Metric Bisimulation Games and Real-Valued Modal Logics for Coalgebras (CONCUR '18)

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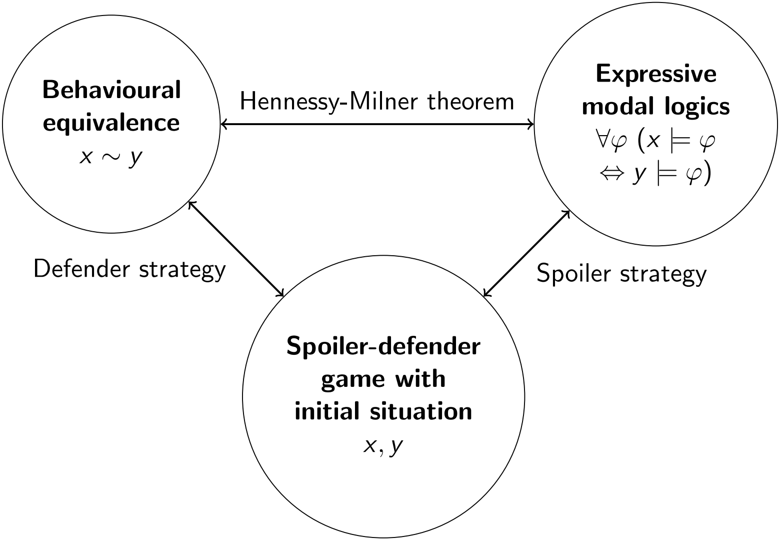
Behavioural Equivalences, Modal Logics & Games



Behavioural Equivalences, Modal Logics & Games

Games are interesting since . . .

- they explain the essence of behavioural equivalences.
- they can be used to explain why two states are not behaviourally equivalent ($x \not\sim y$) by giving the spoiler strategy.
- they can be used to prove interesting theorems (for instance the van Benthem/Rosen theorem: "the bisimulation-invariant fragment of first-order logic corresponds to propositional modal logics").

There is little work on games that goes beyond labelled transition systems [Desharnais et al., Fijalkow et al.] and only few paper on games [Baltag, Kupke] in a coalgebraic setting.

Behavioural Metrics

Find a quantitative notion of behavioural equivalence ...

- Do not insist on the exact same behaviour.
- Measure the behavioural distance between two states.
- Make statements such as "the behaviour of two states differs only by ε".

$\rightsquigarrow$ Behavioural metrics

Applications: differential privacy [Palamidessi], conformance testing for cyber-physical systems [Khakpour, Mousavi]

Here: work out the triad of behavioural distance, modal logics and spoiler-defender games for behavioural metrics.

Behavioural distances

Pseudo-metric space

Let X be a set. A pseudo-metric is a function $d : X \times X \to [0, 1]$ where for all $x, y, z \in X$:

1. $d(x, x) = 0$ (reflexivity) (metric if $d(x, y) = 0 \Rightarrow x = y$)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

A (pseudo-)metric space is a pair (X, d) where X is a set and d is a (pseudo-)metric.

A directed pseudo-metric must only satisfy reflexivity and the triangle inequality.

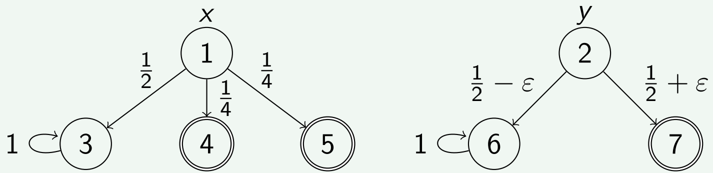
Example: Probabilistic Transition System

Probabilistic transition system

A probabilistic transition system is a tuple $P = (X, T, p)$, where X is a set of states, $T \subseteq X$ is the set of terminal states and every state $x \notin T$ is assigned a probability distribution $p_x : X \to [0, 1]$.

Similar models studied by [Larsen, Skou], [van Breugel, Worrel]

Example: Probabilistic Transition System



States $4, 5, 7$ are terminal.

What is the distance between states x, y?

Example: Probabilistic Transition System

Distance of states in a probabilistic transition system

Compute the smallest fixed-point of

$$d(x,y) = \begin{cases} 1 & \text{if } x \in T, y \notin T \text{ or } x \notin T, y \in T \\ 0 & \text{if } y \in T, y \in T \\ d^P(p_x, p_y) & \text{otherwise} \end{cases}$$

What does it mean to compute the distance between two probability distributions p_x, p_y on a metric space (i.e. to lift the distance to probability distributions)?

Example: Probabilistic Transition System

Non-expansive function

A non-expansive function $f\colon X \to Y$ between two (pseudo-) metric spaces $(X, d_X), (Y, d_Y)$ satisfies for $x, y \in X$:

$$d_X(x, y) \geq d_Y(f(x), f(y))$$

The category of pseudo-metric spaces and non-expansive functions is denoted by **PMet**.

Example: Probabilistic Transition System

We obtain the distance for two probability distributions $p, q : X \to [0,1]$ by lifting a metric $d : X \times X \to [0,1]$ as follows:

Kantorovich lifting for probability distributions

The supremum of $|\sum_{x\in X} f(x)\cdot p(x) - \sum_{x\in X} f(x)\cdot q(x)|$ for all non-expansive functions $f\colon (X,d) \to ([0,1], d_e)$
(where $d_e(a,b) = |a-b|$)

Intuition: we measure the cost of transporting supply p to demand q from the point of view of a logistics company [Villani]

- f is a price function, which specifies buying/selling prices at each location
- Non-expansiveness means $f(x) - f(y) \le d(x,y)$ for all x, y. If this is not satisfied the company is not hired, since it is cheaper for the customer to perform the transport himself.

Coalgebras

Coalgebras & Coalgebra Homomorphism

Let F be a functor. A coalgebra is a function $\alpha : X \to FX$ (where X is the state set).
We call $f \colon X \to Z$ a coalgebra homomorphism from α to $\beta \colon Y \to FY$ whenever $Ff \circ \alpha = \beta \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow{\beta} & FY \end{array}$$

Two states $x, y \in X$ are behaviourally equivalent ($x \sim y$) if there exists a coalgebra homomorphism f with $f(x) = f(y)$.

Probabilistic transition system: $\alpha \colon X \to \mathcal{D}X + 1$ where $\mathcal{D}$ is the probability distribution functor and $1 = \{\surd\}$.

Coalgebras & Behavioural Metrics

Aim: Lift the functor F from **Set** to **PMet**

Evaluation maps

We need one parameter: a set Γ of evaluation maps $\gamma\colon F[0,1] \to [0,1]$.

Example: Probabilistic transition systems, $FX = \mathcal{D}X + 1$

We define two evaluation maps (one for probability distributions, one for termination):

- $\gamma_{\mathcal{D}}\colon \mathcal{D}[0,1] + 1 \to [0,1]$ with
  - $\gamma_{\mathcal{D}}(p) = \sum_{r\in[0,1]} r \cdot p(r)$, $p\colon X \to [0,1]$ (expectation)
  - $\gamma_{\mathcal{D}}(\surd) = 0$
- $\gamma_{\bullet}\colon \mathcal{D}[0,1] + 1 \to [0,1]$ with
  - $\gamma_{\bullet}(p) = 0$
  - $\gamma_{\bullet}(\surd) = 1$

Functor Lifting via Kantorovich

Kantorovich Lifting (for a functor F)

Let $d : X \times X \to [0,1]$ be a pseudo-metric. We define $d^{\uparrow\Gamma} \colon FX \times FX \to [0,1]$. Let $t_1, t_2 \in FX$:

$$d^{\uparrow\Gamma}(t_1, t_2) = \sup\{d_e(\gamma(Ff(t_1)), \gamma(Ff(t_2))) \mid f \colon (X, d) \to ([0,1], d_e) \text{ non-expansive}, \gamma \in \Gamma\}$$

Lifting directed pseudo-metrics: exchange $d_e(a, b) = |a - b|$ with $d_\ominus(a, b) = max\{a - b, 0\}$.

Functor Lifting via Kantorovich

Every evaluation map γ induces a (real-valued) predicate lifting

$$(f\colon X \to [0,1]) \qquad \mapsto \qquad (\gamma \circ Ff\colon FX \to [0,1])$$

We require that this predicate lifting is non-expansive wrt. the supremum metric (local non-expansivenses, see also [Turi, Rutten]):

$$d_e^\infty(\gamma \circ Ff, \gamma \circ Fg) \le d_e^\infty(f,g)$$

for all $f,g\colon X \to [0,1]$. (The same holds if we replace d_e by $d_\ominus$.)

Supremum metric: $d^\infty(f,g) = \sup_{x\in X} d(f(x),g(x))$

In the directed case this condition generalizes monotonicity.

Defining Distances in Coalgebras

Coalgebraic behavioural metrics [Baldan, Bonchi, Kerstan, K.]

Given a set of evaluation maps Γ and coalgebra $\alpha : X \to FX$ in **Set**, compute its associated behavioural metric $d : X \times X \to [0, 1]$ as the smallest fixed-point of:

$$d = d^{\uparrow \Gamma} \circ (\alpha \times \alpha)$$

where $d^{\uparrow \Gamma}$ is the Kantorovich lifting.

This instantiates to the earlier definition of probabilistic behavioural metric.

Real-Valued Modal Logics

Behavioural metrics in the probabilistic case are often associated with real-valued modal logics with formulas that assign real numbers to states (instead of truth values $0, 1$), indicating the "degree" up to which a formula holds.
Given a coalgebra $\alpha\colon X \to FX$, the semantics of a formula φ is given by a function $[\![\varphi]\!]_\alpha\colon X \to [0,1]$.

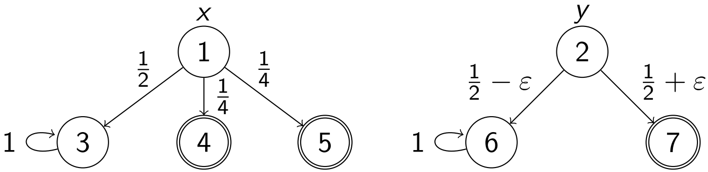
φ:	1	$[\gamma]\psi$, $\gamma \in \Gamma$	$\min(\psi, \psi')$
$[\![\varphi]\!]_\alpha$:	1	$\gamma \circ F[\![\psi]\!]_\alpha \circ \alpha$	$\min\{[\![\psi]\!]_\alpha, [\![\psi']\!]_\alpha\}$

φ:	$\neg\psi$	$\psi \ominus q$
$[\![\varphi]\!]_\alpha$:	$1 - [\![\psi]\!]_\alpha$	$[\![\psi]\!]_\alpha \ominus q$

The modalities $[\gamma]\psi$ are based on the evaluation maps $\gamma \in \Gamma$.

Every function $[\![\varphi]\!]_\alpha$ is non-expansive (analogue of bisimulation-invariant).

Real-Valued Modal Logics



The formula $\varphi = [\gamma_{\mathcal{D}}][\gamma_\bullet]1$ distinguishes the states x, y.

- $\psi = [\gamma_\bullet]1$ assigns 1 to terminating states and 0 to non-terminating states.
- x makes a transition to a terminating state with probability $\frac{1}{2}$ $\Rightarrow [\![\varphi]\!](x) = \gamma_{\mathcal{D}}(\mathcal{D}[\![\psi]\!](\alpha(x))) = \frac{1}{2}$.
  Similarly: $[\![\varphi]\!](y) = \frac{1}{2} + \varepsilon$.

Hence $d_\alpha^L(x, y) \geq d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) = \varepsilon$. (In fact we have equality.)

A Hennessy-Milner Theorem Real-Valued Modal Logics

Logical Distance

Let $\alpha\colon X \to FX$, $x, y \in X$:

$$d_\alpha^L(x, y) = \sup\{d_e([\![\varphi]\!]_\alpha(x), [\![\varphi]\!]_\alpha(y)) \mid \varphi\}.$$

Hennessy-Milner Theorem

Assume that the fixpoint d_α is reached in ω steps.
Then behavioural distance equals logical distance: $d_\alpha = d_\alpha^L$.

Proof strategy (adapted from [van Breugel/Worrell]):

- Show that the lifted functor preserves total boundedness.
- Show that the set $\{[\![\varphi]\!]\colon X \to [0, 1] \mid \varphi\}$ is dense in the non-expansive functions.

Games for behavioural distances

Metric bisimulation game for a coalgebra $\alpha\colon X \to FX$

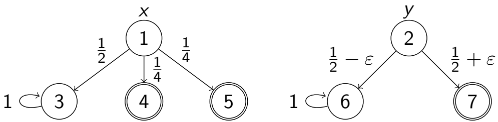
- **Initial situation:** (x, y, ε) where $\varepsilon \in [0, 1]$.
- **Step 1:** Spoiler (S) chooses $s \in \{x, y\}$ and a (real-valued) predicate $p_1\colon X \to [0, 1]$.
- **Step 2:** Defender (D) takes $t \in \{x, y\}\backslash\{s\}$ and has to answer with a predicate $p_2\colon X \to [0, 1]$, which satisfies

$$d_\ominus(\gamma(Fp_1(\alpha(s))), \gamma(Fp_2(\alpha(t)))) \leq \varepsilon \text{ for all } \gamma \in \Gamma.$$

- **Step 3:** S chooses p_i with $i \in \{1, 2\}$ and a state $x' \in X$.
- **Step 4:** D chooses $y' \in X$ with $p_i(x') \leq p_j(y')$ where $j \neq i$
- **Next round:** (x', y', ε') with $\varepsilon' = p_j(y') - p_i(x')$.

D wins the game if the game continues forever or if S has no move at Step 3. If D has no move at Step 2 or Step 4, S wins.

Games for behavioural distances



- **Initial situation:** (x, y, ε)
- **Step 1:** S chooses x with $p_1(4) = 1$ (0 otherwise).
- **Step 2:** D plays p_2 with $p_2(4) = p_2(5) = p_2(7) = 1$ (0 otherwise).

  $$d_\ominus(\gamma_\mathcal{D}(\mathcal{D}p_1(\alpha(x))), \gamma_\mathcal{D}(\mathcal{D}p_2(\alpha(y)))) = d_\ominus(\frac{1}{4}, \frac{1}{2} + \varepsilon) = 0 \leq \varepsilon$$

  (Similar for $\gamma_\bullet$.)
- **Step 3/4:** If S picks a terminating state x' and p_i, D can also pick a terminating state y' and p_j with $p_j(y') - p_i(x') = 0$ (similarly for non-terminating states).
  New situation: $(x', y', 0)$ where $x' \sim y'$.

Games for behavioural distances

Game Distance

For a coalgebra $\alpha\colon X \to FX$ the game distance is defined as:

$$d_\alpha^G(x,y) = \inf\{\varepsilon \mid \text{D has a winning strategy for } (x,y,\varepsilon)\}$$

Behavioural Distance and Game Distance

The behavioural distance equals the game distance: $d_\alpha = d_\alpha^G$.

Games for behavioural distances

Spoiler Strategy

Take a modal formula $\varphi \neq 1$ with $d_\ominus([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$.

- $\varphi = [\gamma]\psi$: S chooses x, $p_1 = [\![\psi]\!]$ at Step 1 and can play in such a way that $d_e([\![\psi]\!](x'), [\![\psi]\!](y')) > \varepsilon'$.
  $\leadsto$ the game continues in the situation (x', y', ε') with the formula ψ.
- $\varphi = \min(\psi, \psi')$: either $d_e([\![\psi]\!](x), [\![\psi]\!](y)) > \varepsilon$ or $d_e([\![\psi']\!](x), [\![\psi']\!](y)) > \varepsilon$ and S picks ψ or ψ'.
- $\varphi = \neg\psi$: S takes ψ.
- $\varphi = \psi \ominus q$: S takes ψ.

Open Questions & Future Work

Open Questions

- Hennessy-Milner theorem if the fixpoint d_α is not reached in ω steps.
- Does the Kantorovich lifting preserve completeness of metrics?
- Can we replace $[0,1]$ by $[0,\infty]$?

Current & Future Work

- Work out the real-valued modal logics for various functors (in the paper: metric transition systems [de Alfaro, Faella, Stoelinga], functor $FX = [0,1] \times \mathcal{P}X$)
- van Benthem/Rosen theorem(s)
  $\rightsquigarrow$ Fuzzy case: [Wild, Schröder, Pattinson, K., LICS '18]
- Up-to techniques for behavioural metrics [Bonchi, K., Petrisan, CONCUR '18]