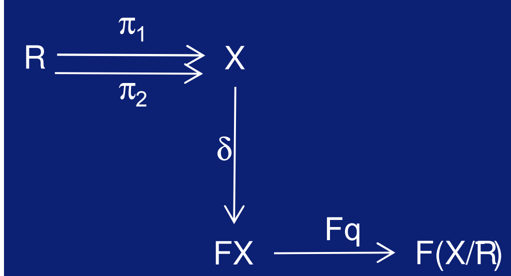
Roma, 16 March 2013



Coalgebraic bisimulation-up-to

Marcello Bonsangue (LIACS)

J. Rot (LIACS), F. Bonchi (ENS), J. Rutten (CWI), D. Pous (ENS), A. Silva (RUN)

Leiden Institute of Advanced Computer Science
Research & Education

Motivation

- *Bisimulation: To prove that x and y are bisimilar, it suffices to show that x and y are related by a relation which is a bisimulation.*

- *Bisimulation-up-to: To prove that x and y are bisimilar, it suffices to show that x and y are related by a relation which is almost a bisimulation.*
  - *Bisimulation up to bisimilarity*
  - *Bisimulation up to equivalence*
  - *Bisimulation up to congruence*
  - *Several combinations of the above*

Modular proofs
Smaller proofs
Equational reasoning

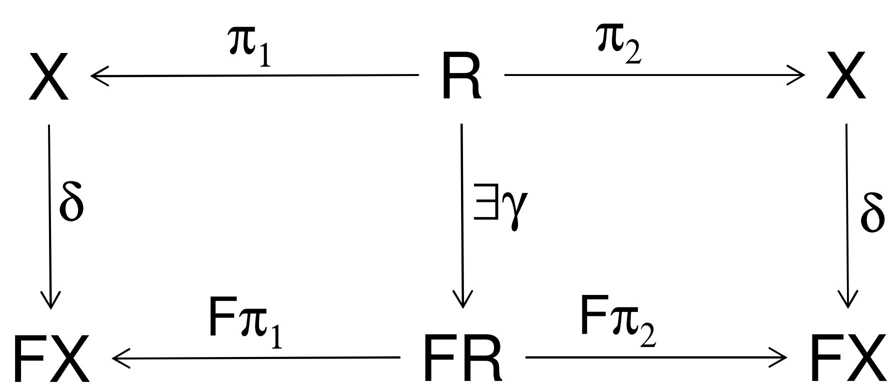
- *Coalgebra: Several notions of bisimilarity*

Historical motivation

1983: Robin Milner	Bisimulation up to bisimilarity for labelled transition systems
1998-2007: Sangiorgi, Pous	Several enhancements of the bisimulation proof method for labelled transition systems
1999: Marina Lenisa	Coalgebraic bisimulation up to: *some results, some problems*
2004: Falk Bartels	Bisimulation up to contextual closure.
2013: Bonchi, Pous	Bisimulation up to congruence for DFA

From bisimulation …

F-bisimulation $R \subseteq X \times X$



Bisimilarity ~ is the largest bisimulation

Example: deterministic automata

$F = 2 \times id^A$ $\langle o,t \rangle : X \to 2 \times X^A$

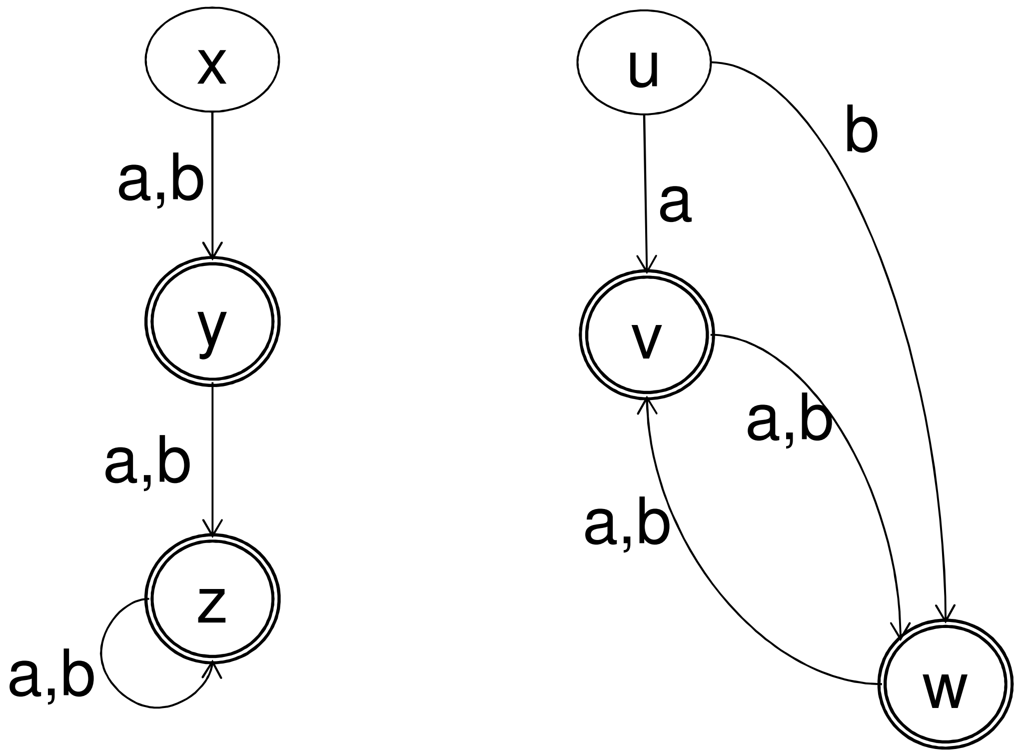
$o(x) = 1$ iff x is accepting

$x_a = t(x)(a)$

F-bisimulation: $R \subseteq X \times X$ such that

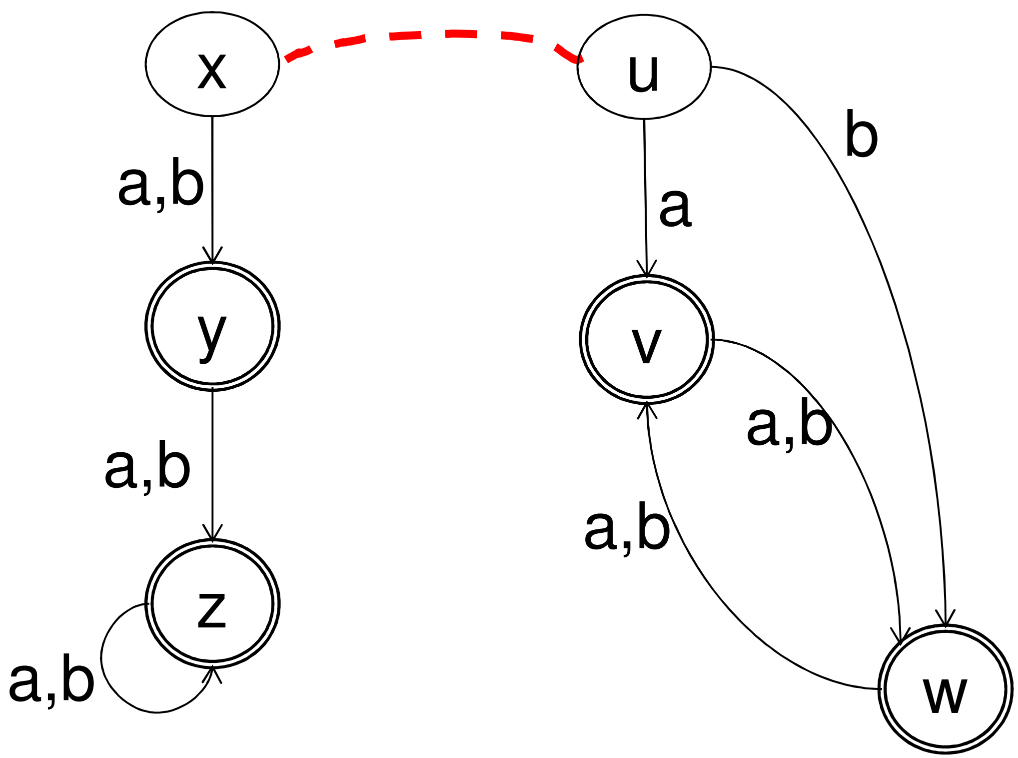
$(x,y) \in R$ implies $o(x) = o(y)$ and $\forall a. (x_a,y_a) \in R$

Example: deterministic automata



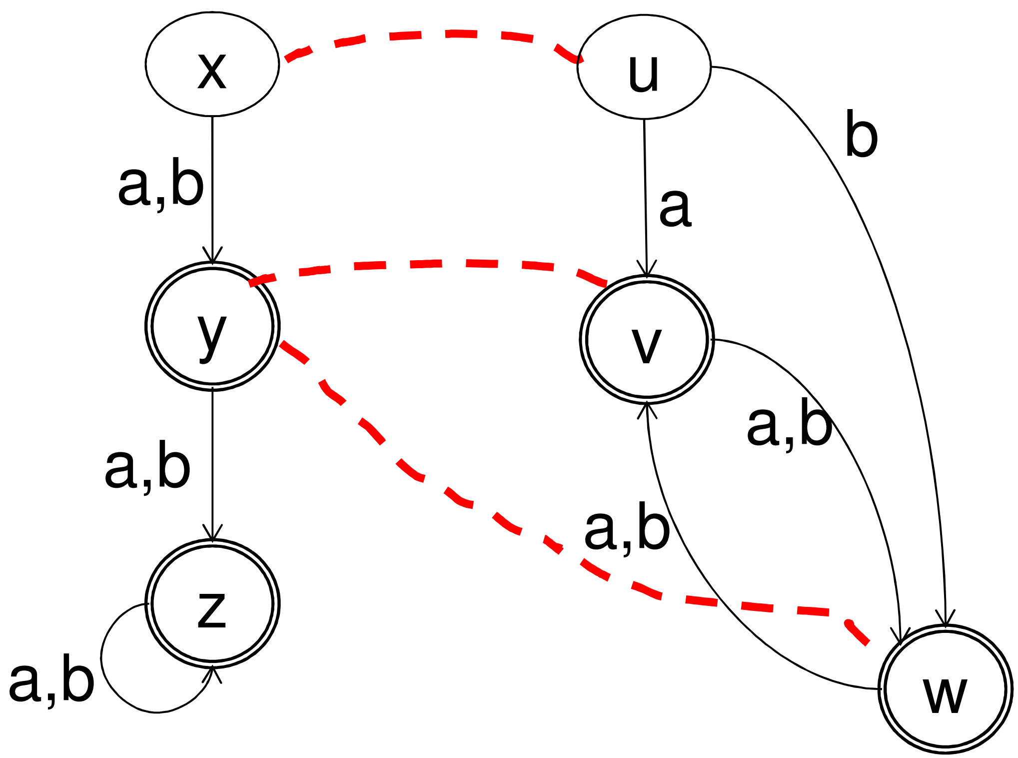
Example: deterministic automata

R = { (x,u) } ?



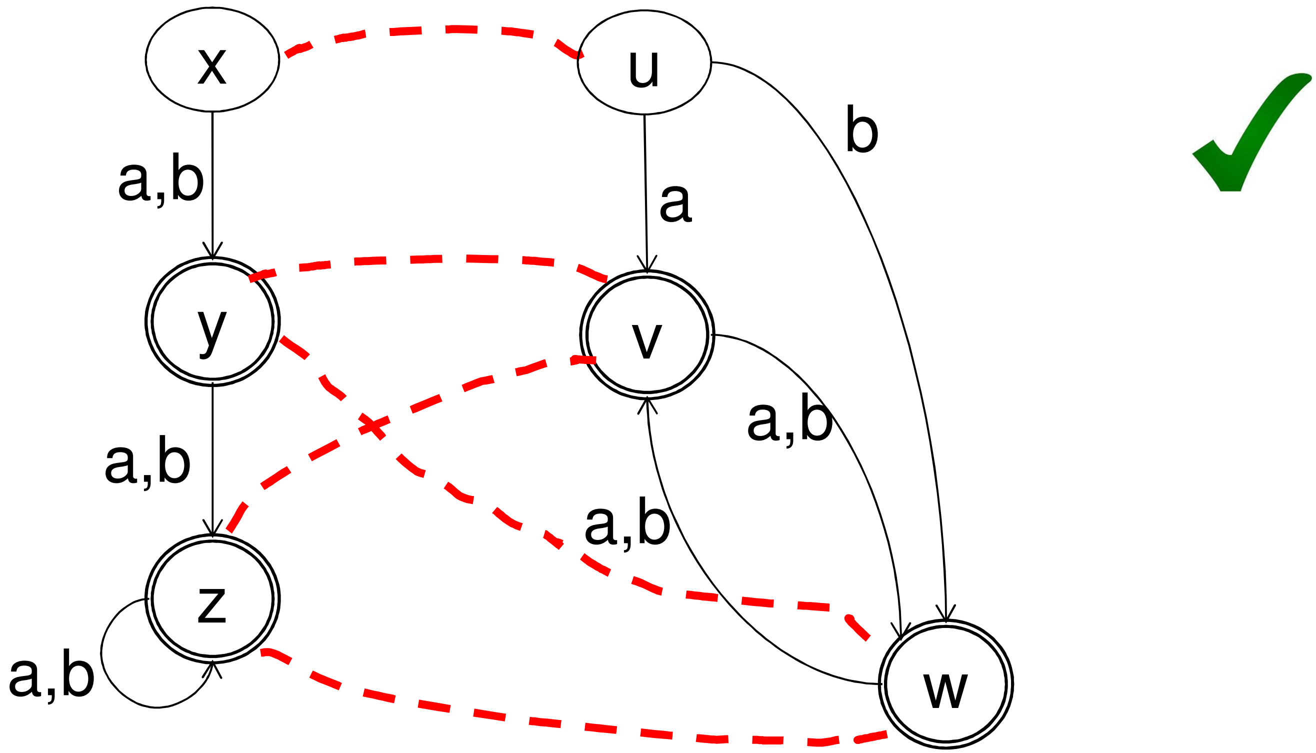
Example: deterministic automata

R = { (x,u),(y,v),(y,w) } ?



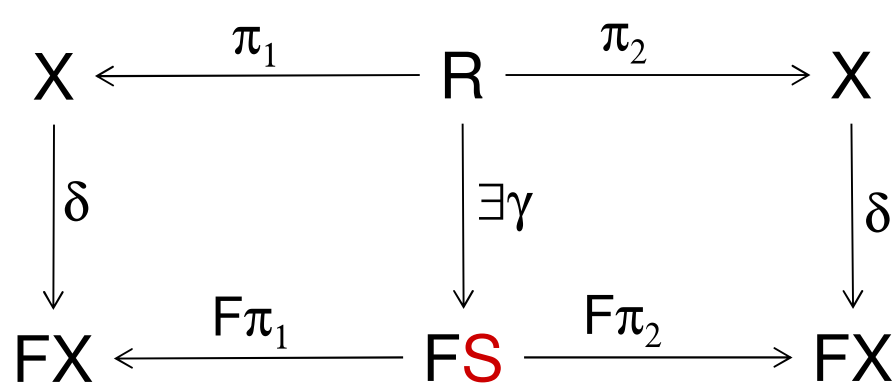
Example: deterministic automata

R = { (x,u),(y,v),(y,w),(z,w),(z,v) }



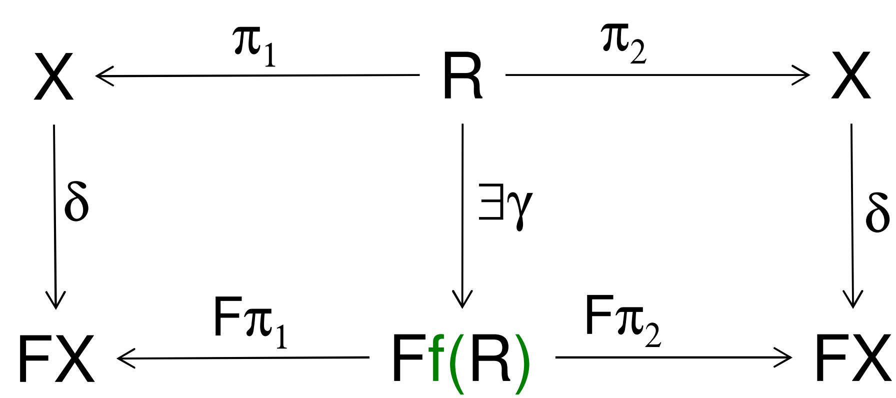
… to progression …

- R progress to S $\quad$ R,S $\subseteq$ X x X



… to bisimulation-up-to

- F-bisimulation up to f $R \subseteq X \times X$



$f : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$

- F-bisimulation up to identity is just F-bisimulation!

Example: deterministic automata

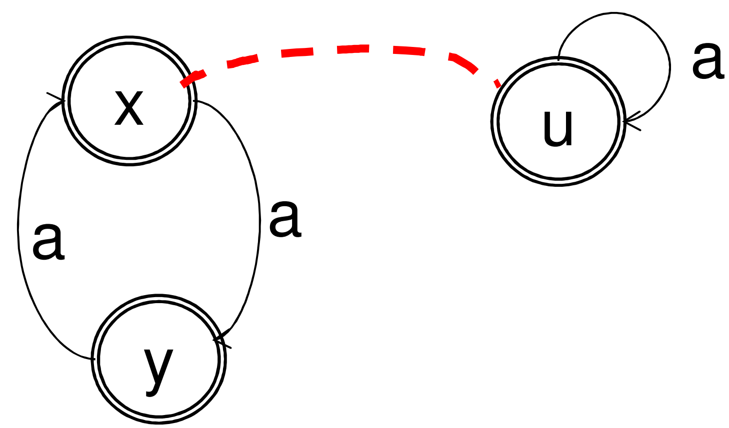
Let $S \subseteq X \times X$ be a set of assumptions

Bisimulation up to union with S: $R \subseteq X \times X$ such that

$(x,y) \in R$ implies $o(x) = o(y)$ and $\forall a. (x_a, y_a) \in R \cup S$

$R = \{(x,u)\}$ is

not a bisimulation!



Example: deterministic automata

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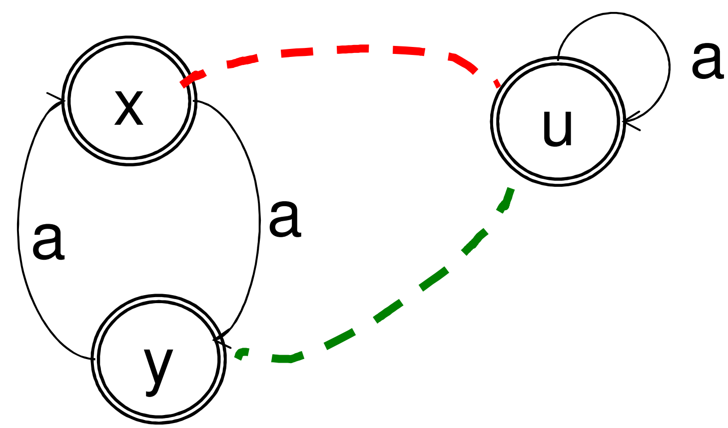
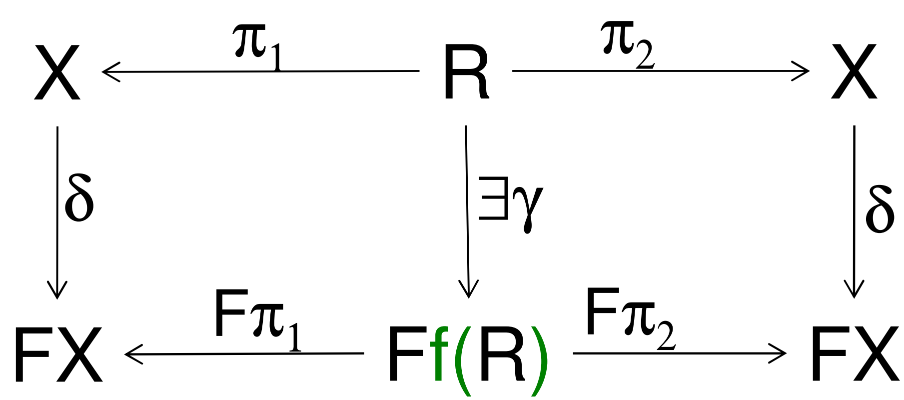
R = {(x,u)} is a bisimulation up to union with S = {(u,y)}

… but not a bisimulation!



Soundness

- A bisimilation up to f is sound for a coalgebra (X, δ) if $R \subseteq \sim_\delta$ for all R that progress to f(R).



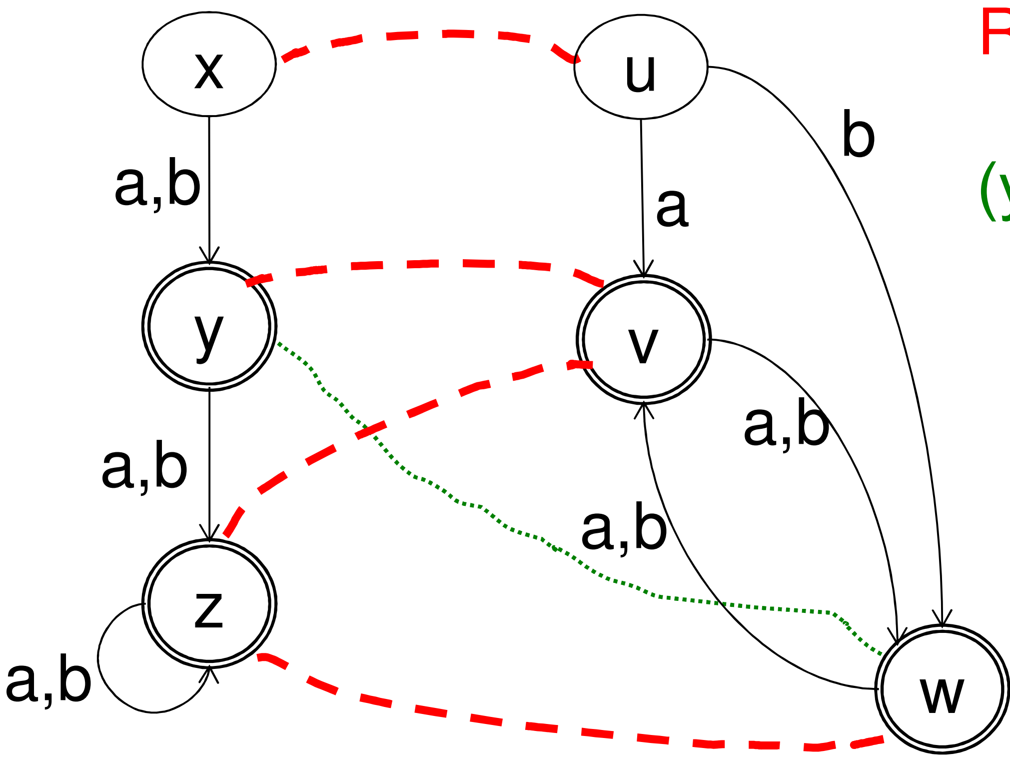
Bisimulation up to union

If S $\subseteq$ ~ then bisimulation up to union with S is sound

i.e. *if our assumptions in S are true than what we can prove using those assumptions is also true.*

Bisimulation up to equivalence

f(R) = e(R) equivalence closure of R



R = { (x,u),(y,v),(z,w),(z,v) }

(y,w) ∈ e(R)

Bisimulation up to union & equivalence

$f(R) = e(R \cup S)$ equivalence closure of $R \cup S$

Bisimulation up to union with ~ and equivalence generalizes bisimulation up to bisimilarity:

$$\sim \circ R \circ \sim \;\subseteq\; \text{equivalence closure of } (R \cup \sim)$$

It allows shorter and modular proofs

Bisimulation up to context

- $F:Set \rightarrow Set$ $\qquad$ $T:Set \rightarrow Set$ monad

- The contextual closure of $R \subseteq X \times X$ for a T-algebra $\alpha:TX \rightarrow X$ is $c_\alpha(R) = <\alpha \circ T\pi_1, \alpha \circ T\pi_2>(TR)$

$t_1\ c_\alpha(R)\ t_2$ if and only if we can obtain one from the other by substituting variables related by R.

$$\frac{s\ R\ t}{s\ c(R)\ t} \qquad \frac{s_i\ R\ t_i \quad \text{for all } i}{op(s_1,\ldots,s_n)\ c(R)\ op(t_1,\ldots,t_n)}$$

Bisimulation up to context

Bisimulation up to contextual closure is sound for λ-bialgebras where λ is a distributive law of a monad T over a functor F. [Bartels 2004]

But bisimulation up to contextual closure becomes interesting only in the final coalgebra or in combination with up-to-bisimilarity or up-to equivalence!

Example: Languages

- FX = = 2 x id^A

- Algebra Union, concatenation, Kleene star

- Coalgebra o(L) = 1 iff $\varepsilon \in L$ $L_a = \{ w \mid aw \in L \}$

Example: Arden's theorem

- Theorem: If $L = K \cdot L \cup M$ and $\varepsilon \notin K$ then $L = K^* \cdot M$.

  Proof: It is enough to prove that $R = \{ (L, K^* \cdot M) \}$ is a bisimulation up to context. Clearly $o(L) = o(K^* \cdot M)$. Further, for every $a \in A$:

$L_a = (K \cdot L \cup M)_a$

$= (K \cdot L)_a \cup M_a$ [because $(X \cup Y)_a = (X)_a \cup (Y)_a$]

$= K_a \cdot L \cup M_a$ [because $(X \cdot Y)_a = X_a \cdot Y$ if $\varepsilon \notin X$]

$c(R)\ K_a \cdot K^* \cdot M \cup M_a$ [coinduction]

$= (K^*)_a \cdot M \cup M_a$ [because $(X^*)_a = X_a \cdot X^*$]

$= (K^* \cdot M)_a$ [because $(X \cdot Y)_a = X_a \cdot Y \cup Y_a$ if $\varepsilon \in X$]

Another example: streams

- $FX = \mathbb{R} \times X$ $\quad$ $TX = \quad t ::= x \mid \underline{r} \mid -t \mid t \oplus t \mid t \otimes t \mid t^{-1}$
- Bialgebras $\quad$ $TTX \to TX \to FTX$
- Behavioral differential equations (i.e. distributive law)

$$\mathbf{o}(\underline{r}) = r \qquad (\underline{r})' = \underline{0}$$

$$\mathbf{o}(\sigma_1 \oplus \sigma_2) = \mathbf{o}(\sigma_1) + \mathbf{o}(\sigma_2) \qquad (\sigma_1 \oplus \sigma_2)' = \sigma_1' \oplus \sigma_2'$$

$$\mathbf{o}(-\sigma) = - \mathbf{o}(\sigma) \qquad (-\sigma)' = - \sigma'$$

$$\mathbf{o}(\sigma_1 \otimes \sigma_2) = \mathbf{o}(\sigma_1) \cdot \mathbf{o}(\sigma_2) \qquad (\sigma_1 \otimes \sigma_2)' = (\sigma_1' \otimes \sigma_2) \oplus (\sigma_1 \otimes \sigma_2')$$

$$\mathbf{o}(\sigma^{-1}) = \mathbf{o}(\sigma)^{-1} \qquad (\sigma^{-1})' = -\sigma' \otimes (\sigma^{-1} \otimes \sigma^{-1})$$

Bisimulation up to union,context & equivalence

- $f(R) = e(c(r(R \cup S))))$ least congruence containing R and S

$R = \{(\sigma \otimes \sigma^{-1}, 1) \mid \sigma \in T(\mathbb{R}^{\omega}),\ \mathbf{o}(\sigma) \neq 0\}$

$S = \{(\sigma_1 \otimes \sigma_2,\ \sigma_2 \otimes \sigma_1)$	commutative
$(\sigma_1 \otimes (\sigma_2 \otimes \sigma_3), (\sigma_1 \otimes \sigma_2) \otimes \sigma_3)$	associative
$(\sigma \otimes 1, \sigma)$	1 as unit
$(\sigma \oplus -\sigma, 0)\}$	sum inverse

Bisimulation up to union,context & equivalence

- $\mathbf{o}(\sigma \otimes \sigma^{-1}) = \mathbf{o}(\sigma) \cdot \mathbf{o}(\sigma^{-1}) = \mathbf{o}(\sigma) \cdot \mathbf{o}(\sigma)^{-1} = 1 = \mathbf{o}(1)$

$(\sigma \otimes \sigma^{-1})' = (\sigma' \otimes \sigma^{-1}) \oplus (\sigma \otimes (\sigma^{-1})')$	definition ⊗
$= (\sigma' \otimes \sigma^{-1}) \oplus (\sigma \otimes (-\sigma' \otimes (\sigma^{-1} \otimes \sigma^{-1})))$	definition $(-)^{-1}$
$c(S)^* \; (\sigma' \otimes \sigma^{-1}) \oplus (-(\sigma' \otimes \sigma^{-1}) \otimes (\sigma \otimes \sigma^{-1}))$	associativity
$c(R) \; (\sigma' \otimes \sigma^{-1}) \oplus (-(\sigma' \otimes \sigma^{-1}) \otimes 1)$	coinduction
$c(S)^* \; (\sigma' \otimes \sigma^{-1}) \oplus -(\sigma' \otimes \sigma^{-1})$	unit of ⊗
$c(S)^* \; 0$	sum inverse
$= 1'$	definition 1

Thus R is a bisimulation-up-to union, context & equivalence but not a bisimulation.

Bisimulation up to union,context & equivalence

- $\mathbf{o}(\sigma\otimes\sigma^{-1}) = \mathbf{o}(\sigma)\cdot\mathbf{o}(\sigma^{-1}) = \mathbf{o}(\sigma)\cdot\mathbf{o}(\sigma)^{-1} = 1 = \mathbf{o}(1)$

- $(\sigma\otimes\sigma^{-1})' = (\sigma'\otimes\sigma^{-1}) \oplus (\sigma\otimes(\sigma^{-1})')$ definition

  $= (\sigma'\otimes\sigma^{-1}) \oplus (\sigma\otimes(-\sigma'\otimes(\sigma^{-1}\otimes\sigma^{-1})))$ definition

  $c(S)^{*}\ (\sigma'\otimes\sigma^{-1}) \oplus (-(\sigma'\otimes\sigma^{-1})\otimes(\sigma\otimes\sigma^{-1}))$ associativity

  $c(R)\ (\sigma'\otimes\sigma^{-1}) \oplus (-(\sigma'\otimes\sigma^{-1})\otimes 1)$ coinduction

  $c(S)^{*}\ (\sigma'\otimes\sigma^{-1}) \oplus -(\sigma'\otimes\sigma^{-1})$

  $c(S)^{*}\ 0$

  $= 1'$

Is it sound?

Thus R is a bisimulation-up-to union, context & equivalence
but not a bisimulation.

Soundness of bisimulation up to techniques

- Bisimulation up to equivalence is not sound in general.
- Bisimulation up to bisimilarity is not sound in general.
- Composition of sound techniques needs not to be sound.

Bisimulation up to equivalence is not sound

- $FX = \{ (x_1,x_2,x_3) \in X^3 | \ x_1=x_2 \text{ or } x_1=x_3 \text{ or } x_2=x_3 \}$

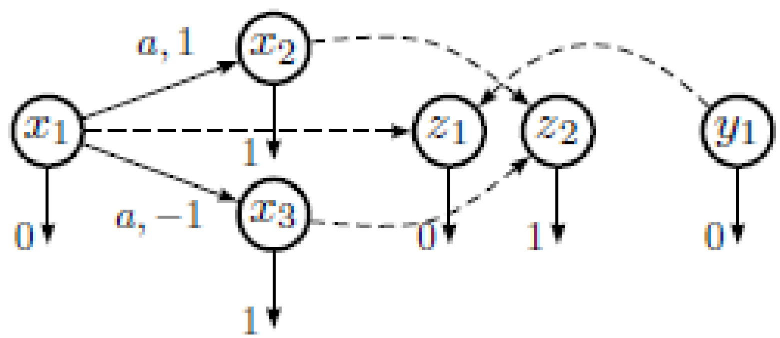
X	$\to$	FX
0	$\mapsto$	(0,1,0)
1	$\mapsto$	(0,0,1)
2	$\mapsto$	(0,0,0)

R	$\to$	$Fe(R)$
(2,0)	$\mapsto$	((0,0),(0,1),(0,0))
(2,1)	$\mapsto$	((0,0),(0,0),(0,1))

- R is a bisimulation up to equivalence because (0,0) and (0,1) are not in R, but in e(R).
- However 0 is not bisimilar to 1 because ((0,0),(1,0),(0,1)) contains three different pairs!

Bisimulation up to bisimilarity

- Bisimulation up to bisimilarity is not sound for weighted automata:



[BBBRS 2012]

$x_2 - x_3 \sim 0$

$\{(x_1,y_1)\}$ is a bisimulation up to bisimilarity

x_1 is not bisimilar to y_1

Soundness

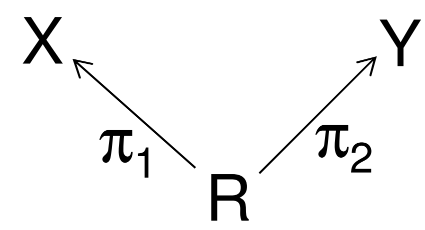
Two directions to prove soundness:

1. using behavioral equivalence
   [Rot,--,Rutten 2012]

2. using the abstract theory of enhancements
   [Rot, Bonchi,--,Rutten, Pous,Silva]

Relators

- $F\text{:Set} \to \text{Set}$
- Relator: $F^{\circledR}\text{:Rel} \to \text{Rel}$



$$F^{\circledR}(X) = X$$

$$F^{\circledR}(R) = F(\pi_2) \circ F(\pi_1)^{-1}$$

- The following are equivalent: [Trnkova, Rutten, Gumm&Schroeder]
  1. F preserves weak pullback
  2. $F^{\circledR}$ is a functor
  3. The composition of two F-bisimulations is a F-bisimulation

Bisimulations as functions

- $\delta: X \to FX$ $\quad\quad$ $R \subseteq X \times X$
- Relational lifting: $\varphi_\delta(R) = \{ (x,y) \mid (\delta(x), \delta(y)) \in F^{®}(R) \}$
- R is a bisimulation iff $R \subseteq \varphi_\delta(R)$
- R is a bisimulation up to f iff $R \subseteq \varphi_\delta(f(R))$
- F preserves weak pullbacks $\Leftrightarrow$ $\varphi_\delta(R) \circ \varphi_\delta(S) = \varphi_\delta(R \circ S)$

Compatible functions

- Let b and f monotone functions on a complete lattice L. The function f is said to be b-compatible if

$$f \circ b \leq b \circ f$$

- b-compatible functions are b-sound if $gfp(b \circ f) \leq gfp(b)$

- b-compatible functions are closed under functional composition and arbitrary unions.

- If $b(R) \circ b(S) \leq b(R \circ S)$ then b-compatible functions are closed also under relational composition (i.e. $(f \bullet g)(R) = f(R) \circ g(R)$).

Soundness via compatibility

If f:$\mathcal{P}$(X x X) $\rightarrow$ $\mathcal{P}$(X x X) is φ_δ-compatible then bisimulation up to f is sound for δ:X $\rightarrow$ FX.

- Equivalence closure is φ_δ-compatible.
- If S $\subseteq$ $\sim_\delta$ then union with S is φ_δ-compatible.
- Contextual closure is φ_δ-compatible for λ-bialgebras, where λ is a distributive law of a monad T over a functor F
- Any combination of the above is φ_δ-compatible.

Soundness

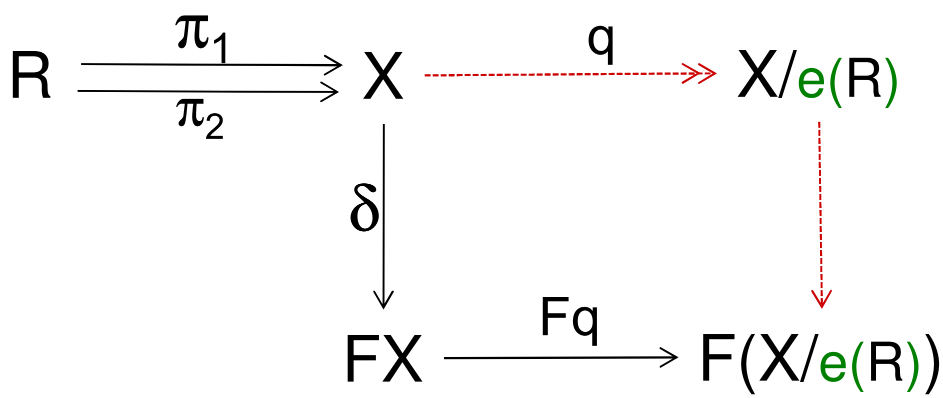
Two directions to prove soundness:

1. using behavioral equivalence
   [Rot,--,Rutten 2012]

2. use the abstract theory of enhancements
   [Rot, Bonchi,--,Rutten, Pous,Silva]

Behavioral equivalence

- F-behavioral equivalence $R \subseteq X \times X$

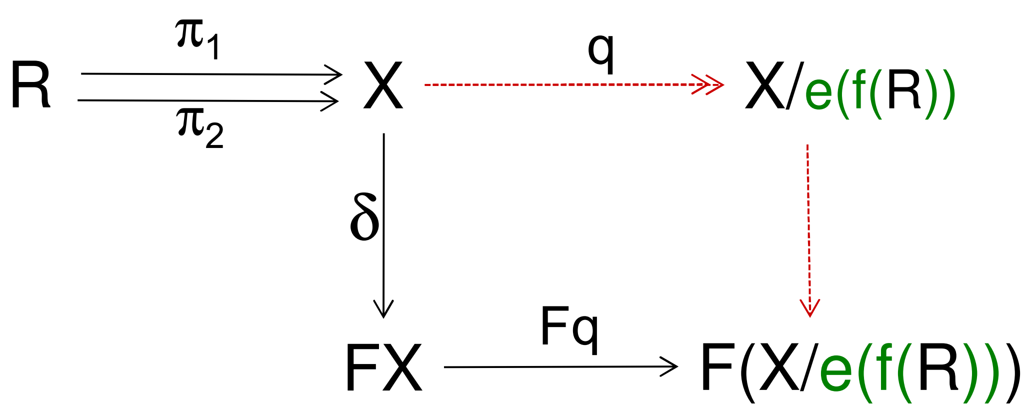


- Any F-bisimulation is a F-behavioral equivalence. If F preserves weak pullbacks then the converse is true.

- Maximal F-behavioral equivalence $\approx_\delta \subseteq X \times X$

Behavioral equivalence up to

- F-behavioral equivalence up to f $\qquad f(R) \subseteq X \times X$



- Behavioral equivalence up to f is sound wrt an F-coalgebra (X,δ) if $R \subseteq \approx_\delta$ for any R behavioral equivalence up to f

Soundness: behavioral equivalence-up-to

- Behavioral equivalence up to equivalence is sound.
- If S $\subseteq \approx$ then behavioral equivalence up to union with S is sound.
- Behavioral equivalence up to contextual closure is sound for λ-bialgebras, where λ is a distributive law of a finitary monad T over a functor F.
- Any combinations of the above is sound.

Soundness: bisimulation-up-to

- Any bisimulation up to f is a behavioral equivalence up to f.

- If F preserves weak pullback then soundness of behavioral equivalence up to (union,context and) equivalence implies soundness of analogous bisimulation up to (union,context and) equivalence.

- In all previous examples about deterministic automata and streams bisimulation up to union, context and equivalence is sound.

Conclusions

- A general theory of bisimulation up to techniques for coinduction
- Interesting proof method for behavioral equivalence.
- Presenting SOS rule formats using up-to context techniques?
- New proof systems for rational behavior?
- Other categories than Set (for example for syntax with bindings)