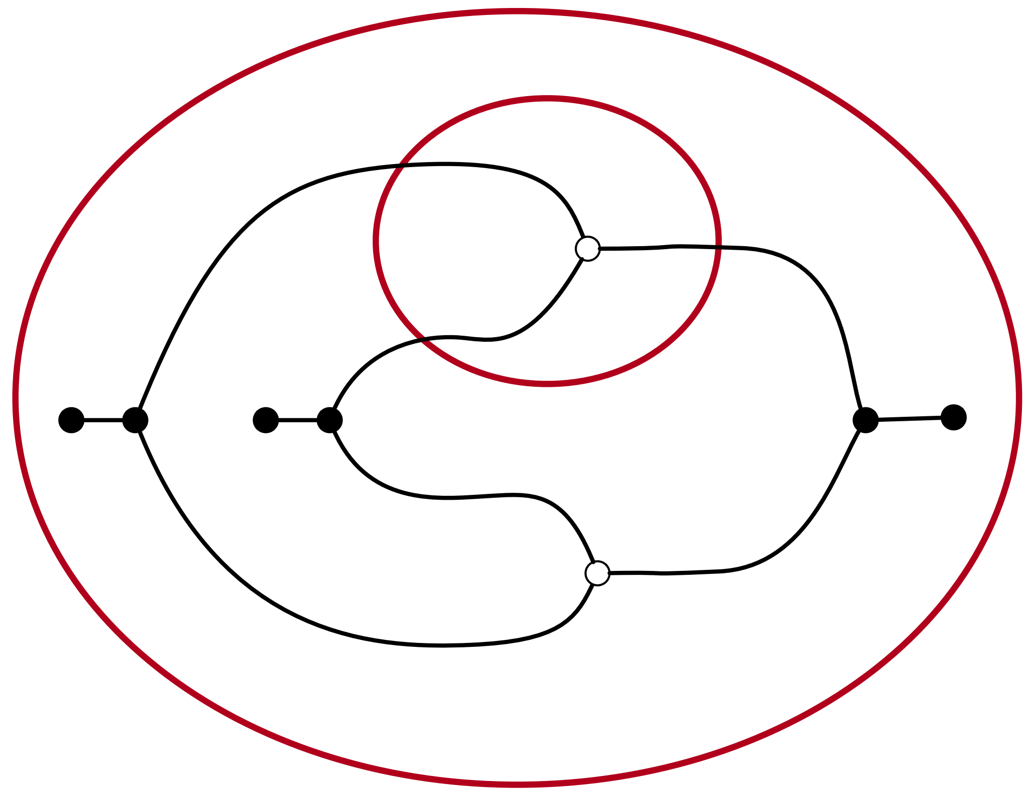
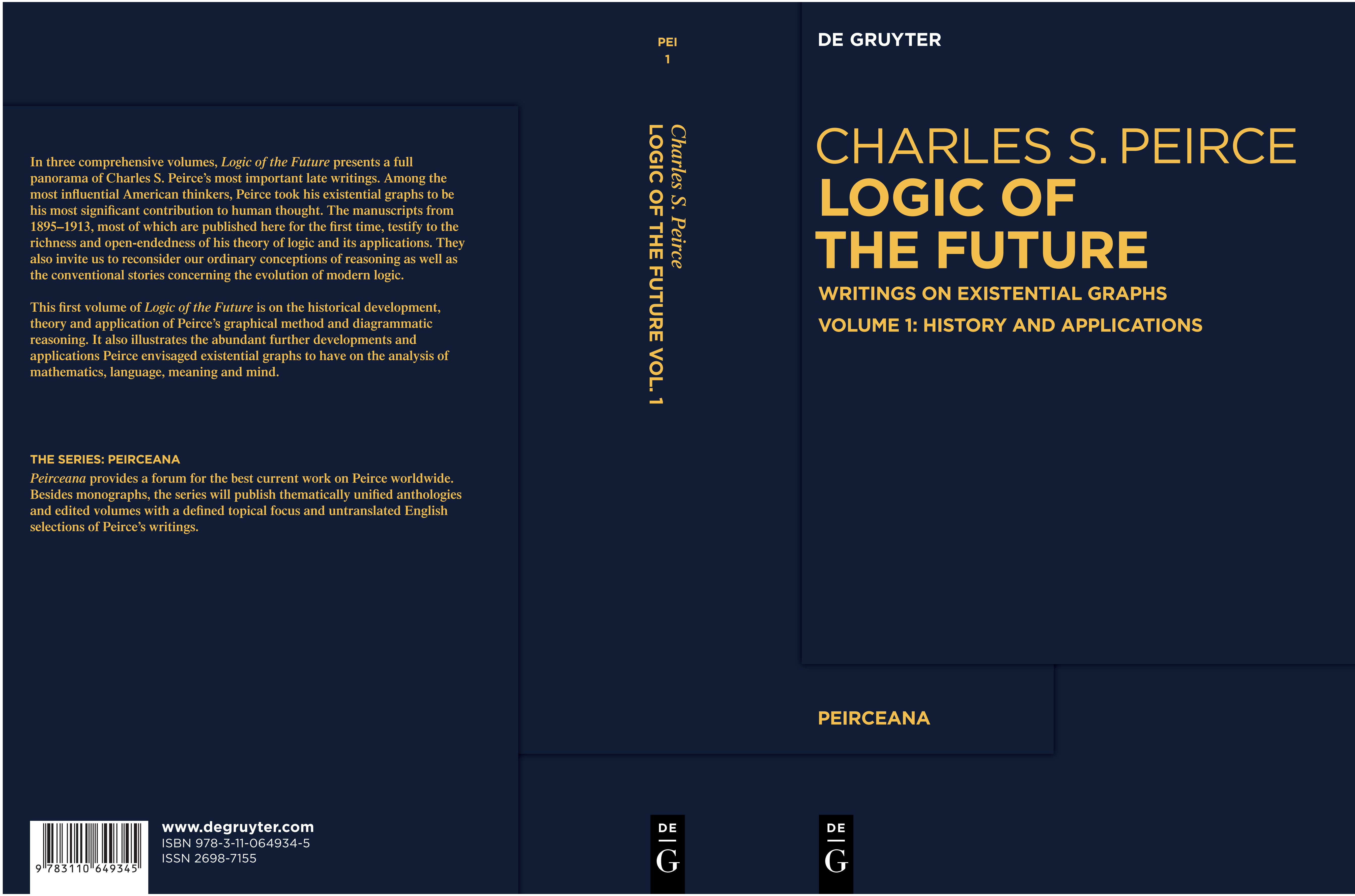
Back to the Future: first order logic and diagrammatic reasoning



Nathan Haydon and Pawel Sobocinski
Tallinn University of Technology, Estonia
IFIP WG1.3 Massa Marittima, January 2020



Hot off the press, December 2019 - order a copy for your library!

In the 1880s Peirce developed independently of Gottlob Frege a system of quantification theory in which quantifiers were treated as variable binding operators; thus he can be regarded, alongside Frege, as a founder of contemporary formal logic. The standard notation used in contemporary logic is a variant of Peirce's notation rather than that adopted by Frege. As a part of his pragmatic theory of meaning, Peirce developed a game-theoretical interpretation of logical

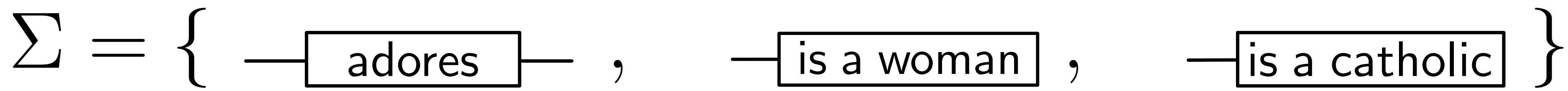
...

In the 1890s Peirce reformulated quantification theory by expressing it in a language of diagrams which he called *existential graphs*. The switch from the algebraic notation to the language of graphs seems to have been motivated by his belief that the latter was more suitable for the purposes of logical analysis. According to Peirce, a system of logic can be used as a calculus which helps to draw inferences as economically as possible, or it can be developed for the purpose of representing and analyzing deductive processes. Peirce also thought that a graphical notation was more suitable for logical analysis than an algebraic notation because of its higher degree of *iconicity*. An iconic sign can be said to show what it means in the sense that it resembles its objects in some respect, that is, some features of the sign itself determine its interpretation. Peirce himself regarded the theory of existential graphs as one of his most important contributions to logic and philosophy.

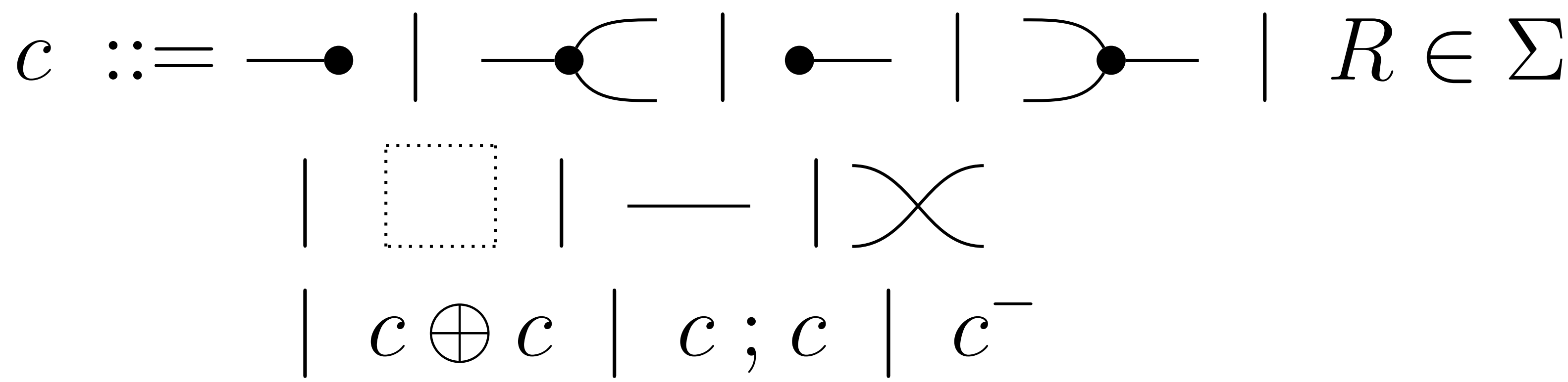
Plan of talk

- **Syntax**
- The algebra of lines of identity & models
- The algebra of negation
- Examples

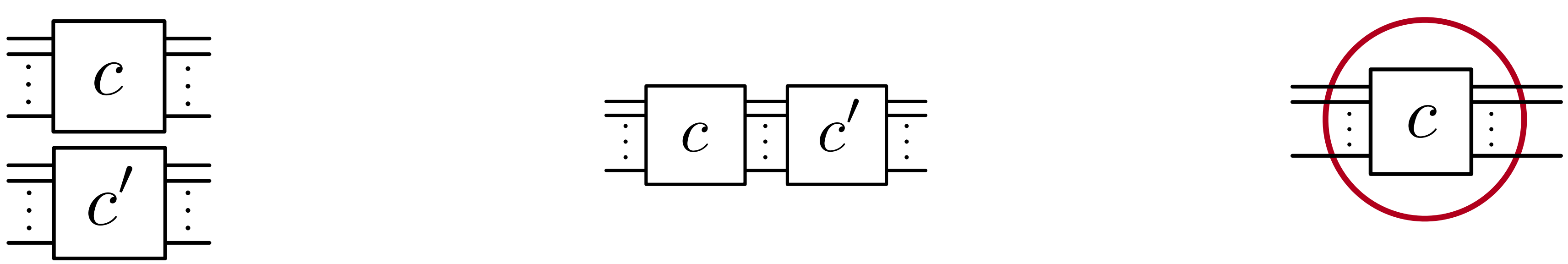
- Monoidal signature (arities and **coarities**)



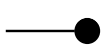
- Terms

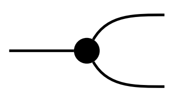


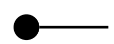
- Diagrammatic conventions

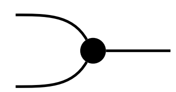


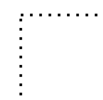
Sort discipline

 $: (1,0)$

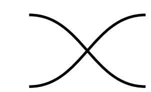
 $: (1,2)$

 $: (0,1)$

 $: (2,1)$

 $: (0,0)$

 $: (1,1)$

 $: (2,2)$

$$\frac{}{R : (ar(R), coar(R))}$$

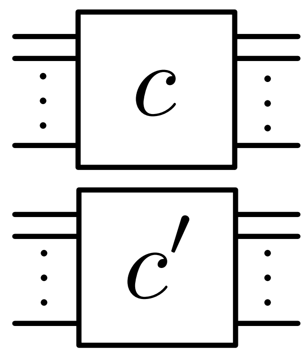
$$\frac{c : (n,z) \qquad d : (z,m)}{c ; d : (n,m)}$$

$$\frac{c : (n,m) \qquad d : (r,z)}{c \oplus d : (n{+}r, m{+}z)}$$

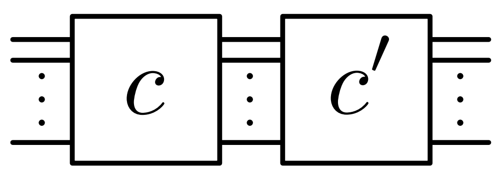
$$\frac{c : (n,m)}{c^{-} : (n,m)}$$

Logical interpretation

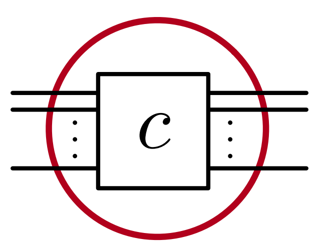
(Carboni and Walters, 1987; Bonchi, S., Seeber, 2018)



$c(x_1,y_1) \wedge c'(x_2,y_2)$

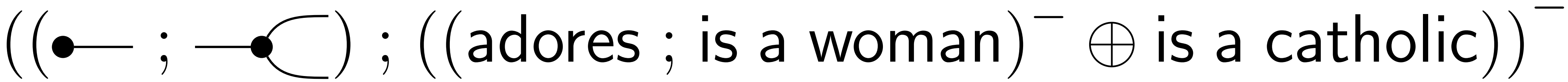


$\exists$ z. $c(x,z) \wedge c'(z,y)$



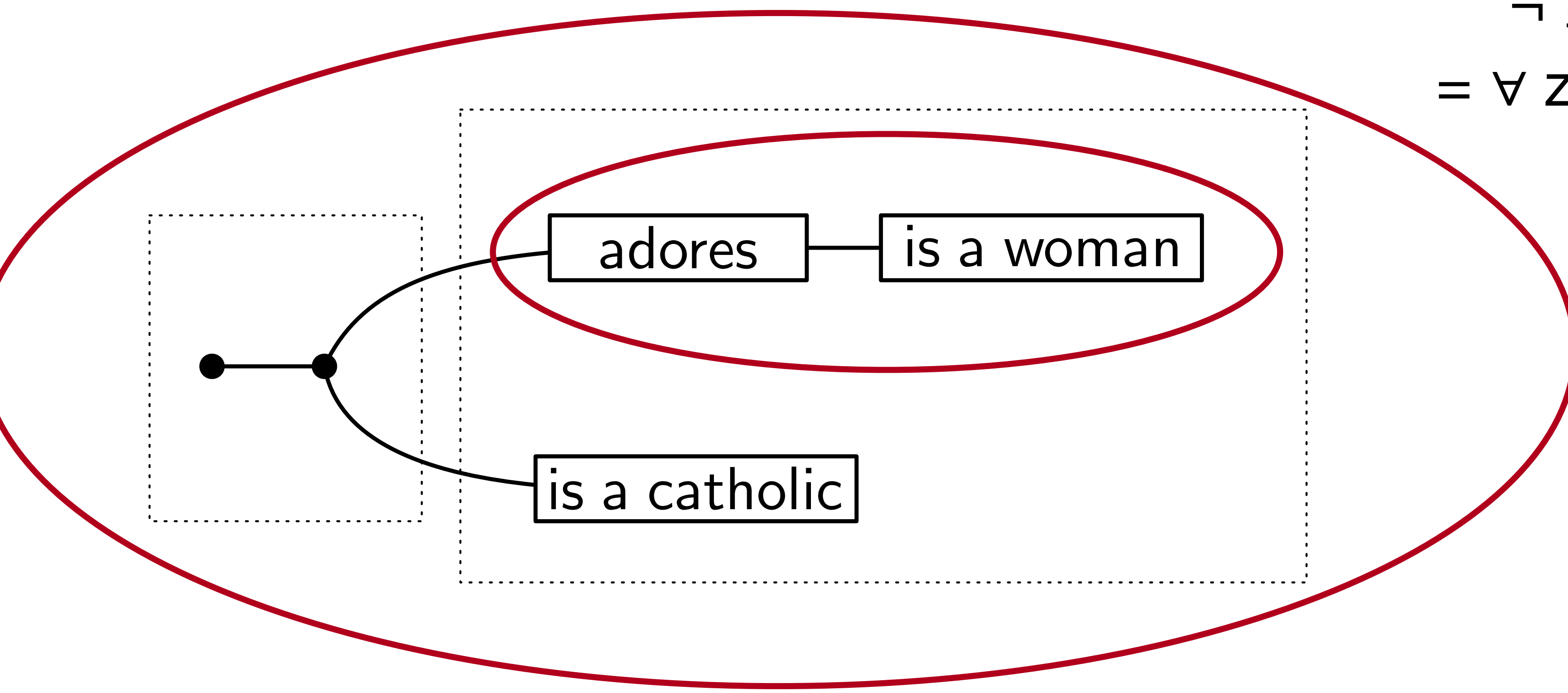
$\neg c(x,y)$

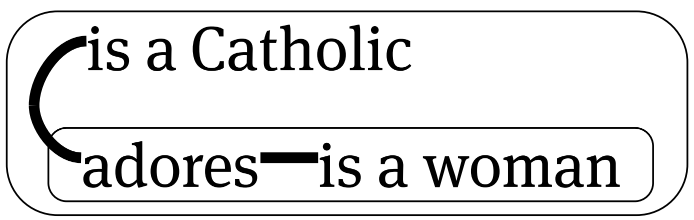
Example



$\neg \exists z.\ \text{adores}(x,z) \wedge \text{is a woman}(z)$

$= \forall z.\ \neg\, \text{adores}(x, z) \vee \neg\, \text{is a woman}(z)$





$\neg \exists x.\ \text{is a catholic}(x) \wedge (\forall z.\ \neg\, \text{adores}(x, z) \vee \neg\, \text{is a woman}(z))$

$= \forall x.\ \neg\, \text{is a catholic}(x) \vee (\exists z.\ \text{adores}(x,z) \wedge \text{is a woman}(z))$

$= \forall x.\ \text{is a catholic}(x) \Rightarrow (\exists z.\ \text{adores}(x,z) \wedge \text{is a woman}(z))$

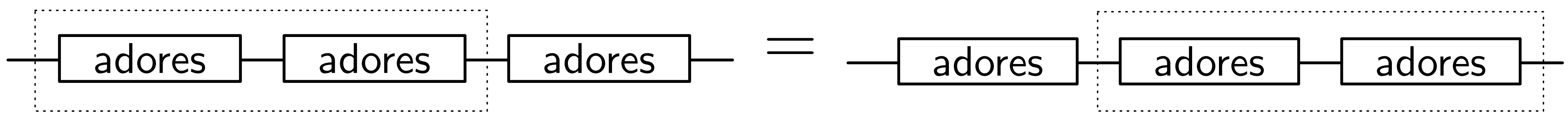
Diag$_\beta$

- **Diag$_\beta$** is the free symmetric strict monoidal category on the syntax. More precisely it is the the free **uoh-prop**

- **Definition.** A prop **X** with a unary operations on homsets is a prop with a family of operations.

$$\overline{}_{m,n} \; : \; \mathbb{X}[m,n] \; \rightarrow \; \mathbb{X}[m,n]$$

There are no additional equations.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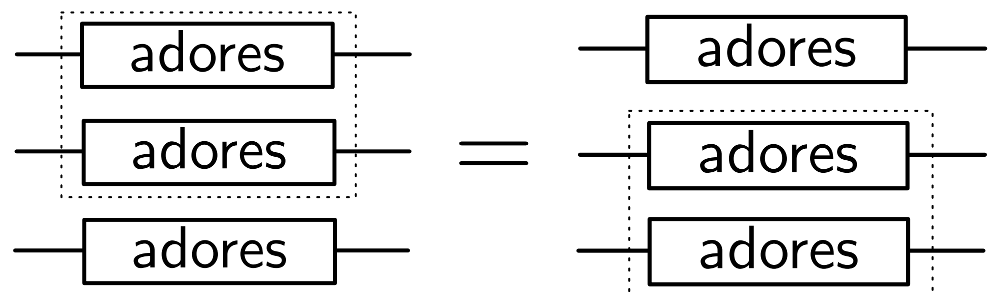
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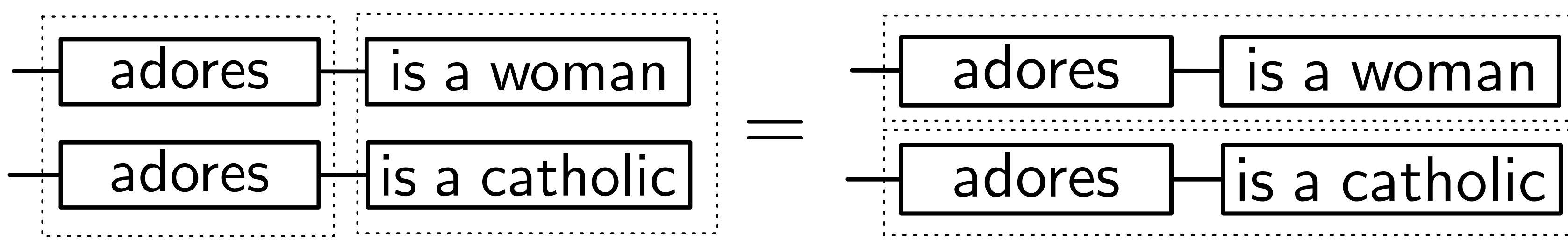
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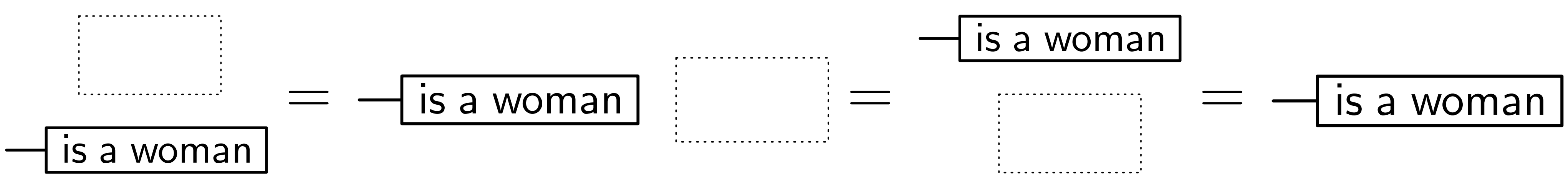
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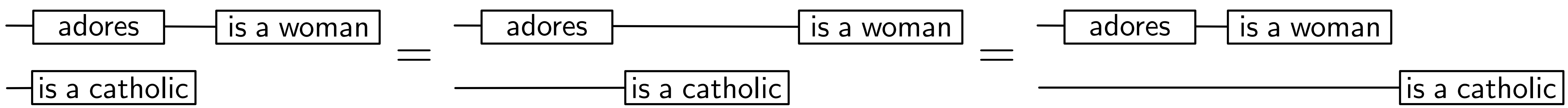
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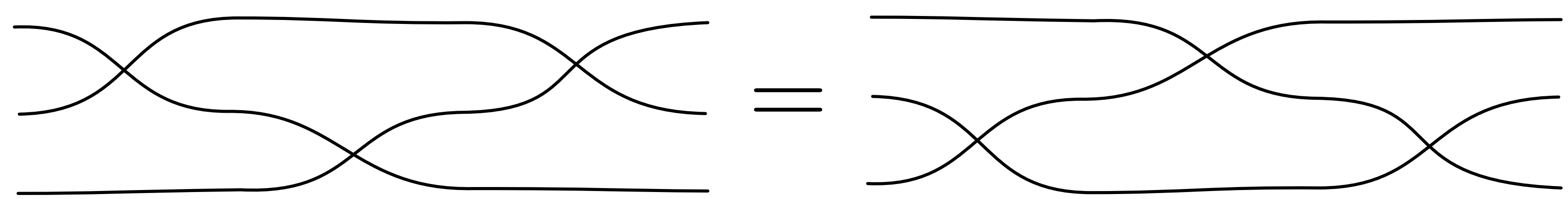

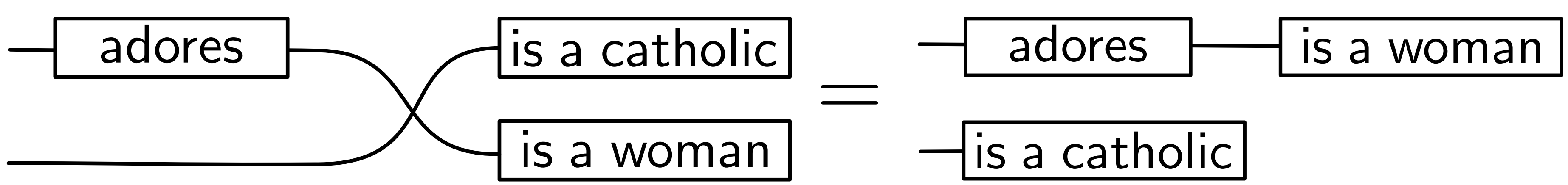
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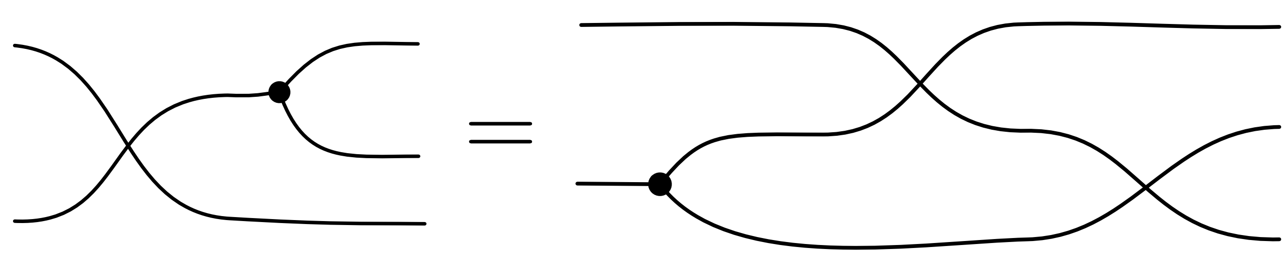


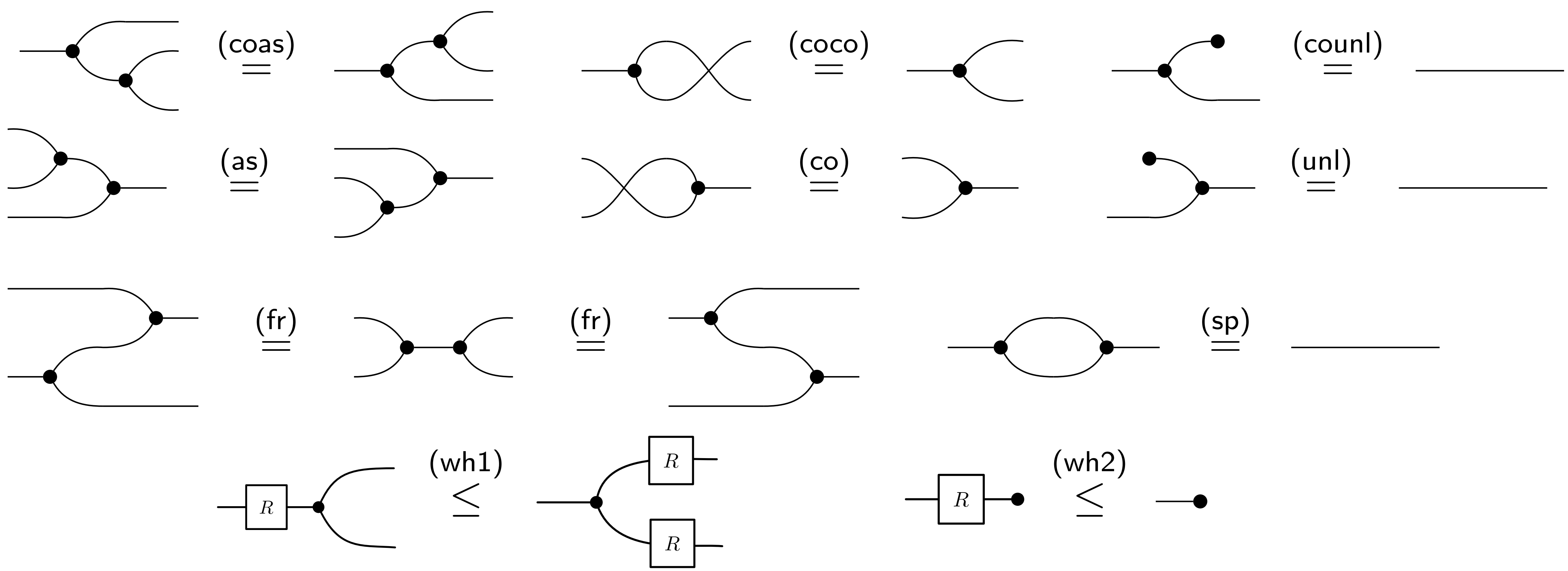
Moral of the story

- **Diag**$_\beta$ is a bona fide syntax, it is a free thing
- its terms are better thought of as string diagrams than traditional syntax trees

Plan of talk

- Syntax
- **The algebra of lines of identity & models**
- The algebra of negation
- Examples

Laws of cartesian bicategories



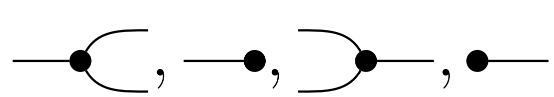
Rel$_X$

Definition 5.1. *Let X be a set. The uoh-prop* Rel_X *has, as arrows $m \to n$, relations $X^m \to X^n$ (subsets of $X^m \times X^n$), where X^m is the m-fold cartesian product of X. Given a relation $R : X^m \to X^n$, R^- is the (set-theoretical) complement of R as a subset of $X^m \times X^n$.*

But **Rel**$_X$ is also a cartesian bicategory in a canonical way!

Definition 5.2 (Model)**.** *A* model *for* Diag_β *consists of a set X and a morphism of uoh-props*

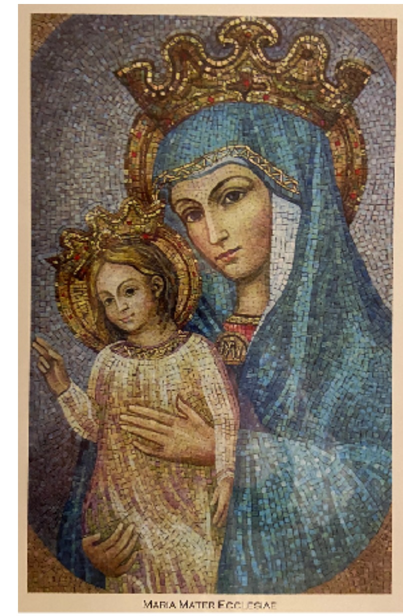
$$[\![-]\!] : \mathsf{Diag}_\beta \to \mathsf{Rel}_X$$

that maps {} *to the canonical Frobenius structure of* Rel_X.

Truth values

Remark 5.3. Note that *closed* diagrams, that is those of sort (0, 0) map to relations of type $0 \to 0$, that is, subsets of $X^0 \times X^0$. Since X^0 is a singleton, there are precisely two such relations – the empty relation $\varnothing$ and the full relation $\{(\star, \star)\}$. It is convenient to identify these with truth values – $\varnothing$ with $\bot$ (false) and $\{(\star, \star)\}$ with $\top$ (true).

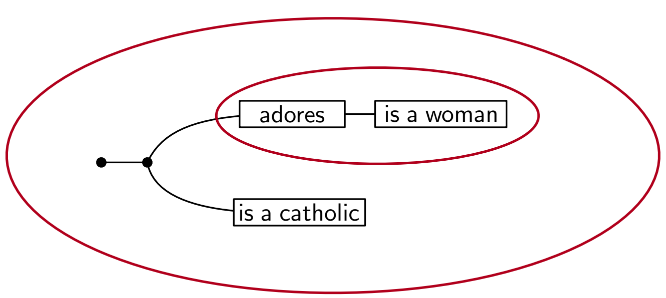
Example



mme

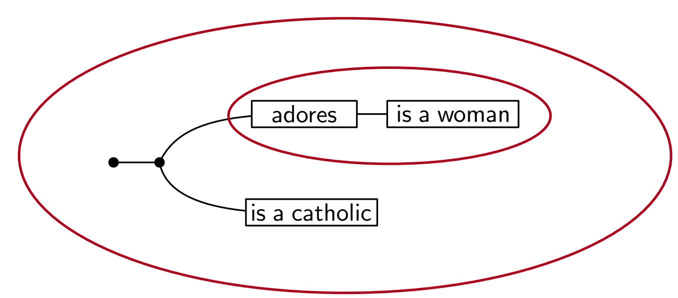
X = {fabio, mme} [[is a woman]] = {mme} [[is a catholic]] = {fabio}

[[adores]] = {(fabio, mme)}

[[

]] = (*,*) = T

[[adores]] = {(fabio, fabio)}

[[

]] = ø = F

Plan of talk

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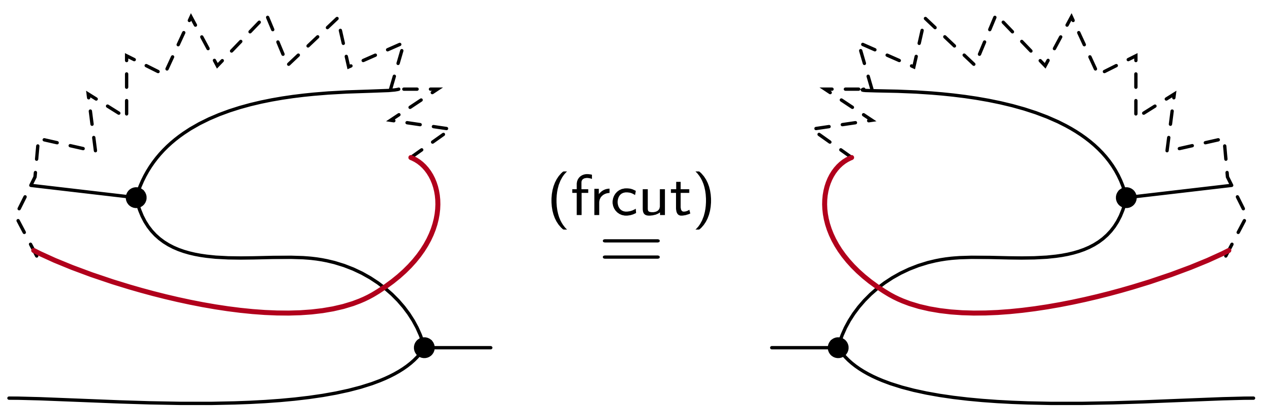
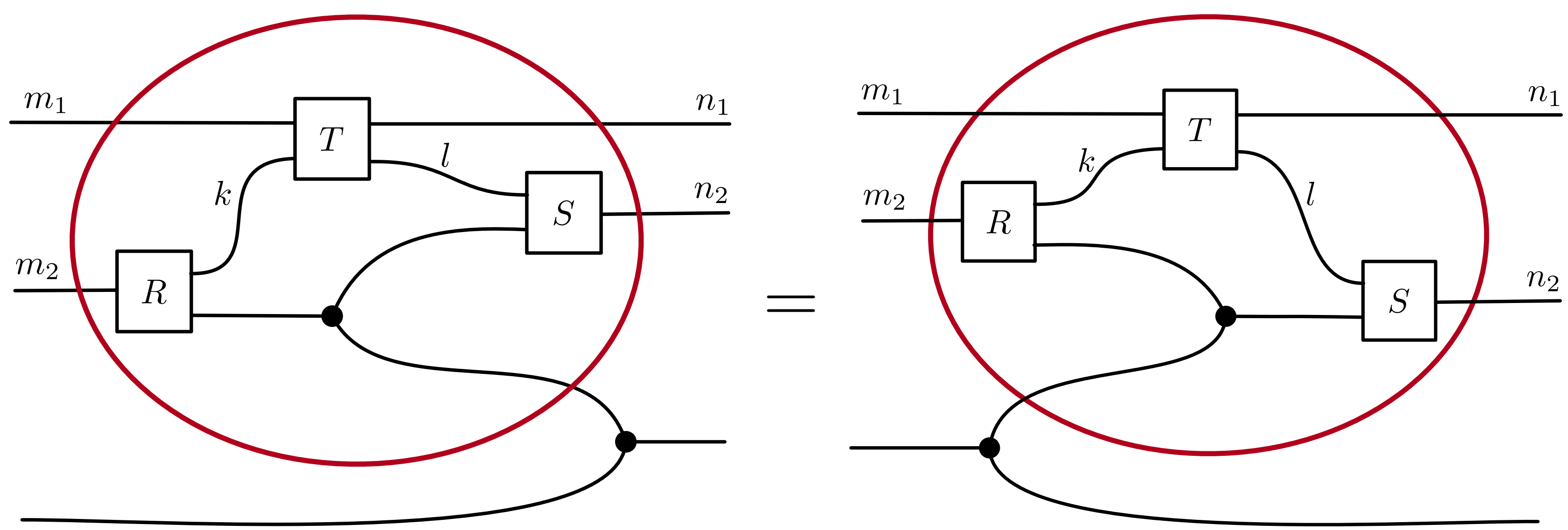
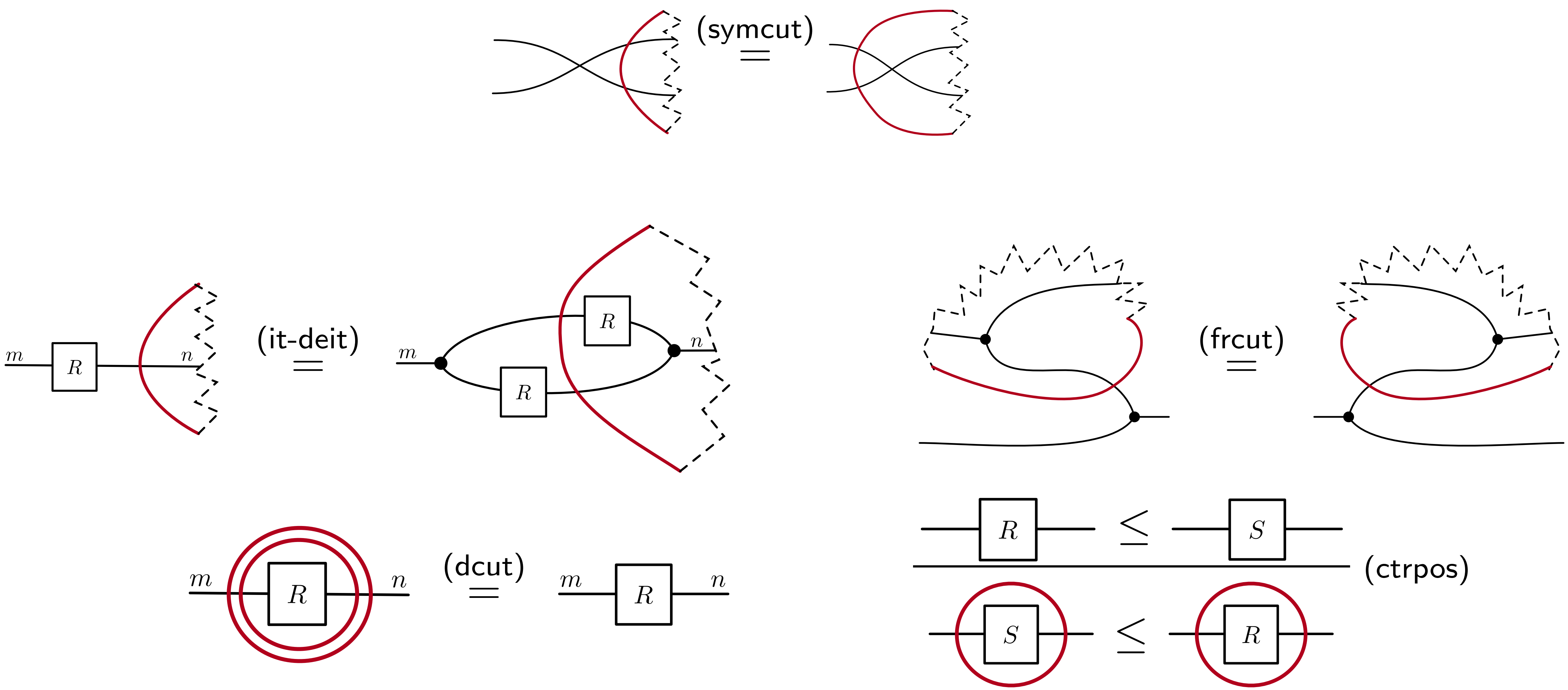
“eggshell” notation

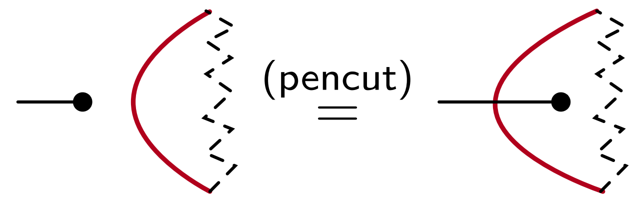




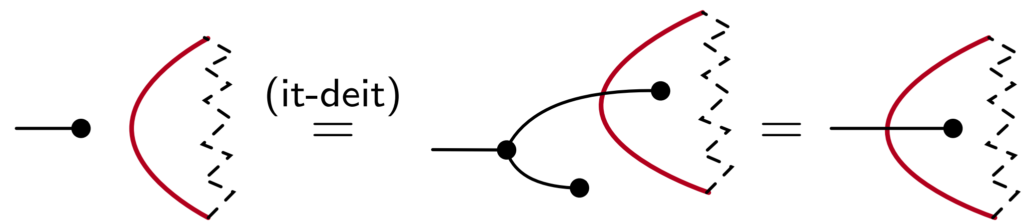
Rules for diagrammatic reasoning



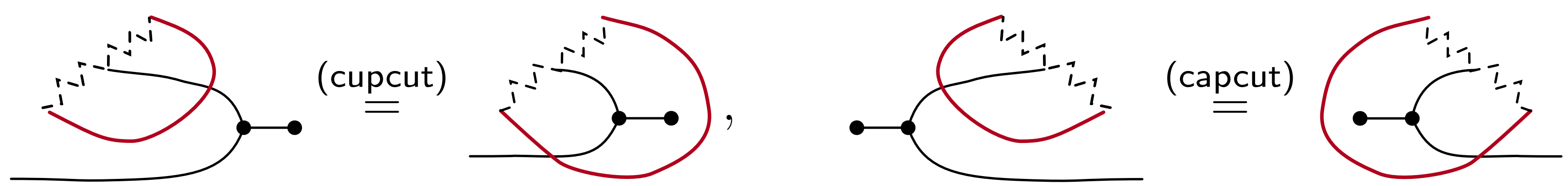
Lemma 6.1.



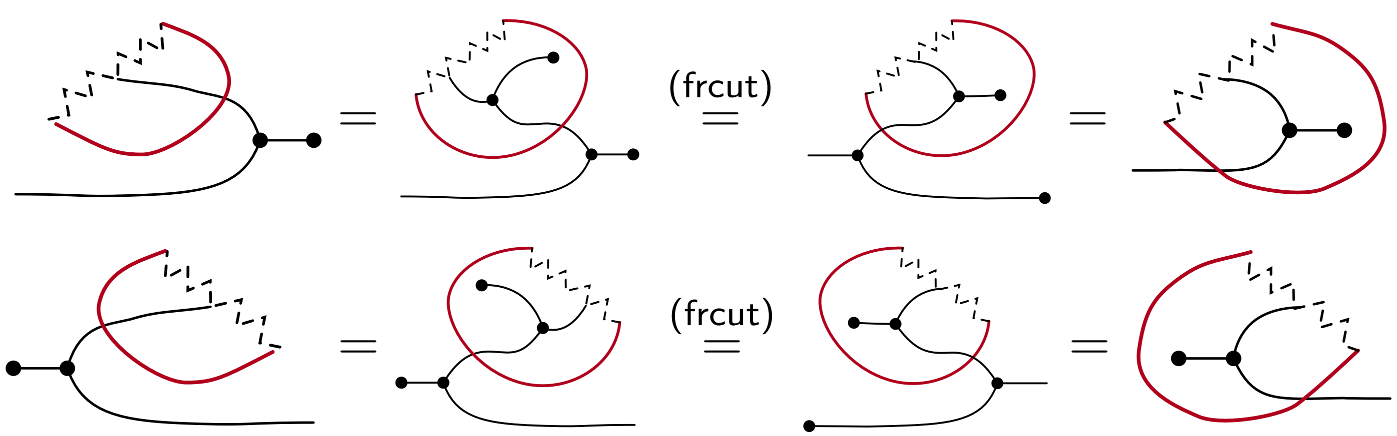
Proof.



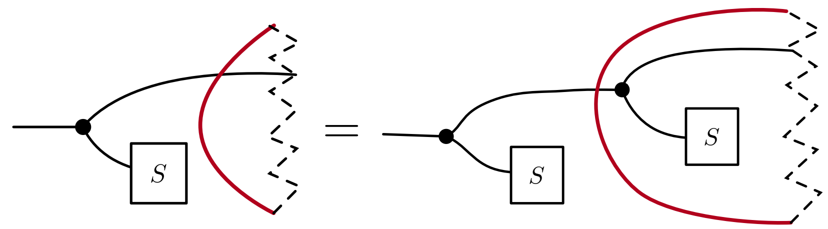
Lemma 6.3.



Proof.



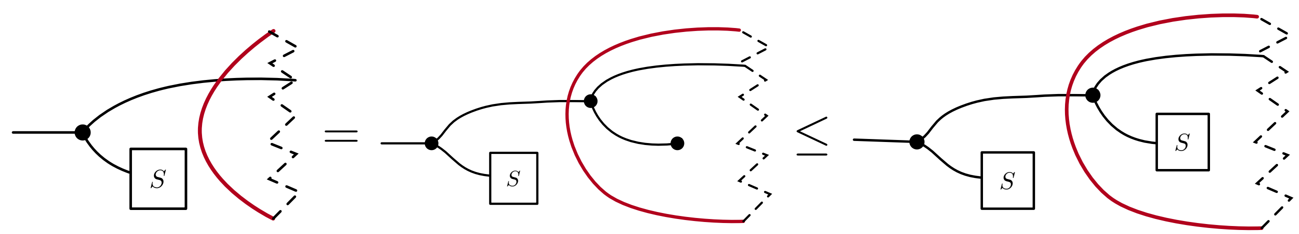
Lemma 6.6.



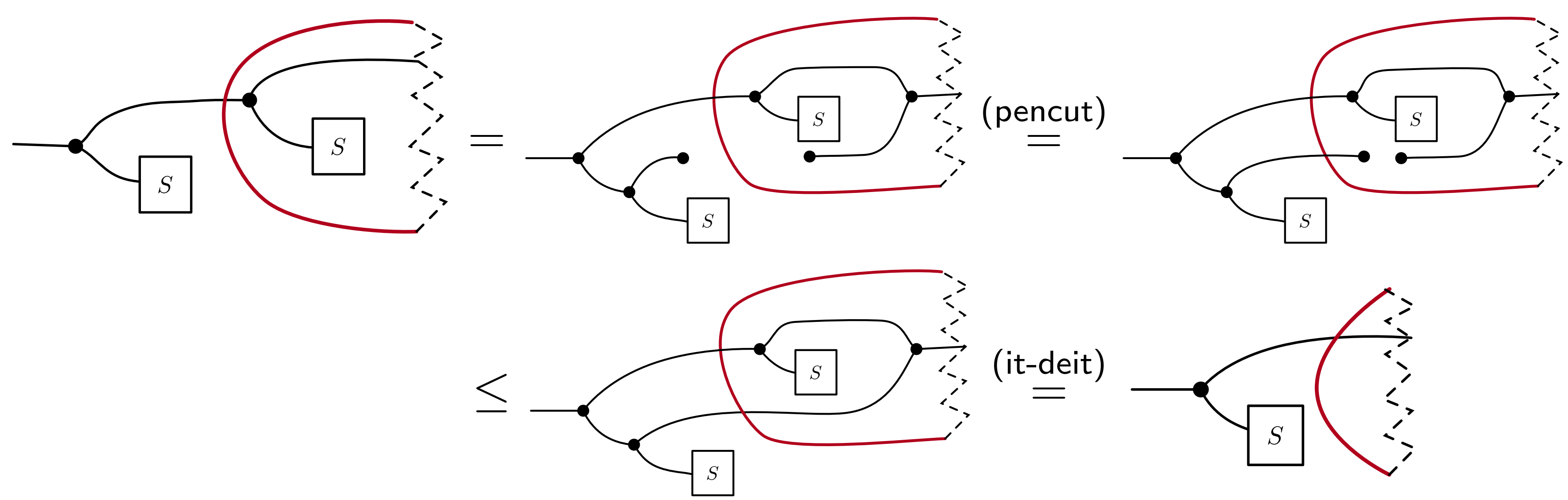
(I/D2)

Proof.

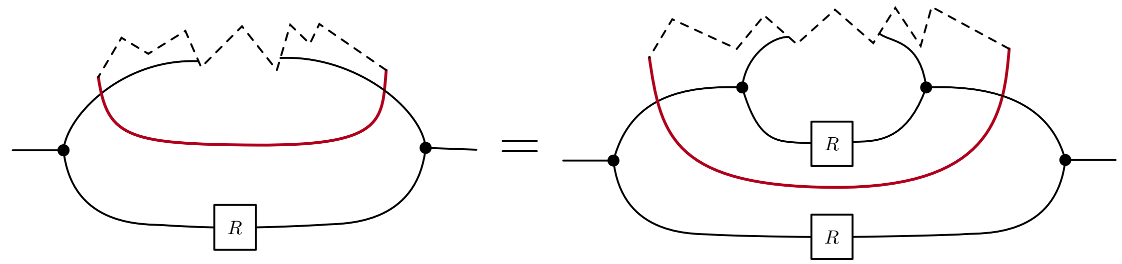
(i)



(ii)

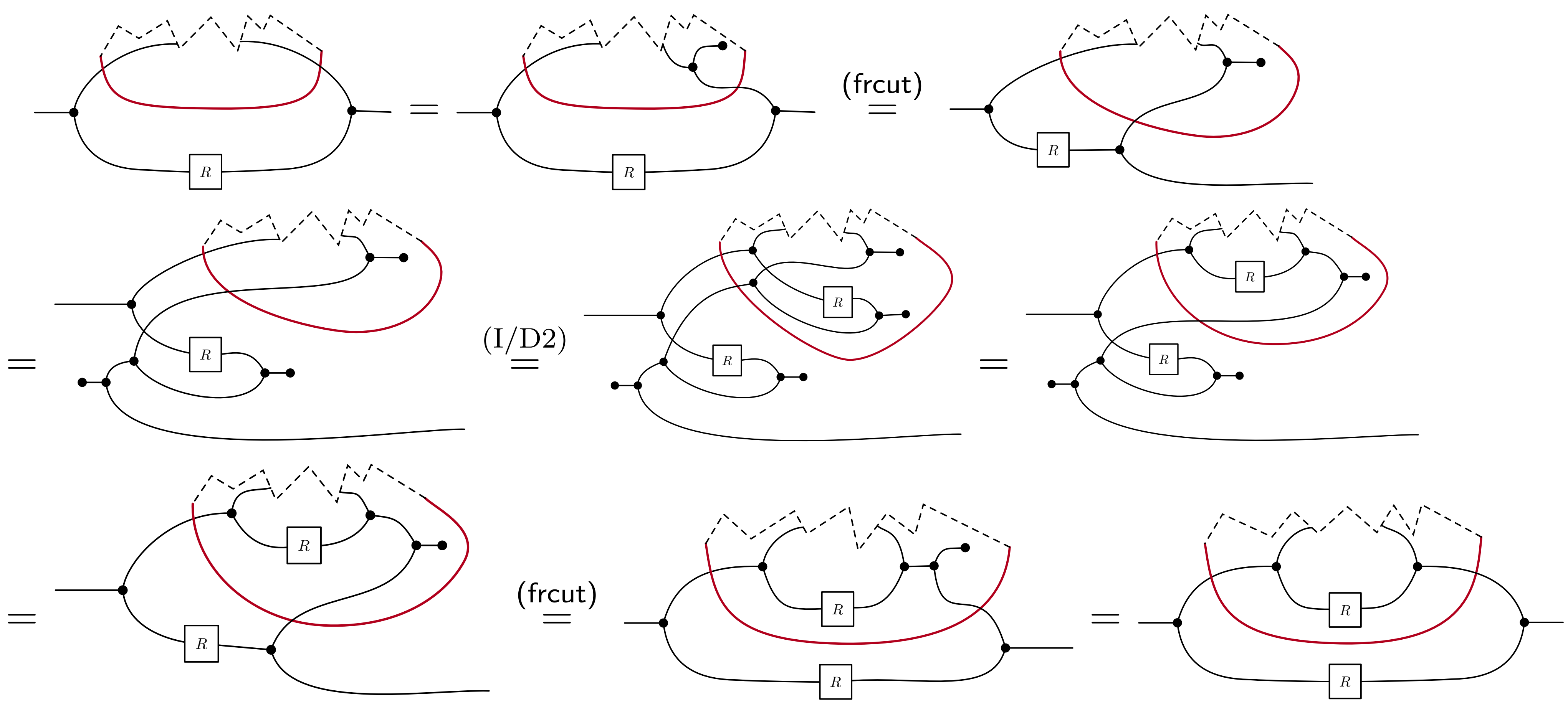


Lemma 6.7.



(I/D3)

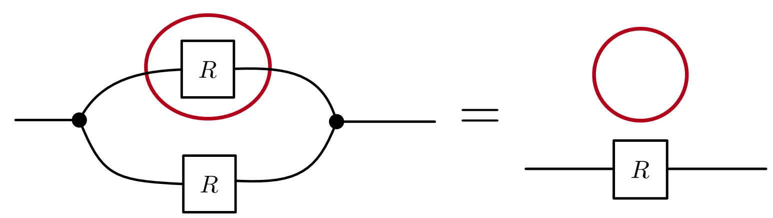
Proof.



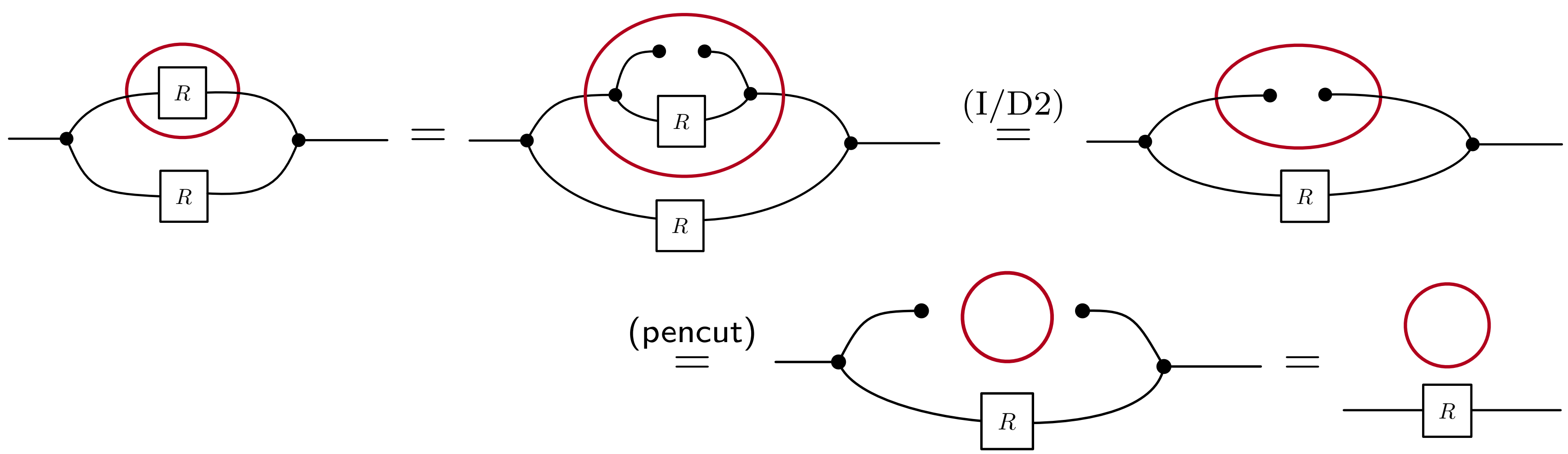
Plan of talk

- Syntax
- The algebra of lines of identity
- Models
- The algebra of negation
- **Examples**

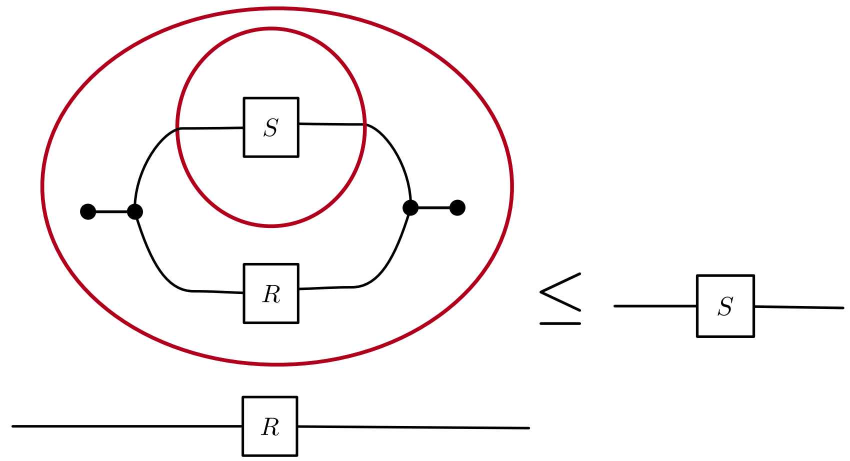
Lemma 6.8.



Proof.



Lemma 6.12.



$$\left(\frac{\varphi_R(x,y)\wedge\forall xy.\varphi_R(x,y)\to\varphi_S(x,y)}{\varphi_S(x,y)}\right) \qquad \text{(MP)}$$

Proof.