \mathsf{T}(Z \to T1) \arrow[d, "\alpha"] \\ (Z \to \mathsf{T}1) \end{tikzcd}$

Maximal Traces from Coind. Predicates

For $\gamma : X \to \mathsf{T}FX$, take the maximal trace map to be the $\psi \circ \varphi$-coinductive predicate in γ:

$$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{\psi_{FX}} \mathbb{P}_{\mathsf{T}FX} \xrightarrow{\gamma *} \mathbb{P}_X$$

Theorem (Hasuo et al 2013)

For well-founded p, finitary F, coinductive predicate φ for F and F-coalgebra γ, the fixpoint is reached at ω.

Cannot *directly* use above result as our fibrations are not well-founded.

Future Work

- further exploit results in [Hasuo et al 2013]
- other monads (e.g. Giry, multiset)
- connection to automata
- trace logics
- ...