# Graded Monads for the Linear-Time / Branching-Time Spectrum

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# Introduction

- Coalgebra does bisimilarity
- Traces need algebra: impose additional equational laws
- Graded monads separate equations by depth
  - control over trace length
- Graded semantics by transformation into graded monad
  - (More or less) generalizes all previous approaches to coalgebraic finite trace semantics
- Graded logics
  - Invariant under graded equivalence (Milius/Pattinson/Schröder CALCO 2015)
  - Expressive under separation conditions
  - Key: Choice of propositional operators

# Coalgebra

Coalgebras = generic reactive systems

- Set (for now) $X$ of states
- Transition structure $X \to GX$
- Functor $G$ is the type of the system.
- E.g. $G = \mathcal{P}$: Non-deterministic branching
- Other examples: Automata, Markov chains, Segala systems / Markov decision processes, concurrent game frames, ...

# Final Coalgebras and the Final Chain

$(Z, \zeta)$ final if

$$\forall (X, \xi). \exists! f : (X, \xi) \to (Z, \zeta)$$

E.g. $GX = A \times X$: $\quad Z = A^\omega$, $\zeta = \langle \mathsf{hd}, \mathsf{tl} \rangle$

- Exists if $G$ is accessible, e.g. finitary ($GX = \bigcup_{Y \subseteq_{fin} X} GY$)
- Then approximated by final chain

$$1 \xleftarrow{\;!\;} G1 \xleftarrow{\;G!\;} G^2 1 \qquad \cdots \qquad G^n 1 \xleftarrow{\;G^n!\;} G^{n+1} 1 \qquad \cdots$$

- $G$ finitary $\implies$
  behavioural equivalence = equality in final chain below $\omega$

# Coalgebraic Modal Logic

Syntactic parameter: modal similarity type $\Lambda$

$$\phi, \psi ::= \bot \mid \neg\phi \mid \phi \wedge \psi \mid L\phi \qquad (L \in \Lambda)$$

Semantic parameters:

- Functor $G$
  - determines models: $G$-coalgebras
- For each $L \in \Lambda$, predicate lifting

$$[\![L]\!] \in \mathrm{Nat}(2^{(-)} \to 2^{G^{op}}) \overset{\text{Yoneda}}{\cong} G2 \to 2$$

Then given $\xi : X \to GX$,

$$x \models L\phi \iff \xi(x) \in [\![L]\!]_X [\![\phi]\!]$$

Base example: relational modal logic ($G = \mathcal{P}$)

$$[\![\Box]\!]_X(A) = \{(U,B) \in \mathcal{P}(\mathrm{At}) \times \mathcal{P}(X) \mid B \subseteq A\}$$

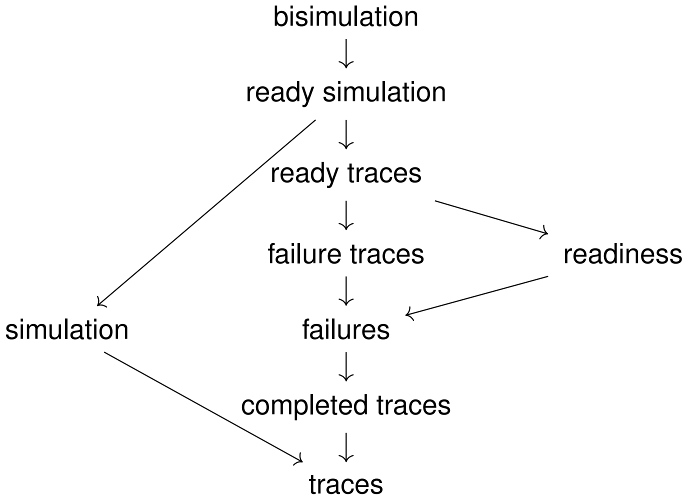
# Branching-Time Expressiveness

- Coalgebraic modal logic is invariant under behavioural equivalence
  - Thanks to naturality of predicate liftings
- $\Lambda$ is separating if

$$\Lambda(2^X) = \left\{ [\![L]\!](f) \mid L \in \Lambda, f \in 2^X \right\} \subseteq GX \to 2$$

  is jointly injective (for finite $X$)
  - trivial when true (e.g. $G = \mathcal{P}$, $\Lambda = \{\Box\}$)
  - Finitary functors admit separating sets of *polyadic* modalities (Schröder 2005)
- **Coalgebraic Hennessy-Milner Theorem**
  $G$ finitary, $\Lambda$ separating $\implies$
  modal indistinguishability = behavioural equivalence
  (Pattinson 2003, Schröder 2005)

# The Linear-Time / Branching-Time Spectrum (Excerpt)



# Recall: Monads and Theories

(Algebraic) theories $(\Sigma, E)$ consist of

- (algebraic) signature $\Sigma$ – operations with arities
- equations $E$.

Correspond to monads $M$ (on **Set**); on set $X$:

- $MX = \Sigma$-Terms with variables in $X$ / equations
- $\eta : X \to MX$ variables-as-terms (unit)
- $\mu : MMX \to MX$ substitution (multiplication)

# Graded Monads and Theories

(Smirnoff 2008)

Graded theories $(\Sigma, d, E)$ consist of

- $d : \Sigma \to \mathbb{N}$ depth
  - $\to$ terms of uniform depth
- equations $E$ of uniform depth

Correspond to graded monads $(M_n)_{n<\omega}$:

- $M_n X = \Sigma$-terms of uniform depth $n$ over $X$
- $\eta : X \to M_0 X$
- $\mu^{nk} : M_n M_k X \to M_{n+k} X$

# Graded Semantics

of $G$-coalgebras = graded monad $(M_n)$ + natural transformation

$$\alpha_X : GX \to M_1 X$$

- Inductively defined pretrace sequence

$$\gamma^{(n)} : X \to M_n X$$

  of $\gamma : X \to GX$

- Trace sequence:

$$X \xrightarrow{\gamma^{(n)}} M_n X \xrightarrow{M_n !} M_n 1$$

# Basic Examples

- Finite-depth bisimilarity (= bisimilarity for $G$ finitary):

$$M_n X = G^n X$$

- Trace semantics of LTS ($G = \mathcal{P}(\mathcal{A} \times X)$):

$$M_n X = \mathcal{P}(\mathcal{A}^n \times X)$$

# Graded Theories for the Linear-Time/Branching-Time Spectrum

$\Sigma$ contains $0, +$ (depth 0), unary $a \in \mathcal{A}$ (depth 1)

$E$ contains join semilattice equations; plus:

- Bisimilarity: –
- Similarity: $a(x+y)+a(x)=a(x+y)$ ($a$ is monotone)
- Traces: $a(0)=0$, $a(x+y)=a(x)+a(y)$
- Completed traces: depth-1 constant $\star$
- Ready traces/simulation: unary operations $\langle A,a\rangle$, $A \subseteq \mathcal{A}$ ready set
- Failure traces:

$$\langle A,a\rangle(x) \leq \langle A \cup B,a\rangle(x)$$

- Readiness/failures: Similarly with depth-1 constants $A$

(Dorsch et al. 2019)

# Graded Algebras

$M_n$-algebra for $n < \omega$ (similarly for $n = \omega$):

- Objects $A_k$, $k \leq n$
- Maps

$$a^{mk} : M_m A_k \to A_{m+k} \qquad (m + k \leq n)$$

# Depth-1 Graded Monads and Canonicity

- Depth-1 graded theory:
  - all operations and equations have depth $\leq 1$
- E.g. all the above
- $M_1$-algebra $(A_0, A_1)$ canonical if free over $M_0$-algebra $A_0$
  - Equivalently $a^{10} : M_1 A_0 \to A_1$ coequalizer (of $\mu^{10,}, M_1 a^{00}$)
  - E.g. $(M_{n+1}X, M_n X)$ if $M$ is depth-1

# Generic Trace Logics

Depth 0:

$$\phi ::= c \mid p(\phi_1, \dots, \phi_k),\ c \text{ truth constant, } p \in \mathcal{O} \text{ propositional operator}$$

Depth $n+1$:

$$\phi ::= L\psi \mid p(\phi_1, \dots, \phi_k), \quad L \in \Lambda \text{ modal operator, } p \in \mathcal{O},\ \psi \text{ depth } n$$

Semantics:

- $M_0$-algebra $\Omega$ of truth values
- Truth values $[\![c]\!] : 1 \to \Omega$
- $M_1$-algebras $[\![L]\!] : M_1\Omega \to \Omega$
- Propositional operators $[\![p]\!] : \Omega^n \to \Omega$ preserving $M_1$-algebras
  - e.g. $M_0$-algebra morphisms

# Semantics of Modalities

For $(A_0, A_1)$ canonical:

$$A_0 \xrightarrow{\;f\;} \Omega$$

$$\begin{array}{ccc} M_1 A_0 & \xrightarrow{M_1 f} & M_1 \Omega \\ {\scriptstyle a^{10}}\big\downarrow & & \big\downarrow{\scriptstyle [\![L]\!]} \\ A_1 & \xrightarrow[{[\![L]\!](f)}]{} & \Omega \end{array}$$

# Expressiveness

**Theorem** Expressiveness holds under depth-0 separation (enough truth constants) and depth-1 separation: For $(A_1, A_0)$ canonical and $\mathfrak{A} \subseteq A_0 \to \Omega$ jointly injective and closed under $\mathcal{O}$,

$$\Lambda(\mathfrak{A}) = \{[\![L]\!](f) \mid L \in \Lambda, f \in \mathfrak{A}\} \subseteq A_1 \to \Omega$$

is jointly injective.

- Subsumes coalgebraic Hennessy-Milner, probabilistic modal logic with only $\wedge$ (Desharnais et al. LICS 1998)

# Characteristic Graded Logics: Examples

- Bisimilarity: $\langle a \rangle$, $\vee$, $\neg$
- Similarity: $\langle a \rangle$, $\vee$, $\wedge$
- Traces: $\langle a \rangle$, $\vee$
- Completed Traces: $\langle a \rangle$, $\vee$, $\star$ (depth 1)
- Readiness / Failures: Constants $A \subseteq \mathcal{A}$ (depth 1) (with different semantics!)
- Ready / Failure Traces: Modalities $\langle A, a \rangle$, $A \subseteq \mathcal{A}$

# Conclusions

- Graded monads cover all finite-depth semantics
- Depth-1 graded monads allow for systematic extraction of characteristic modal logics
- New:
  - Systematic treatment of propositional operators
  - Expressiveness criterion generalizing branching-time coalgebraic Hennessy-Milner theorem
- Future work: Temporal extensions, axiomatizations, model checking, behavioural metrics

# Examples: Coalgebraic Trace / Language Semantics

- Kleisli-style coalgebraic traces:
  - $G = TF$, $T$ monad (e.g. $T = \mathcal{P}$, $F = \mathcal{A} \times (-)$)
  - $M_n = TF^n$
  - Canonical forgetting
- Eilenberg-Moore-style coalgebraic traces:
  - $G = FT$ (e.g. $T = \mathcal{P}$, $F = (-)^{\mathcal{A}}$)
  - $M_n = F^n T$
  - Canonical forgetting