# Nawrotzki's Algorithm, for the Countable Splitting Lemma, Constructively

IFIP WG 1.3 Meeting "Salzburg" 19.1.22

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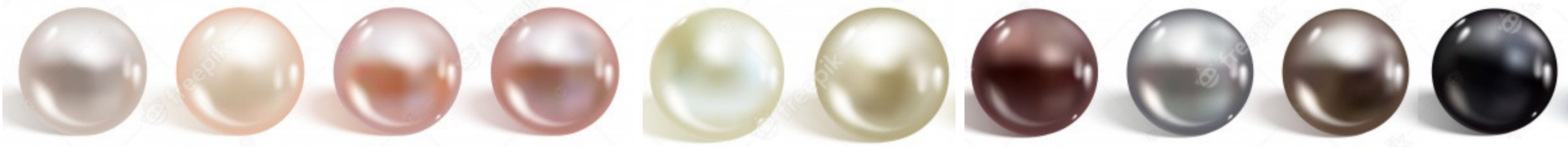

# Foundations of System Specification

Probabilistic Systems - Analysis and Verification

coupling as a proof technique [Barthe, Hsu,…]

**In particular:** Probabilistic Programming …

**Interesting connection to** economy, games, financial mathematics via

**stochastic dominance**

But in essence, this talk presents a purely mathematical result - functional analysis - on constructing couplings in the countable case

new (constructive) proof of an old result

# Question 1:

**Given:** Two measures $\mu, \nu$ on $X, Y$, respectively

**Goal:** Find a measure on $X \times Y$ with marginals $\mu, \nu$

**Answer:** Easy, just take the product measure $\mu \times \nu$

**What if we require additional properties?**

# Question 2:

topological, with Borel σ-algebra

D(X) - probability measures on X

**Given:** Two probability measures $\mu, \nu$ on $X, Y$, respectively, $\Lambda \subseteq D(X \times Y)$

**Goal:** Find a measure $\lambda \in \Lambda$ with marginals $\mu, \nu$

convex and closed, models the additional conditions

**Strassen '65 :** It is possible iff ★

$$\int_X f d\mu + \int_Y g d\nu \leqslant \sup \left\{ \int_{X \times Y} (f \oplus g) d\lambda \mid \lambda \in \Lambda \right\}$$

$$(f \oplus g)(x, y) = f(x) + g(y)$$

for bounded, measurable f,g

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**Strassen '65 :** It is possible iff ★

**one particular special case…**

# Question 3:

and a relation

$R \subseteq X \times Y$

**Given:** Two probability measures $\mu, \nu$ on $X, Y$, respectively, $\Lambda = \{\lambda \mid \lambda(R^c) = 0\}$

**Goal:** Find a measure $\lambda \in \Lambda$ with marginals $\mu, \nu$

**further, one particular special case…**

# Question 4:

and a partial order $R \subseteq X \times X$

**Given:** Two probability measures $\mu, \nu$ on countable $X$, $\Lambda = \{\lambda \mid \lambda(R^c) = 0\}$

**Goal:** Find a measure $\lambda \in \Lambda$ with marginals $\mu, \nu$

**Kellerer '61, Nawrotzki '62 :** It is possible iff $\mu \preceq \nu$, i.e., they satisfy stochastic dominance.

Countable Splitting Lemma (Jones,Levy)

# Stochastic dominance ?

**Example:** A politician can pick up strategies for the elections. One strategy $\nu$ stochastically dominates another strategy $\mu$ iff the outcome under $\nu$ is always better.

$$\forall x \in X.\ \ \mu(\{y \mid x \leqslant y\}) \leqslant \nu(\{y \mid x \leqslant y\}) \quad ★$$

**Def:** $\nu$ stochastically dominates $\mu$ if for every upward closed set $U \subseteq X (= \mathbb{N})$

$$\mu(U) \leqslant \nu(U)$$

wrt a preorder R

This amounts to ★ for the set of natural numbers and any well-order on it,

and to ★ from Strassen's theorem, in this special case.

# Question 4:

and a partial order $R \subseteq X \times X$

**Given:** Two probability measures $\mu, \nu$ on countable $X$, $\Lambda = \{\lambda \mid \lambda(R^c) = 0\}$

**Goal:** Find a measure $\lambda \in \Lambda$ with marginals $\mu, \nu$

**Kellerer '61, Nawrotzki '62 :** It is possible iff $\mu \leqslant \nu$, i.e., they satisfy stochastic dominance.

Countable Splitting Lemma (Levy)

# Kellerer's proof

1. Proves the finite case

2. Considers cutoffs $\mu_n, \nu_n$

3. Produces $\lambda_n$ on $\{1, \ldots, n\} \times \{1, \ldots, n\}$ with marginals $\mu_n, \nu_n$

4. Takes the (pointwise) limit … but it does not necessarily exist.

**Way out:** Choose a subsequence $(n_l)_{l=0}^{\infty}$ such that the limit

$$\lambda(i, j) = \lim_{l \to \infty} \lambda_{n_l}(i, j)$$ exists for all $i, j$

# Nawrotzki's proof

Also produces approximations, but differently - not with cutoffs

Produces $\lambda_n$ with the monotonicity property:

$$i = j \Rightarrow \lambda_{n+1}(i,j) \leqslant \lambda_n(i,j)$$
$$i \neq j \Rightarrow \lambda_{n+1}(i,j) \geqslant \lambda_n(i,j)$$

non increasing on the diagonal
non decreasing off the diagonal

These approximations do not have "correct" marginals (in general).

Defines $\lambda(i,j) = \lim_{n\to\infty} \lambda_n(i,j)$

exists by monotonicity

Proves that this limit has the correct marginals.

# Nonconstructiveness

**Kellerer:** Each approximation $\lambda_n$ is computable.

Nonconstructiveness due to limit by compactness argument.

> Heine-Borel: On a compact subset of real numbers, every sequence has a converging subsequence…
> but how to find it ?

**Nawrotzki:** Nonconstructiveness is in the definition of the approximations $\lambda_n$

> requires computing a sum of an infinite series and evaluating suprema of infinite sets

Only $\lambda_1$ is computable, the others not.

# Nonconstructiveness

Strassen’s proof is super-nonconstructive — compactness comes in on every corner !

Banach-Alaoglu, Riesz-Markov representation, Krein-Milman

# Our proof

follows Nawrotzki,
uses ideas of cutoffs

**4/5 Nawrotzki + 1/5 Kellerer**

Each approximation is computable.

Still does not have ”correct” marginals.

Has computable error estimate, for fixed position $i, j$

The sequence of approximation converges to the solution in $\ell^1$ -norm

But we have no computable bound for the error $\|\lambda_n - \lambda\|_1$

Thank You !