Reasoning about space: a logical framework based on Simplicial Complexes

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Among those techniques model checking is one of the most successful.

From time to space. . .

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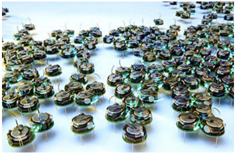
A huge amount of devices that interact with users and with each other (data exchange) within a physical space...



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  - reachability: there is a route connecting a point to others.
- efficient model checking algorithms to reason about space.

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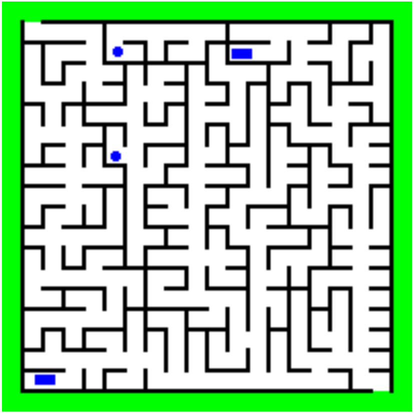
Reasoning about space. . .

An Example



Question: Can starting points reach the exit area?

Reasoning about space. . .

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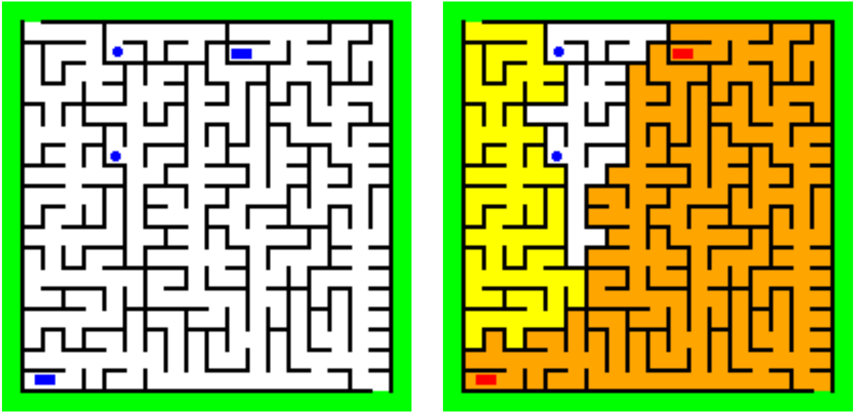

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From physical to logical space. . .

The same tools can be used to reason about different type of structures where spatiality is defined in terms of a given relation like:

- (Social) Networks;
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Spatial logic is often used as a query language to select points/data from a model.

Limitations. . .

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Our Choice!

Our proposal...

1. Use of simplicial complexes to model physical and logical space;

2. Interpretation of spatial logic on simplicial complexes;

3. Study of expressiveness of the logic and its operators in terms of spatial equivalence relations.

Example: Network of Scientific Collaborations

Let us consider

- $\mathcal{A} = \{a_1, a_2, \ldots a_n\}$, a set of authors
- $\mathcal{P} = \{p_1, p_2, \ldots p_k\}$, a set of publications

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We can use spacial properties to find...

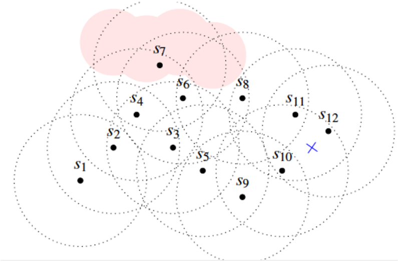
Q1 the groups containing authors of at least a paper over topic A;

Q2 chains of collaborations on a topic A that leads to a work on topic B.

Example: Emergency Rescue

We consider a rescue scenario where. . .

- an accident occurred that caused the emission of dangerous gasses
- some sensors are spread in area to identify dangerous zones



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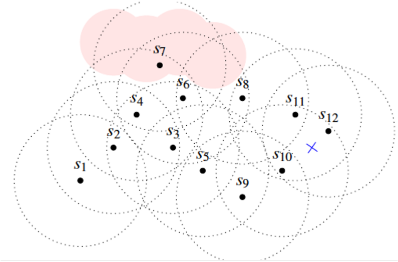
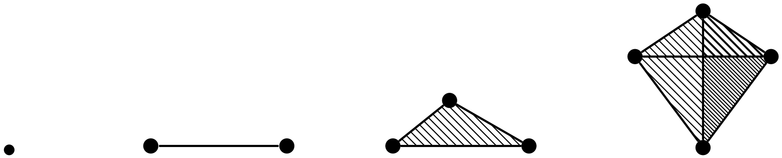


We want to identify safe paths and surfaces and allows us to reach victims.

Simplicial compex. . .

A Simplicial Complex is a set of simplices. . .



node 0-simplex

links 1-simplex

triangle 2-simplex

tetrahedra 3-simplex

Any k-simplex

- is a face of the simplicial complex
- is generalization of the notion of a triangle to arbitrary dimensions
- is characterized by a dimension k, the number of vertices minus one

Simplicial compex. . .

A simplicial complex $\mathcal{K}$ is a collection of simplices, such that:

1. every face of a simplex of $\mathcal{K}$ is also in $\mathcal{K}$
2. the intersection of any two simplices σ_i, σ_j of $\mathcal{K}$ is either $\emptyset$ or a face of both σ_i and σ_j

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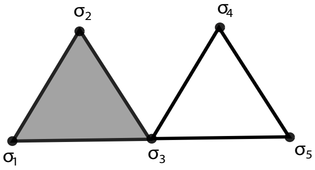
A simplicial model $\mathcal{M} = (\mathcal{K}, \mathsf{P}, \nu)$ consists of:

- a simplicial complex $\mathcal{K}$;
- a set of atomic propositions P;
- a labeling function $\nu : \mathsf{P} \rightarrow \mathcal{K}$.

Simplicial Complex for Scientific Collaborations

A geometric interpretation of the relationships between actors and events

- 0-simplexes (nodes) identify the authors
- 1-simplexes (links) represent pairs of co-authors
- ...
- k-simplex formalizes the relations ”co-authorship group of $k + 1$ researchers”

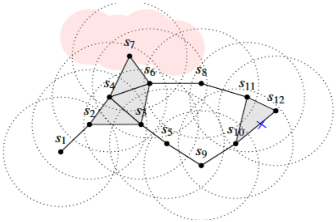


Four groups of co-authors:

- $[\sigma_1, \sigma_2, \sigma_3]$ is composed of 3 authors
- $[\sigma_3, \sigma_4]$, $[\sigma_3, \sigma_5]$, $[\sigma_5, \sigma_4]$ are 3 groups composed of 2 authors

Simplicial Complexes for Emergency Rescue

- 0-simplexes correspond to sensors
- 1-simplexes consist of the set of $[s_i, s_j]$ such that $\{s_i, s_j\} \subseteq A_{s_i} \cap A_{s_j}$
- 2-simplexes consist of the set of $[s_i, s_j, s_k]$ such that $\{s_i, s_k, s_k\} \subseteq A_{s_i} \cap A_{s_j} \cap A_{s_k}$

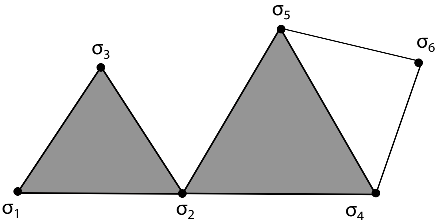


A_{s_i} denotes the area covered by sensor s_i

Adjacency of simplicial complexes

Relations characterising the adjacency of simplicial complexes

- lower adjacency $\smile$
  two k-simplices share a common $(k-1)$-face
- upper adjacency $\frown$
  two k-simplices are both faces of the same common $(k+1)$-simplex
- spacial adjacency $\bullet\!\!-\!\!\bullet$
  two simplices share a common face



- $[\sigma_5, \sigma_6] \smile [\sigma_4, \sigma_6]$
- $[\sigma_1, \sigma_3] \frown [\sigma_3, \sigma_2]$
- $[\sigma_1, \sigma_3, \sigma_3] \bullet\!\!-\!\!\bullet [\sigma_2, \sigma_4, \sigma_5]$

Upper adjacency of 0-simplices corresponds to the graph adjacency

Spatial logics for Simplicial Complexes

The Spatial Logics for Simplicial Complexes (SLSC) consists of

- boolean operators: true ($\top$), negation ($\neg$), and conjunction ($\wedge$)
- **Neighbourhood**, $\mathcal{N}$
- **Reachability**, $\mathcal{R}$

The syntax of SLSC is

$$\phi ::= \mathsf{p} \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathcal{N}\varphi_1 \mid \varphi_1 \mathcal{R} \varphi_2 \ .$$

where p is an atomic propositions.

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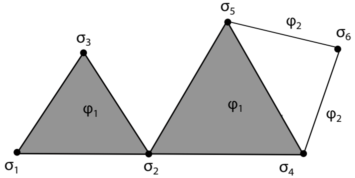

- $[\sigma_1, \sigma_2, \sigma_3]$ does not satisfy $\mathcal{N}\varphi_1$ considering the lower adjacency
- $[\sigma_1, \sigma_2, \sigma_3]$ satisfies $\mathcal{N}\varphi_1$ considering the spatial adjacency

Reachability Operator

A simplex σ satisfies $\varphi_1 \mathcal{R} \varphi_2$ if it satisfies φ_2 or it is satisfies φ_1 and it is adjacent to a simplex satisfying $\varphi_1 \mathcal{R} \varphi_2$.

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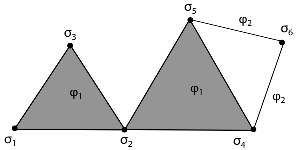


- $[\sigma_4, \sigma_6]$ satisfies $\varphi_1 \mathcal{R} \varphi_2$ considering the spatial adjacency
- $[\sigma_1, \sigma_2, \sigma_3]$ does not satisfy $\varphi_1 \mathcal{R} \varphi_2$ considering the upper or lower adjacency

Semantics of SLSC

Let $\mathscr{C}$ be an element of $\{\smile, \frown, \bullet\!\!-\!\!\bullet\}$. The set of simplexes of $\mathcal{K}$ satisfying formula φ that is in simplicial model $\mathcal{M} = (\mathcal{K}, \mathsf{P}, \nu)$ is defined by

$$\llbracket\, \mathsf{a} \,\rrbracket_{\mathscr{C}} = \nu(\mathsf{a}) \tag{1}$$

$$\llbracket\, \top \,\rrbracket_{\mathscr{C}} = \mathcal{K} \tag{2}$$

$$\llbracket\, \varphi_1 \wedge \varphi_2 \,\rrbracket_{\mathscr{C}} = \llbracket\, \varphi_1 \,\rrbracket_{\mathscr{C}} \cap \llbracket\, \varphi_2 \,\rrbracket_{\mathscr{C}} \tag{3}$$

$$\llbracket\, \neg\varphi \,\rrbracket_{\mathscr{C}} = \mathcal{K} \setminus \llbracket\, \varphi \,\rrbracket_{\mathscr{C}} \tag{4}$$

$$\llbracket\, \mathcal{N}\varphi \,\rrbracket_{\mathscr{C}} = \{\sigma_1 \in \mathcal{K} : \exists\, \sigma_2 \in \llbracket\, \varphi \,\rrbracket_{\mathscr{C}} \text{ and } \sigma_1 \;\mathscr{C}\; \sigma_2\} \tag{5}$$

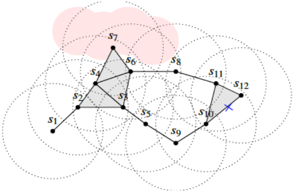
$$\llbracket\, \varphi_1 \mathcal{R} \varphi_2 \,\rrbracket_{\mathscr{C}} = \bigcup_{\mathsf{i}=0}^{\infty} \llbracket\, \varphi_1 \,\mathcal{R}^{\mathsf{i}}\, \varphi_2 \,\rrbracket_{\mathscr{C}} \tag{6}$$

where

$$\llbracket\, \varphi_1 \mathcal{R}^0 \varphi_2 \,\rrbracket_{\mathscr{C}} = \llbracket\, \varphi_2 \,\rrbracket_{\mathscr{C}} \tag{7}$$

$$\llbracket\, \varphi_1 \mathcal{R}^{\mathsf{n}+1} \varphi_2 \,\rrbracket_{\mathscr{C}} = \{\sigma_1 \in \llbracket\, \varphi_1 \,\rrbracket_{\mathscr{C}} : \exists \sigma_2 \in \llbracket\, \varphi_1 \mathcal{R}^{\mathsf{n}} \varphi_2 \,\rrbracket_{\mathscr{C}} \sigma_1 \;\mathscr{C}\; \sigma_2\}$$

A use of SLSC formulas in Emergency Rescue



A simplex is `safer` if it is `safe` and it is not adjacent with an `unsafe` simplex

$$\varphi_{\texttt{safer}} = \texttt{safe} \wedge \neg \mathcal{N}(\texttt{unsafe})$$

To select the areas that the rescue team can use to reach a victim

$$\varphi_{\texttt{safer}} \mathcal{R} \texttt{victim}$$

Spatial Model Checking

The Model Checking Algorithm for SLSC...

- takes as input a simplicial model $\mathcal{M} = (\mathcal{K}, \mathsf{P}, \nu)$ and a formula φ
- returns the set $\{\sigma \in \mathcal{K} : \sigma \in [\![\varphi]\!]_{\mathscr{C}}\}$
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The algorithm is standard:

- $\varphi = \mathcal{N}\varphi_1$: selection of the simplices adjacent to the ones satisfying φ
- $\varphi = \varphi_1 \mathcal{R} \varphi_2$: selection of the simplices
  - from all the simplexes that satisfying φ_2
  - iteratively the algorithm consider the adjacent simplex that satisfying φ_1

Logic expressiveness. . .

We introduce two space equivalences that are indexed with respect to an adjacency relations $\mathscr{C} \in \{\smile, \frown, \bullet\!\!-\!\!\bullet\}$:

- a $\mathscr{C}$-spatial bisimulation, denoted by $\Leftrightarrow^{\mathscr{C}}$;
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We have that:

- $\sigma_1 \leftrightarrows^{\mathscr{C}} \sigma_2$ if and only if σ_1 and σ_2 satisfy the same set of formulas ϕ;
- $\sigma_1 \leftrightarrows_{\mathrm{b}}^{\mathscr{C}} \sigma_2$ if and only if σ_1 and σ_2 satisfy the same set of formulas ψ (ψ is not using operator $\mathcal{N}$).

Concluding remarks and Future directions. . .

We have seen how simplicial complexes can be used to model both physical and logical space and the use of a spatial logic to specify and verify their properties.

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The proposed formalism in a conservative extension of the previous work based on closure spaces.

We have introduced two spatial equivalence relations that permits identifying the set of simplicial complexes that satisfy the same property.

Future directions. . .

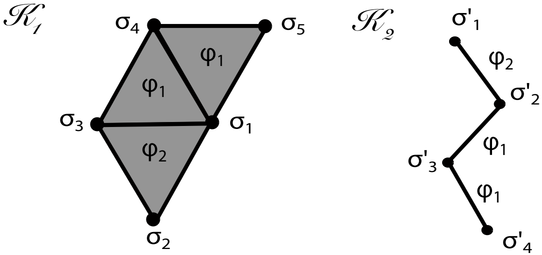
In the future we plan to

- study the interaction of space and time in dynamic evolving models;
- use the proposed formalism to describe some algebraic topology concepts, such as Betti Numbers;
- use the formalism to study complex scenarios (integration with engines for graph databases is under development).

Thank you for the attention!

Spatial equivalences. . .

Two lower-spatial bisimilar simplicial complex models:



Spatial equivalence. . .

Below two lower-spatial branching bisimilar simplicial complex models: