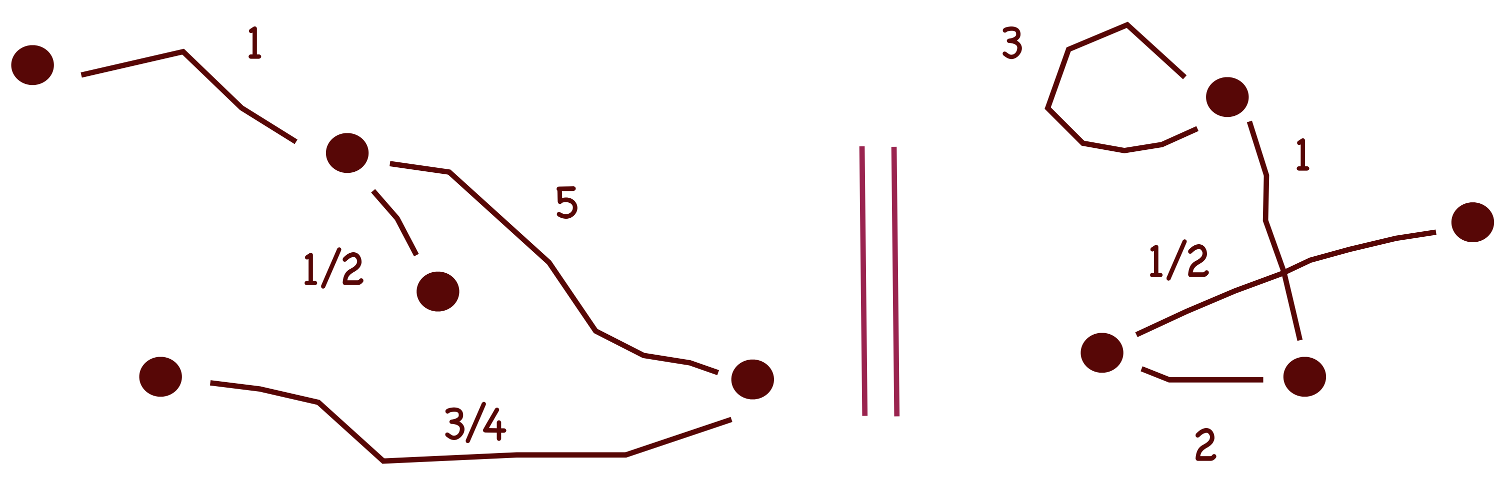
Semantics for Probability and Concurrency

Ana Sokolova
UNIVERSITY of SALZBURG



IFIP WG 1.3 meeting Berlin, 4-9-17

Rigorous methods
for
engineering of
and
reasoning about
reactive systems

Rigorous methods
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and
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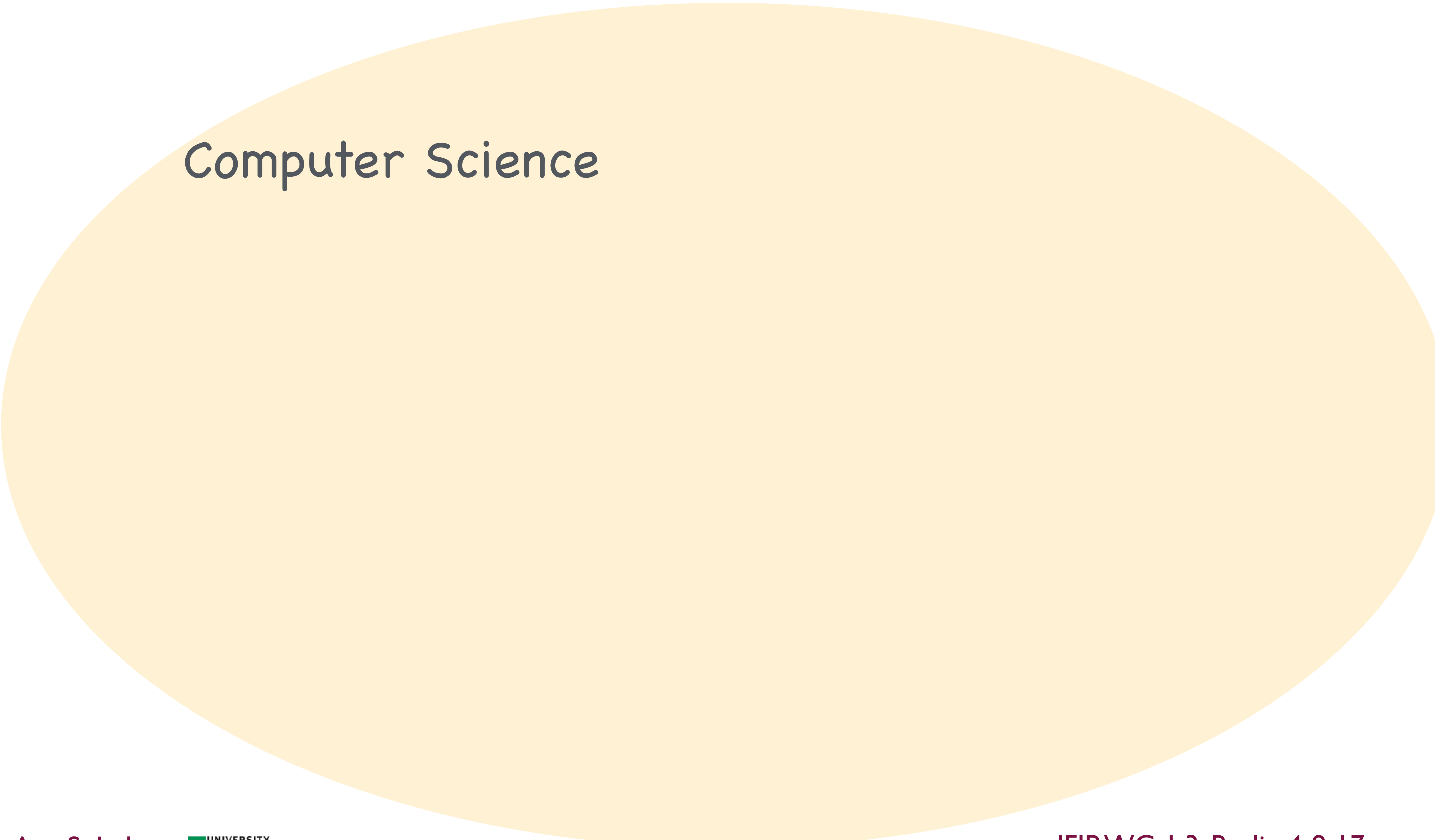
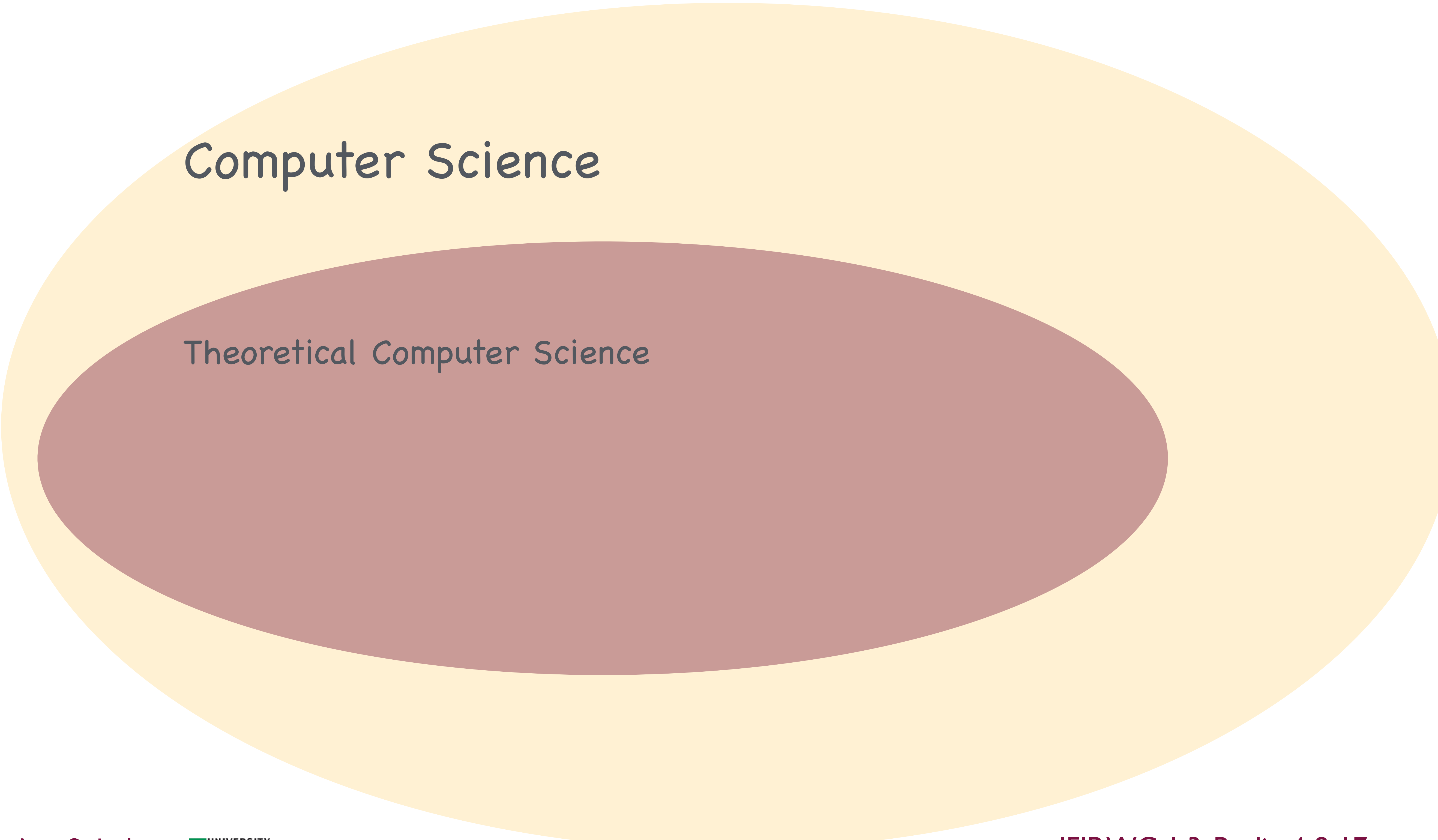
concurrent

Background big picture

Background big picture



Background big picture



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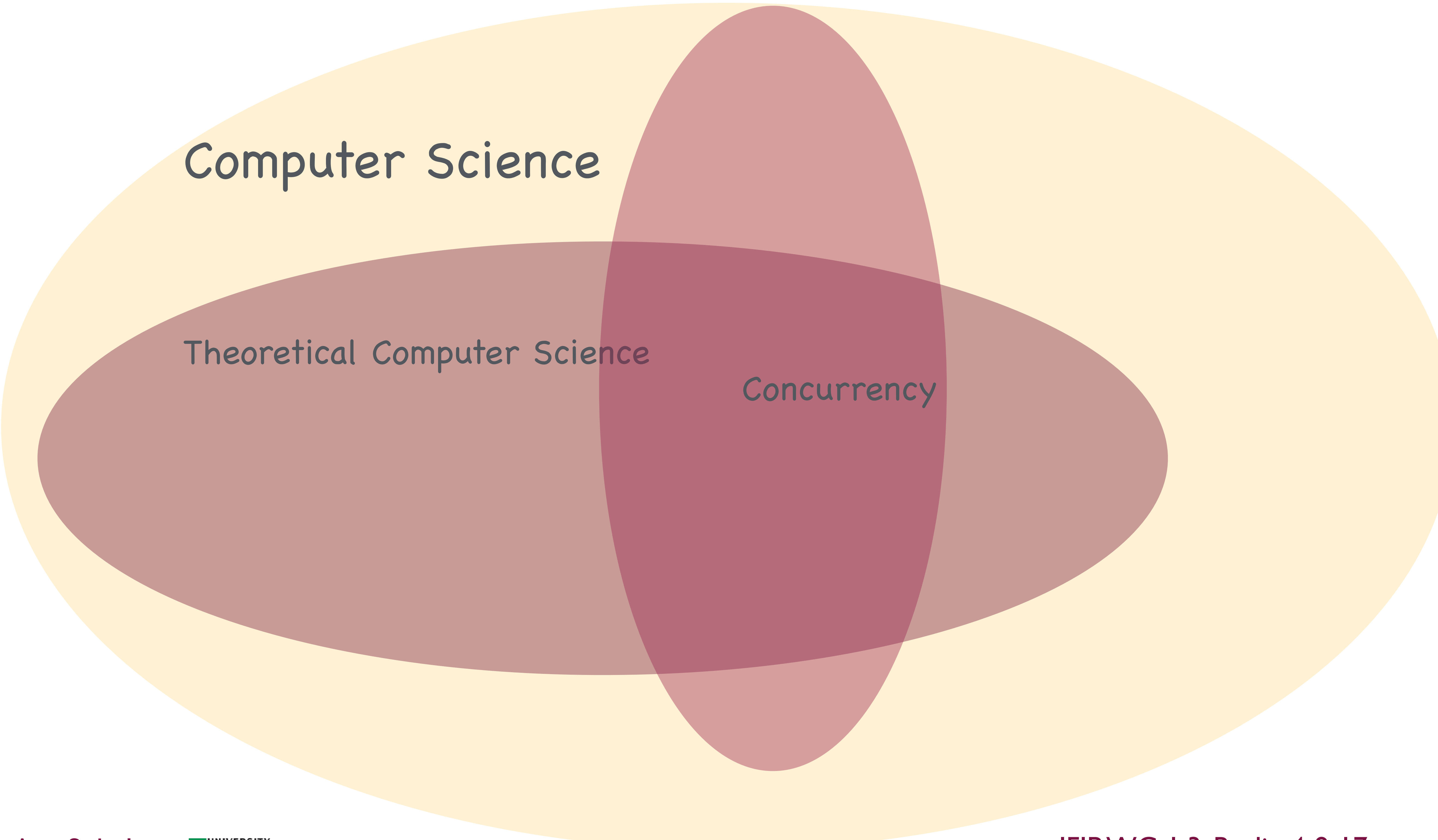

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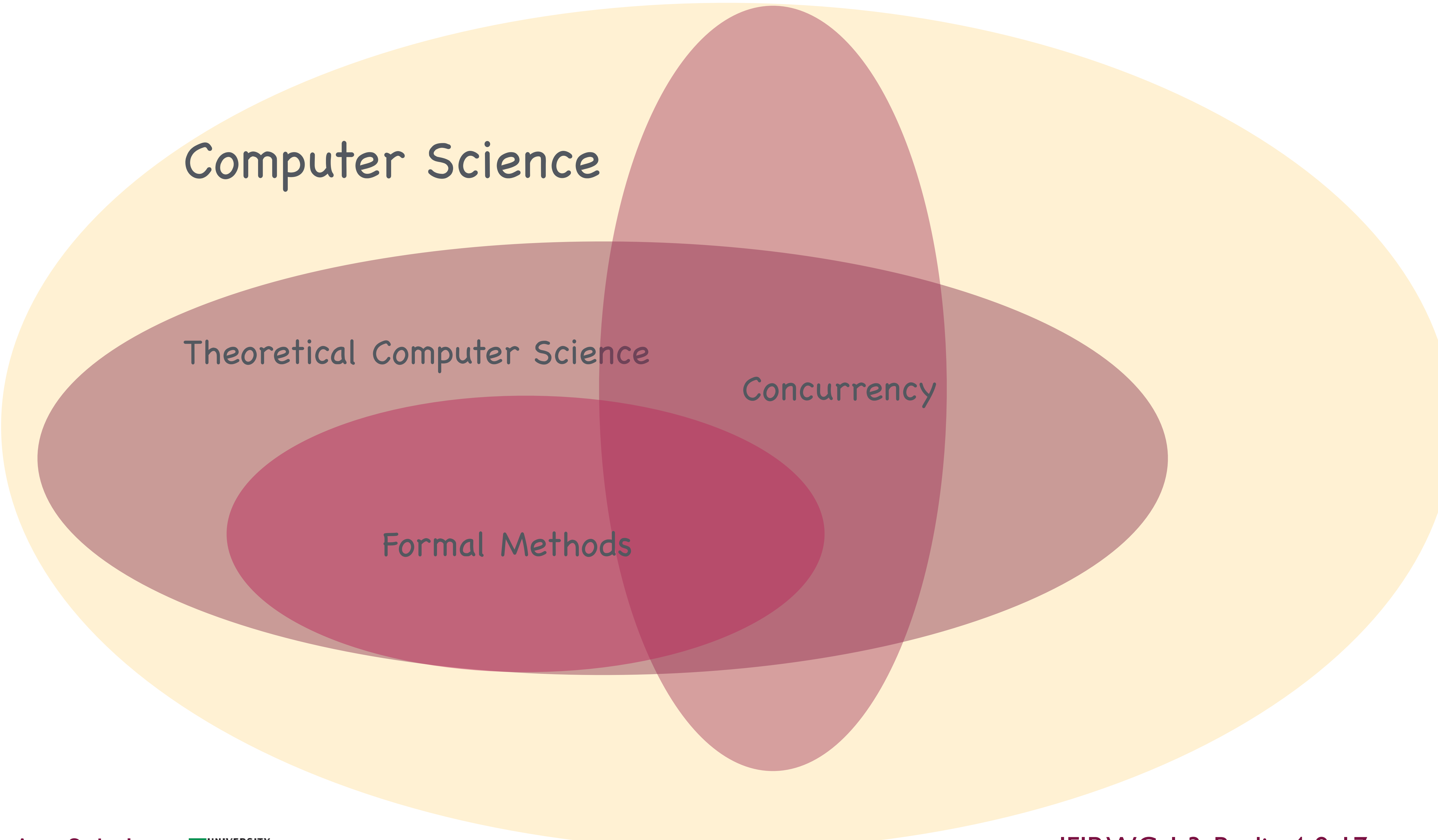

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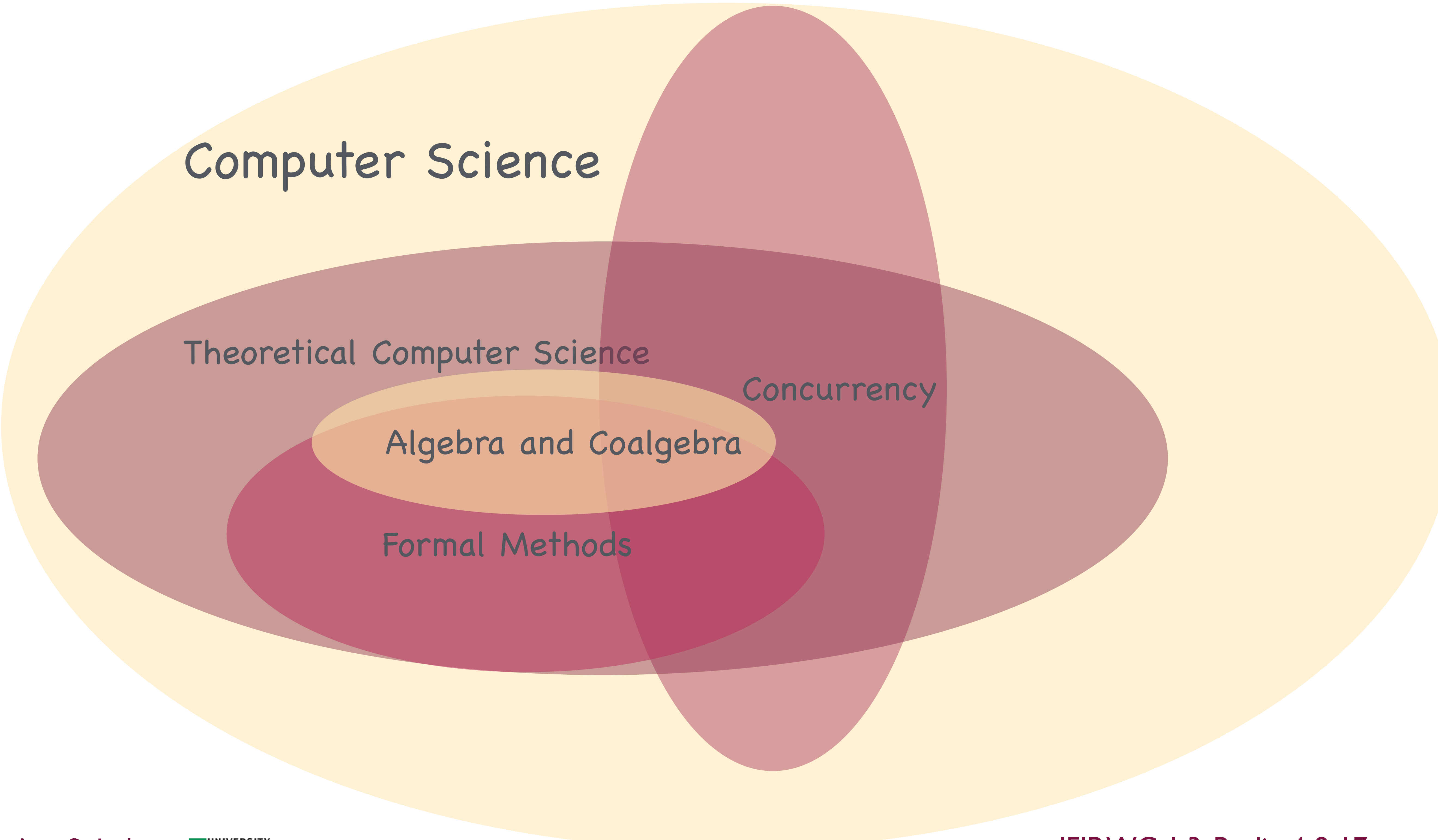

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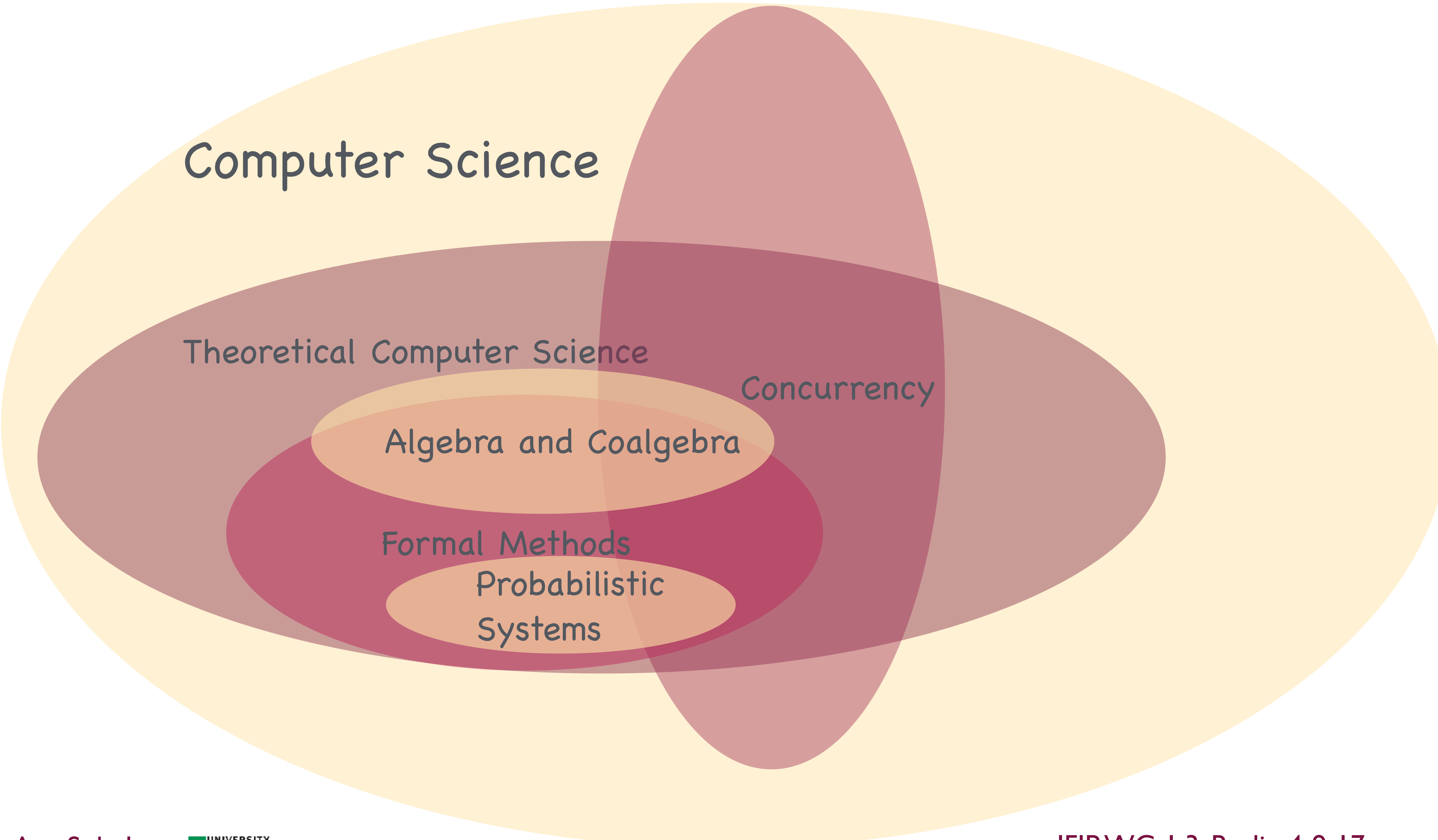

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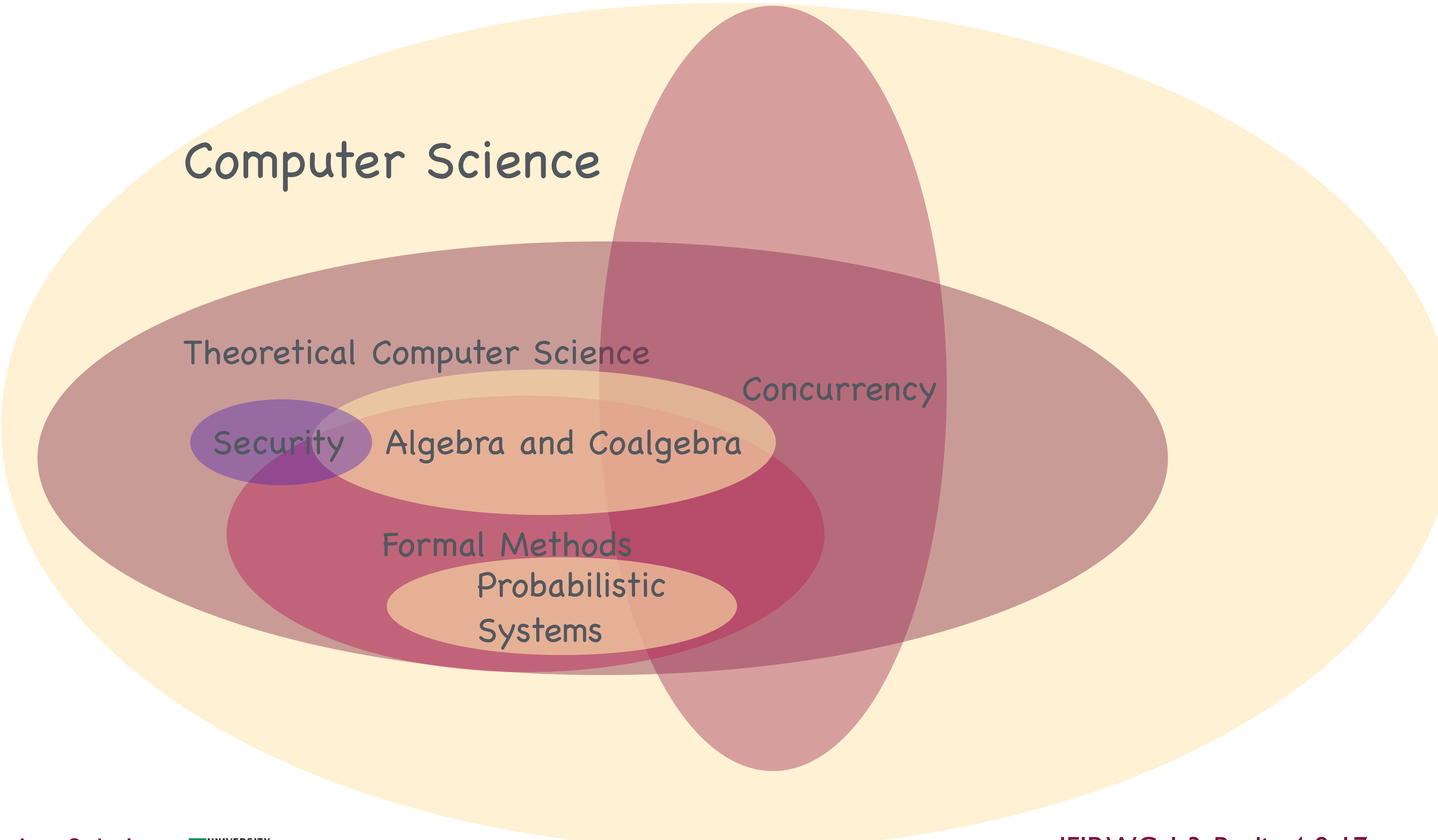

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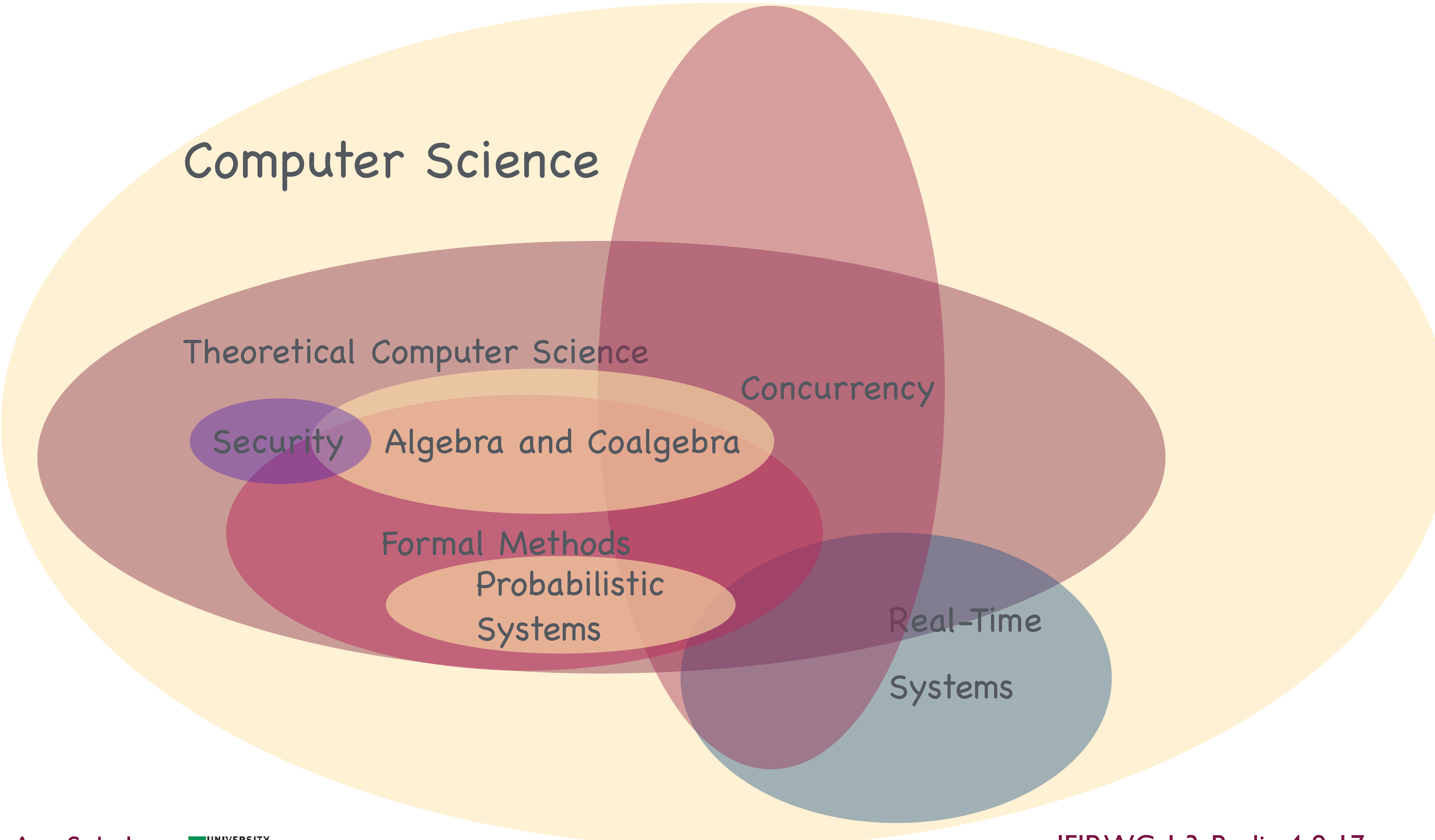

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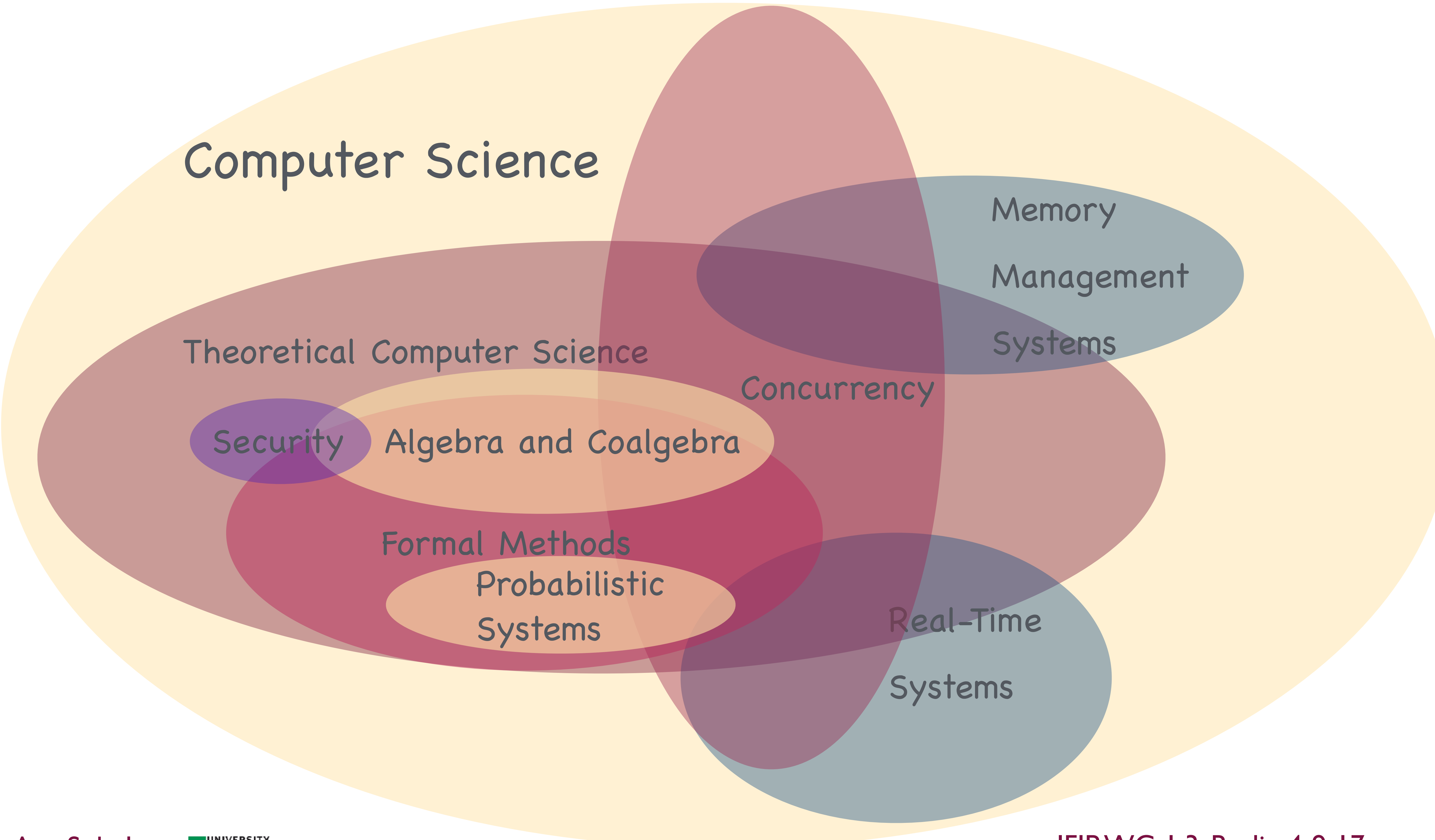

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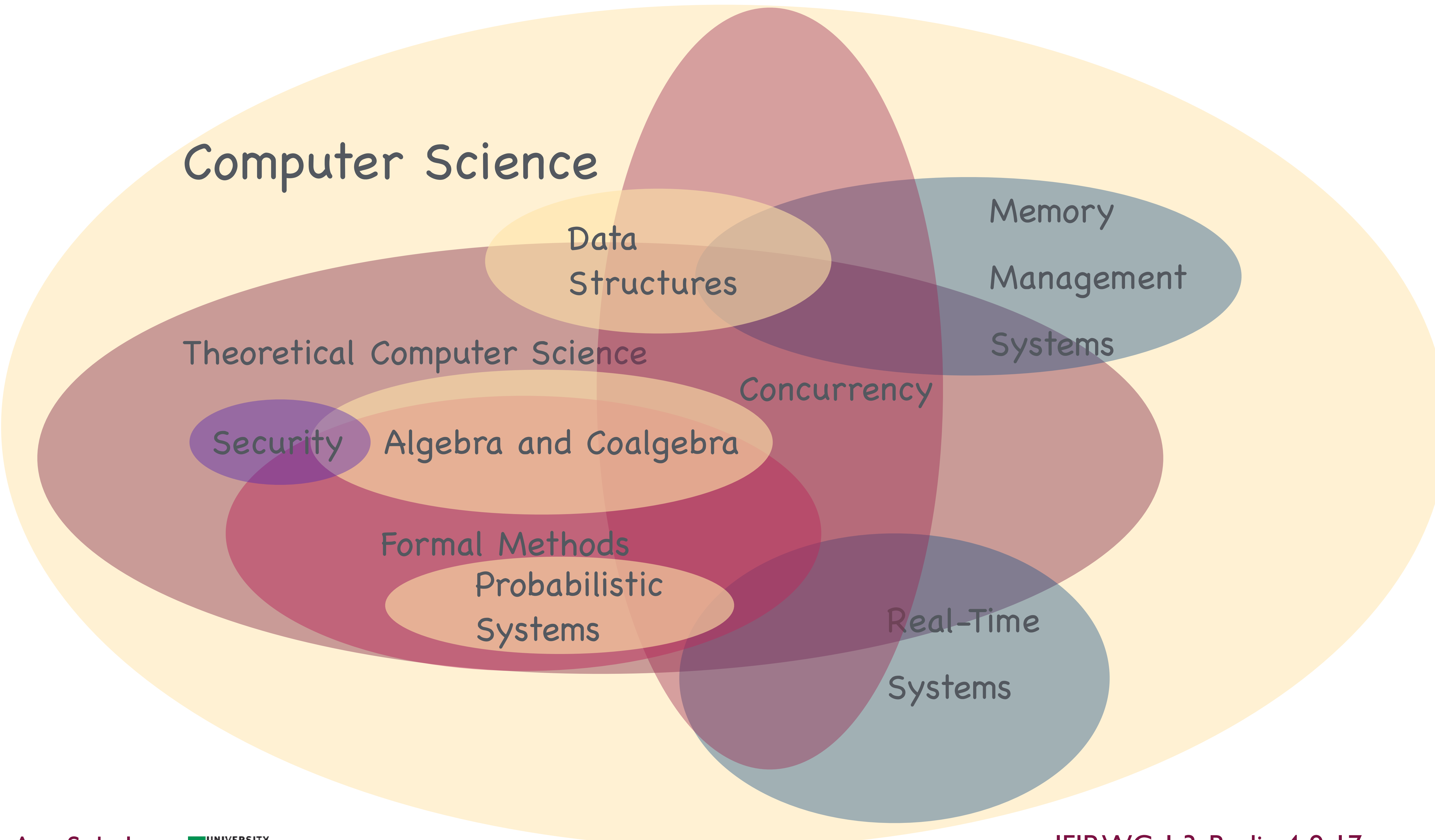

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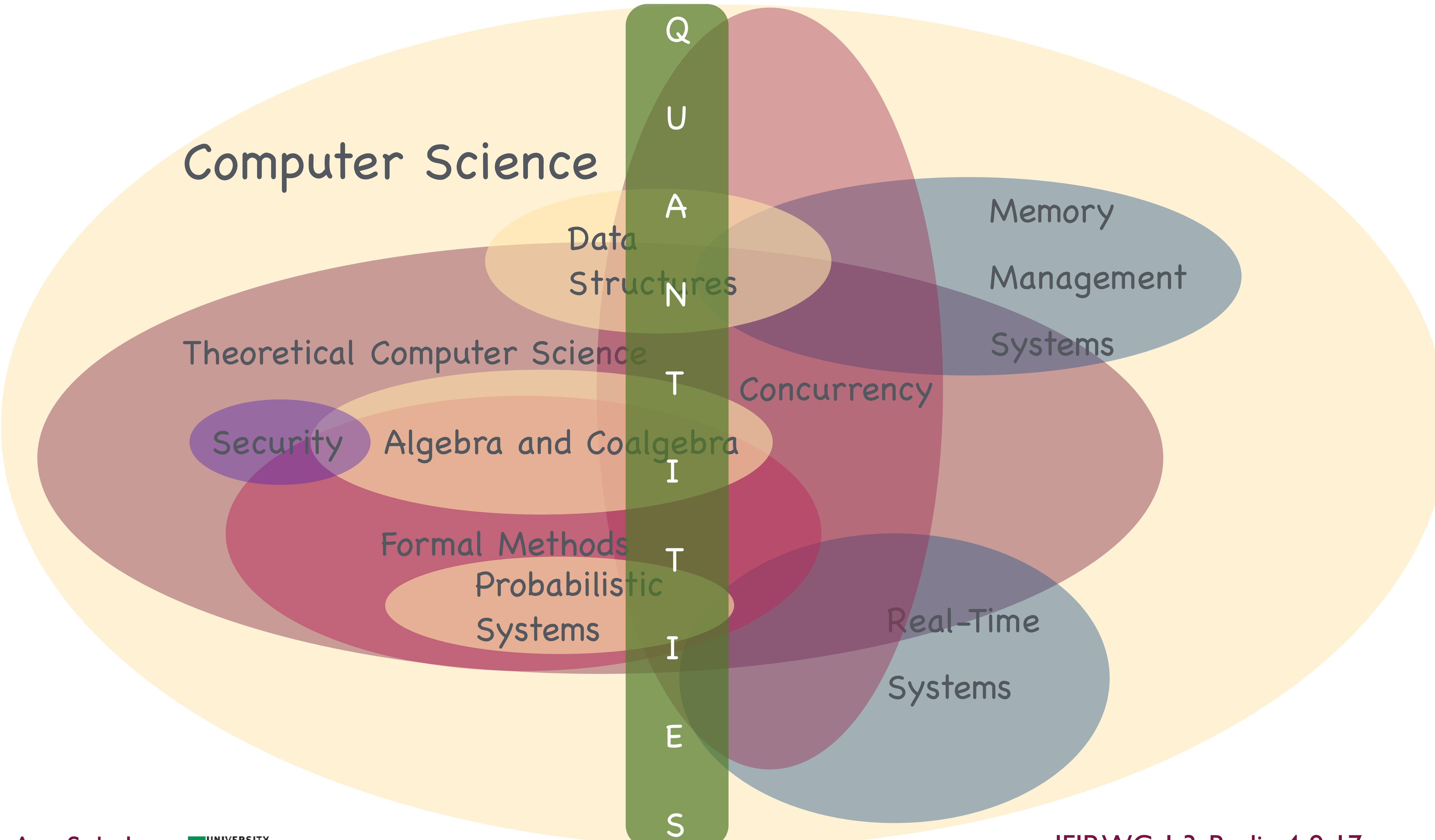
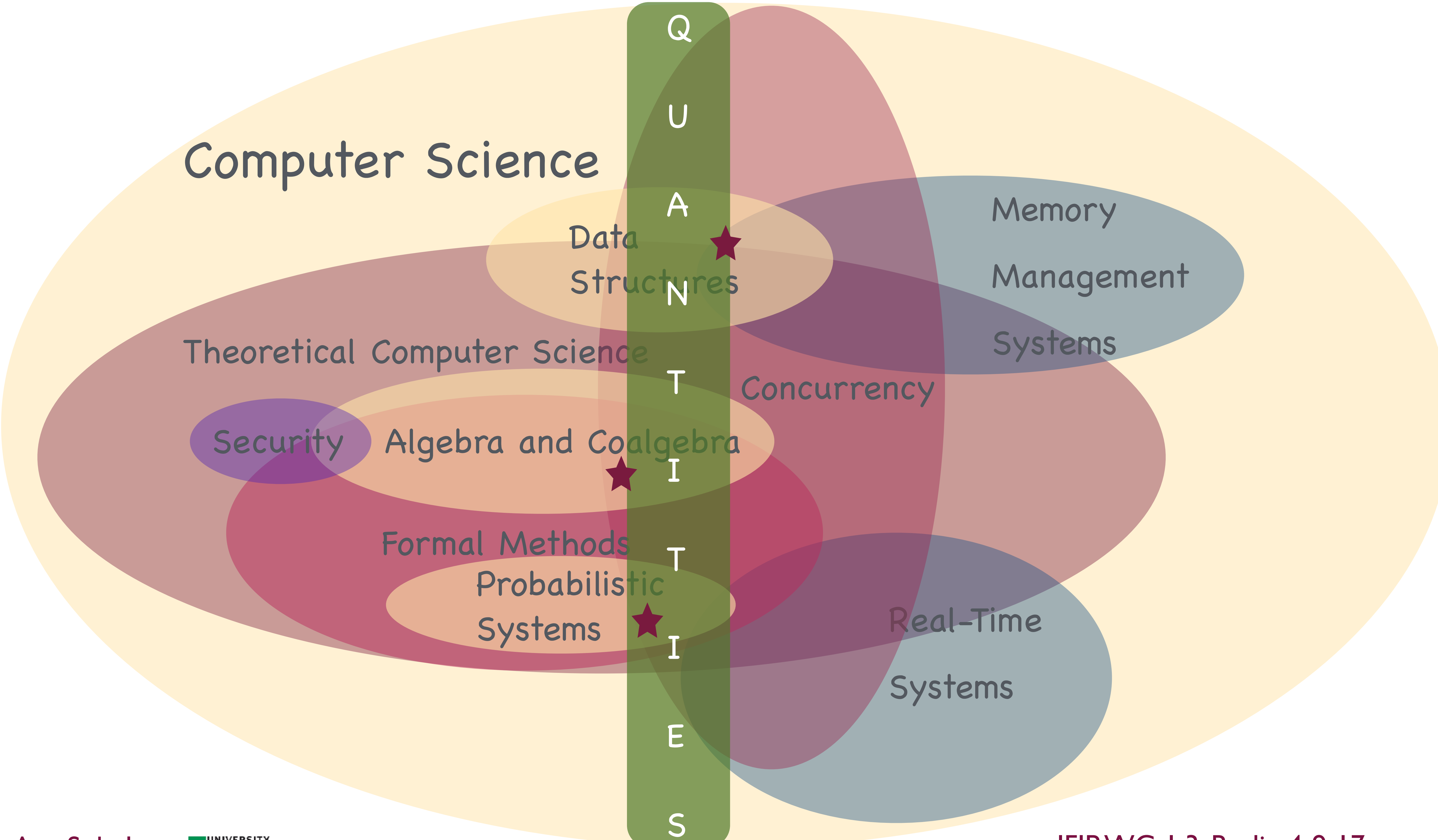
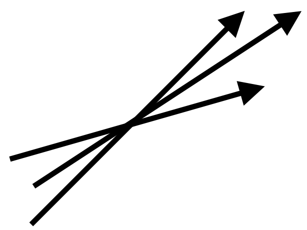
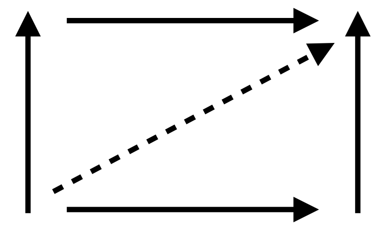


Current favourites







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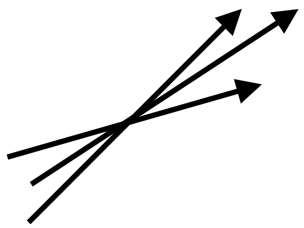
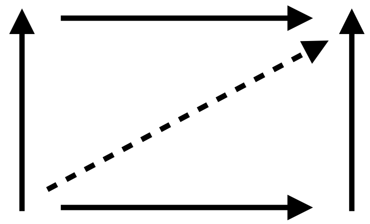
Part I

Coalgebra/algebra + probability highlights





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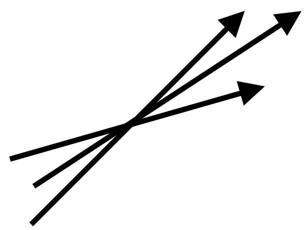
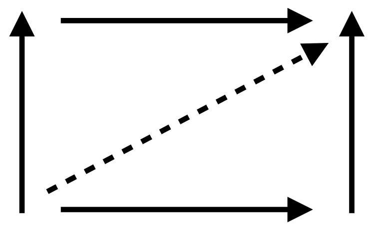
Part I
Coalgebra/algebra + probability highlights

Mathematical framework based on category theory for state-based systems semantics





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Coalgebra/algebra + probability highlights

Mathematical framework based on category theory for state-based systems semantics

A. S. Probabilistic systems coalgebraically TCS'11

B. Jacobs, I. Hasuo, A. S. Generic trace semantics via coinduction LMCS'07

B. Jacobs, I. Hasuo, A. S. The microcosm principle and concurrency in coalgebra FoSSaCS'08

A. Silva, A. S. Sound and complete axiomatisation of trace semantics for probabilistic systems MFPS'11

B. Jacobs, A. Silva, A. S. Trace semantics via determinization JSS'15

A. S., H. Woracek Congruences of convex algebras JPAA'15

A. S., H. Woracek Termination in convex sets of distributions CALCO'17

F. Bonchi, A. Silva, A. S. The power of convex algebras CONCUR'17

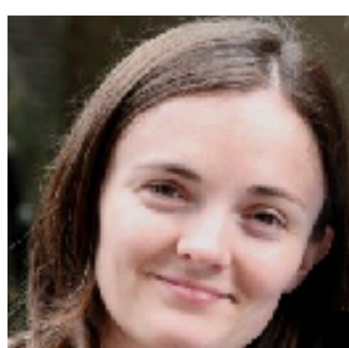
Joint work with

Erik de Vink **TU/e**

Bart Jacobs
Radboud University

Ichiro Hasuo
The University of Tokyo

Harald Woracek TU WIEN



Alexandra Silva UCL

Filippo Bonchi ENS DE LYON

Modelling discrete probabilistic systems

Probability distribution functor on $\mathbf{Sets}$

$$\mathcal{D}X = \{\mu \colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

for $f \colon X \to Y$ we have $\mathcal{D}f \colon \mathcal{D}X \to \mathcal{D}Y$ by

$$\mathcal{D}f(\mu)(y) = \sum_{x \in f^{-1}(y)} \mu(x) = \mu(f^{-1}(y))$$

Modelling discrete probabilistic systems

Probability distribution functor on $\mathbf{Sets}$

$$\mathcal{D}X = \{\mu\colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

and its variants

$$\mathcal{D}_{\leqslant 1}X = \{\mu\colon X \to [0,1] \mid \sum_{x \in X} \mu(x) \leqslant 1\}$$

$$\mathcal{D}_f X = \{\mu\colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1, \mathrm{supp}(\mu) \text{ is finite}\}$$

Modelling discrete probabilistic systems

Almost all known probabilistic systems can be modelled as coalgebras on $\mathbf{Sets}$ for functors given by the following grammar:

$$F ::= - \mid A \mid \mathcal{D} \mid \mathcal{P} \mid F^A \mid F + F \mid F \circ F \mid F \times F$$

in all cases concrete and coalgebraic bisimilarity (and behavioural equivalence) coincide

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$$X \xrightarrow{c} FX$$

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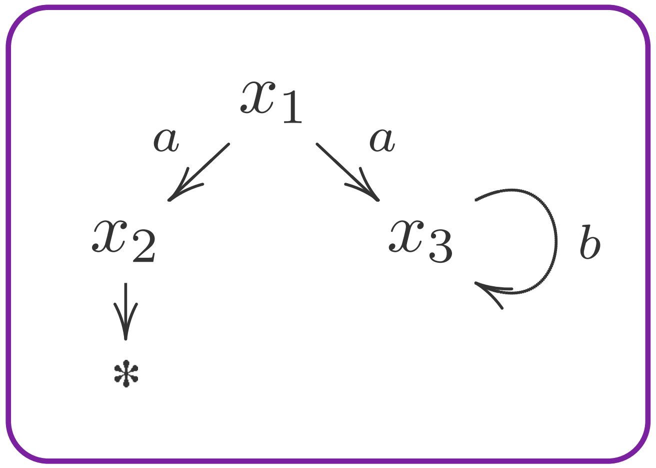
Examples

Examples

NFA

$2 \times (\mathcal{P}(-))^A \cong \mathcal{P}(1 + A \times (-))$



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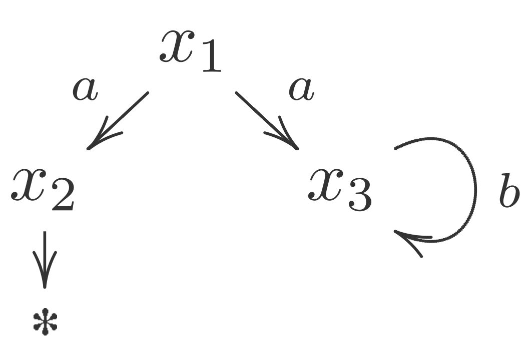
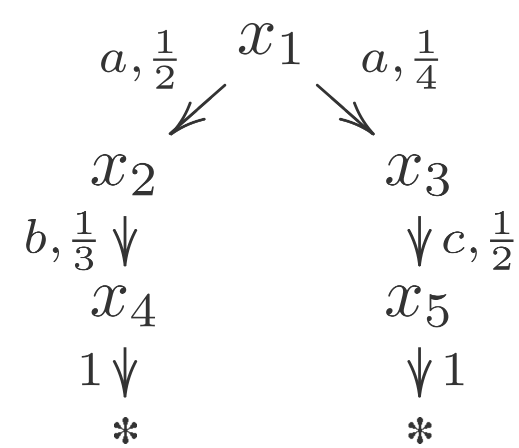


Generative PTS

$\mathcal{D}(1 + A \times (-))$



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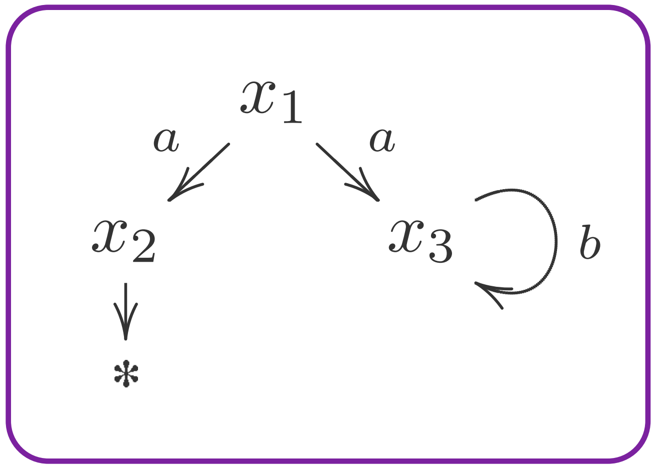

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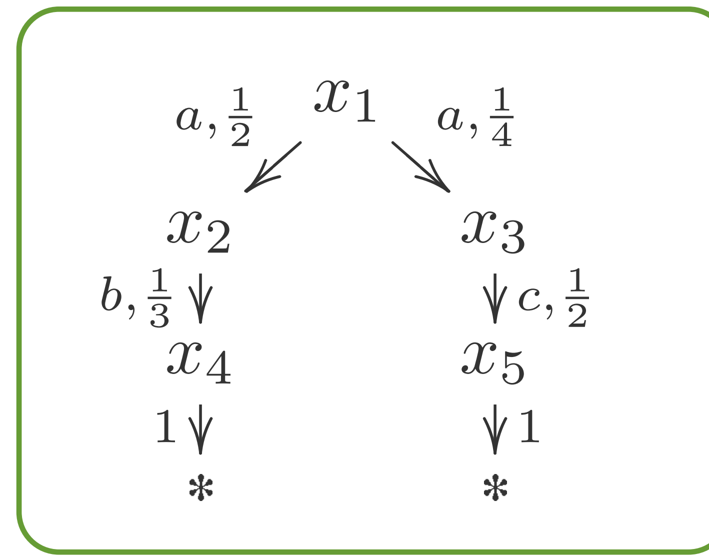
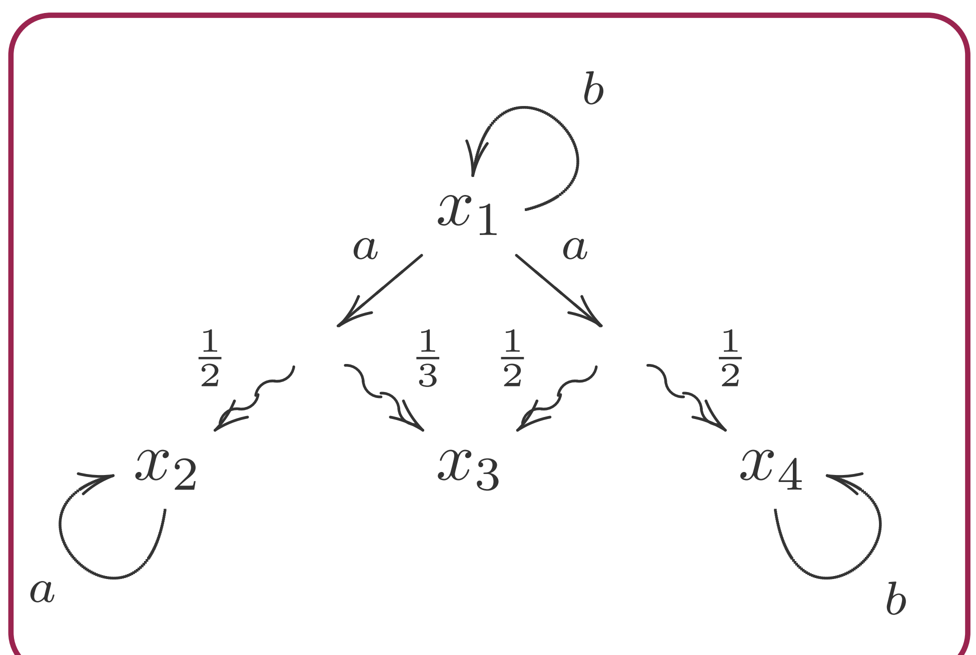


Simple PA

$\mathcal{P}(A \times \mathcal{D}(-))$



Examples

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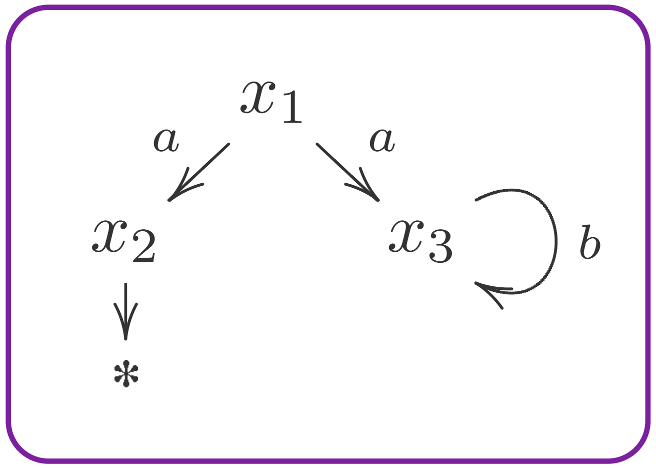

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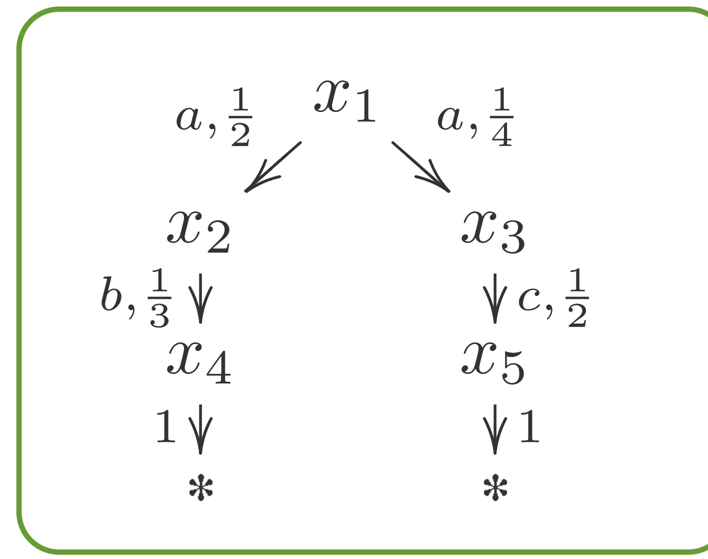

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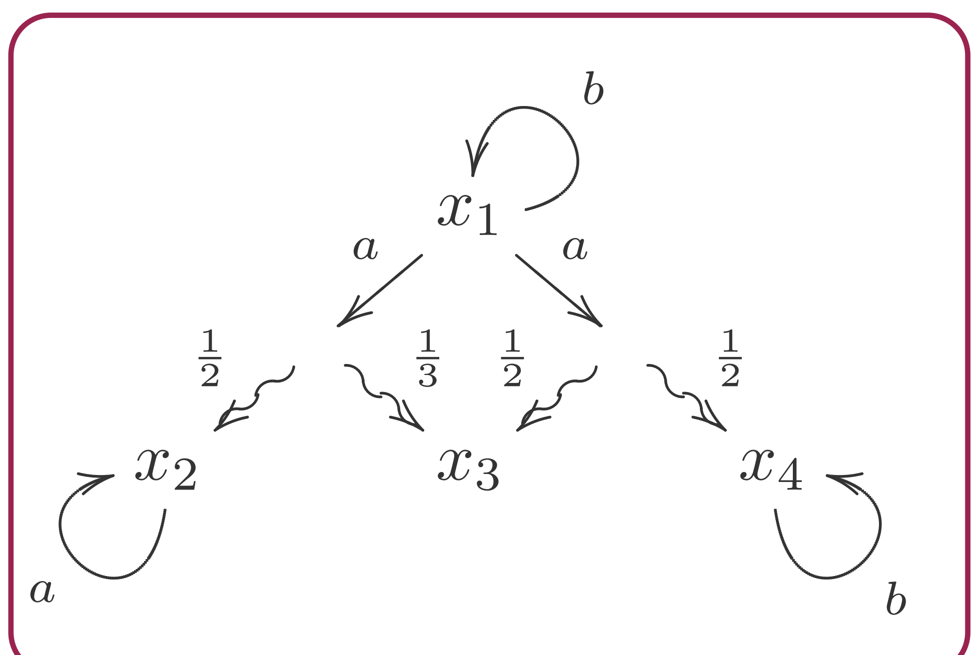
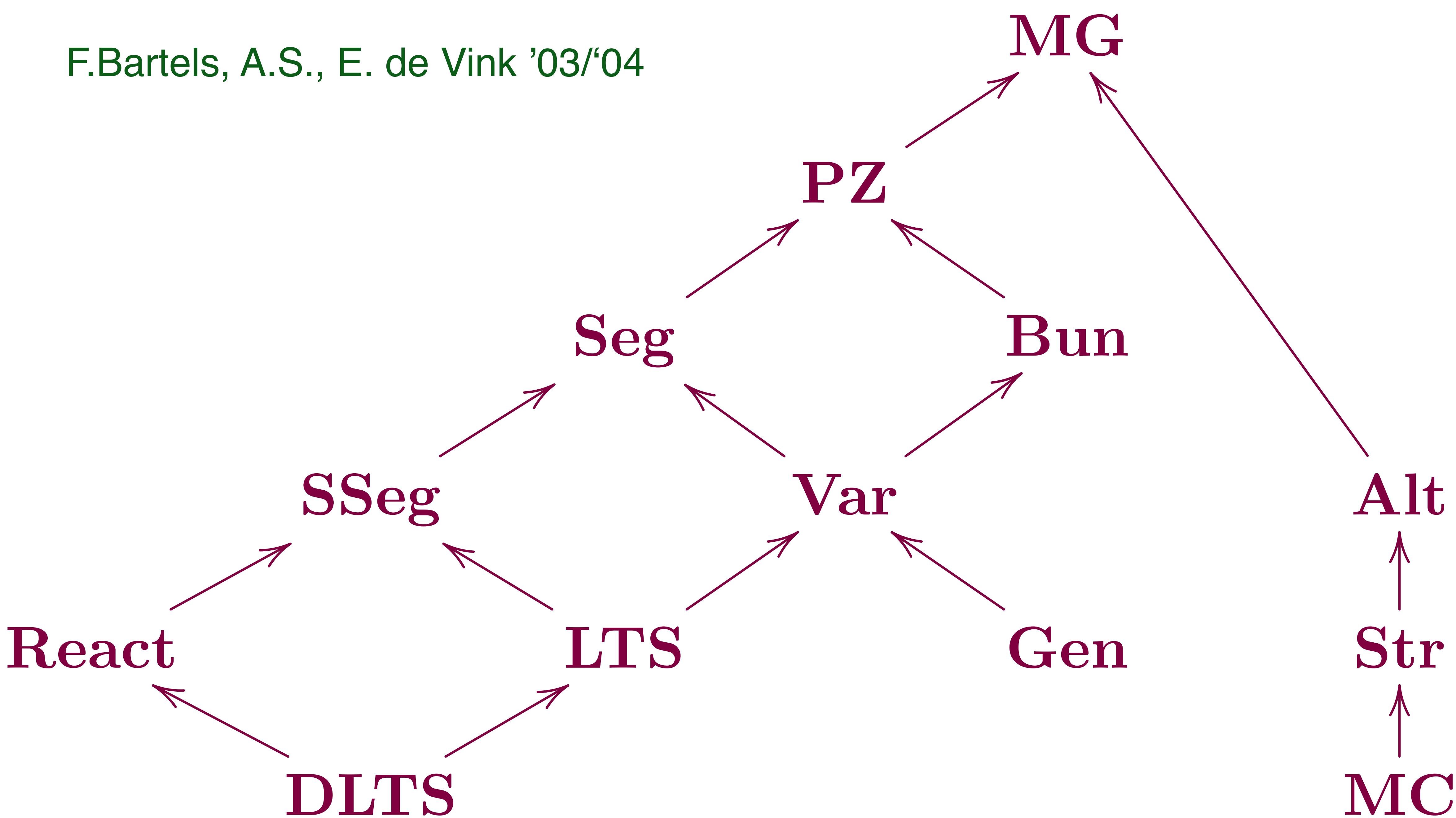


Here $\mathcal{D}$ for $\mathcal{D}_{\leq 1}$

Expressiveness hierarchy

F.Bartels, A.S., E. de Vink ’03/‘04



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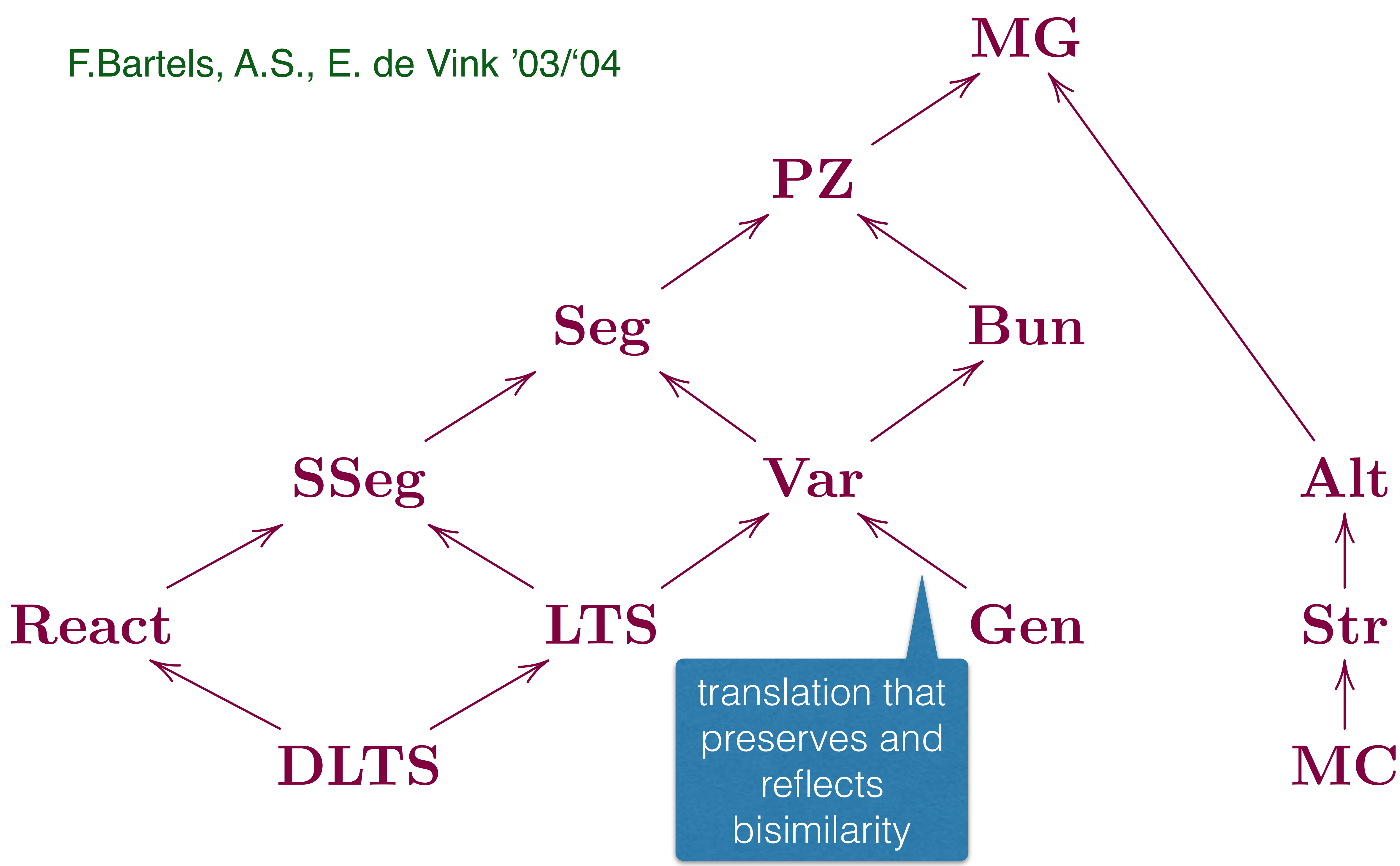

Traces ?

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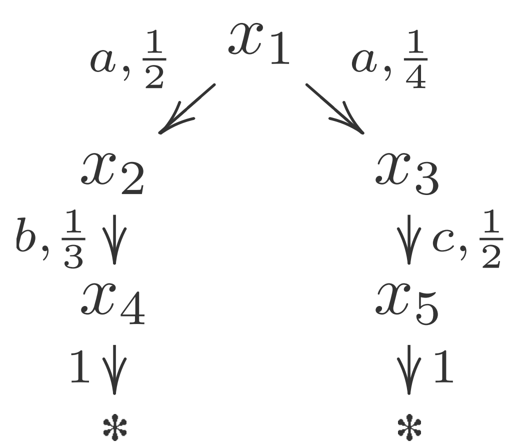

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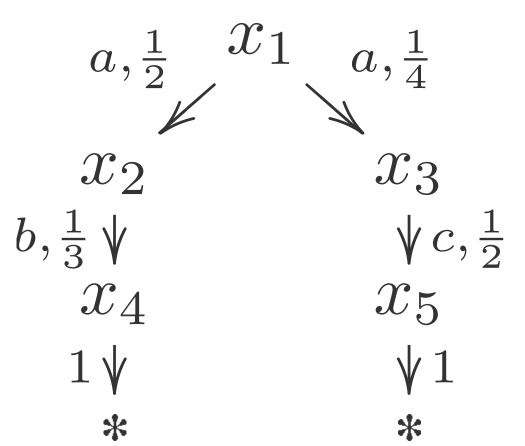

$$\mathrm{tr}(x_1)(ab) = \frac{1}{6} \quad \mathrm{tr}(x_1)(ac) = \frac{1}{8}$$

Traces ?

Generative PTS

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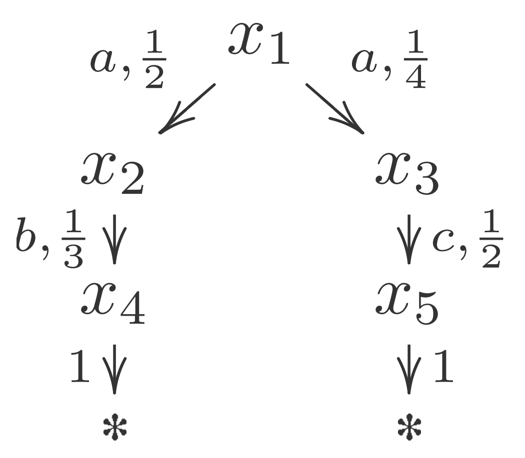

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Traces ?

Generative PTS

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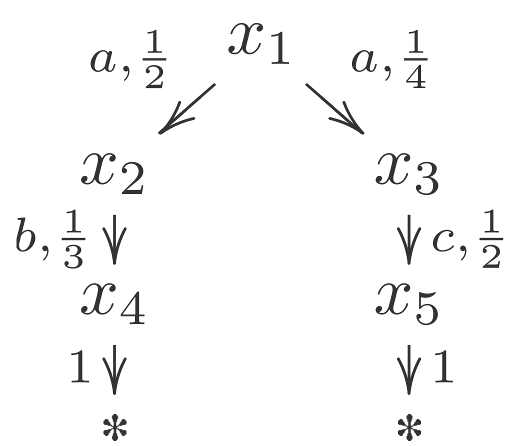

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$\mathcal{D}$ is a monad

Traces ?

Generative PTS

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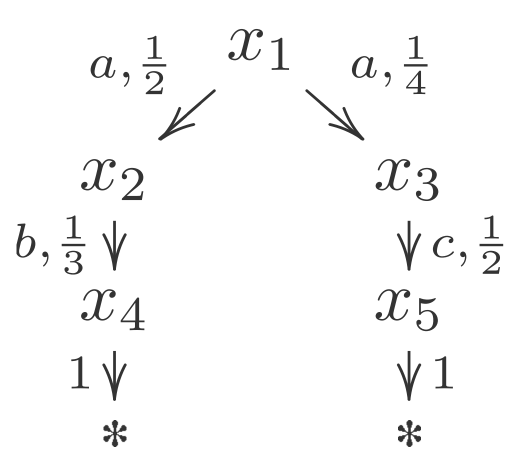

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arrow in $\mathcal{K}\ell(\mathcal{D})$

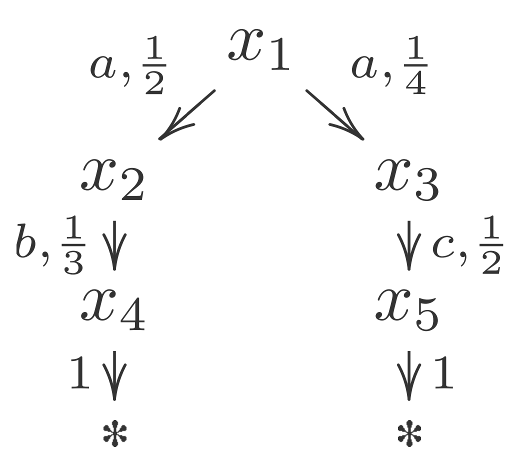
$\mathcal{D}$ is a monad

Traces ?

Generative PTS

$\mathcal{D}(1 + A \times (-))$

lifts to $\mathcal{K}\ell(\mathcal{D})$ via a distributive law



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Traces ?

Generative PTS

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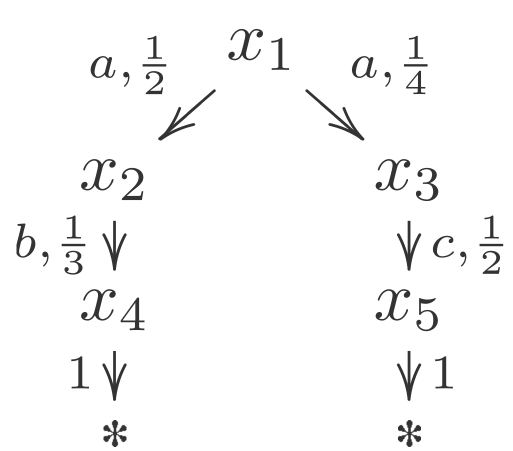

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$\mathcal{D}$ is a monad

arrow in $\mathcal{K}\ell(\mathcal{D})$

$$X \to \mathcal{D}(1 + A \times X) \to \mathcal{D}(1 + A \times \mathcal{D}(1 + A \times X)) \to \mathcal{D}^2(1 + A \times (1 + A \times X)) \to \mathcal{D}(1 + A \times X + A^2 \times X) \cdots$$

Traces via determinisation

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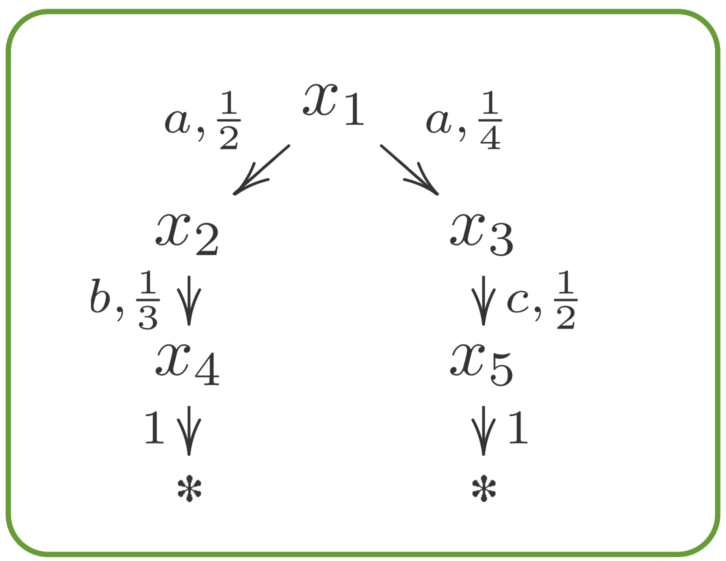
Generative PTS

$\mathcal{D}(1 + A \times (-))$



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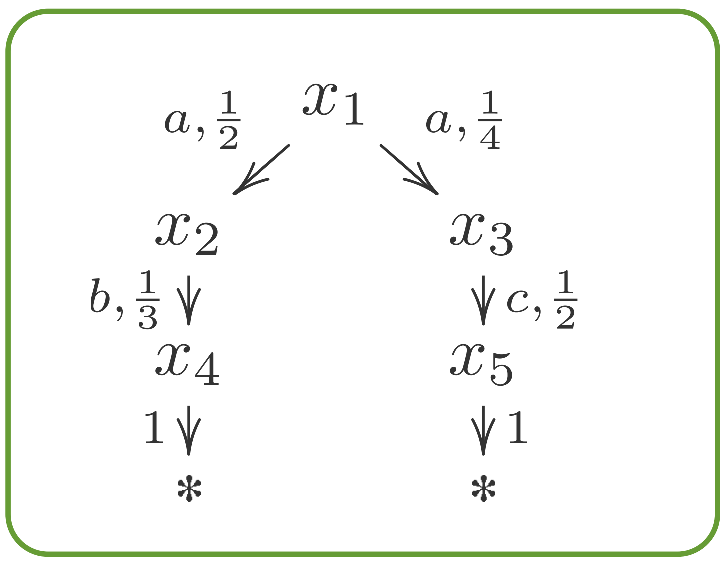

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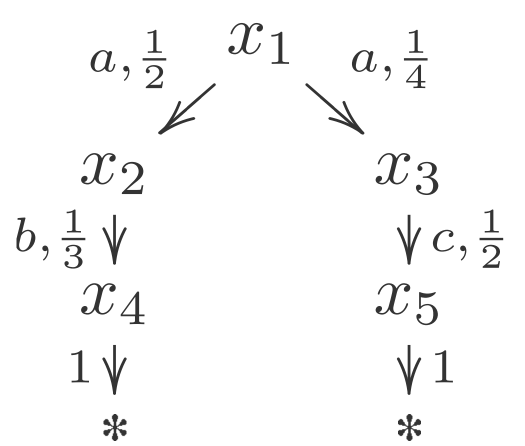

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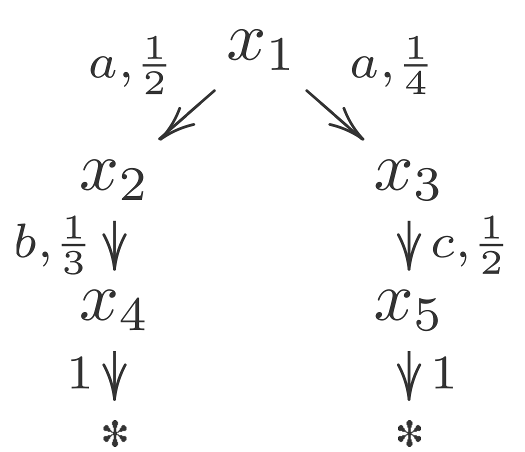

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trace = bisimilarity after determinisation

Traces via determinisation

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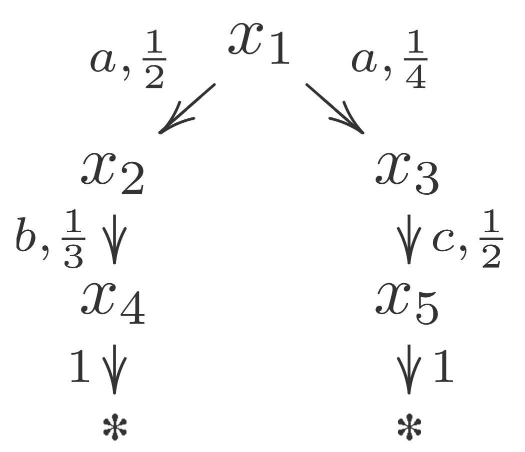
Generative PTS

$\mathcal{D}\,(1 + A \times (-))$



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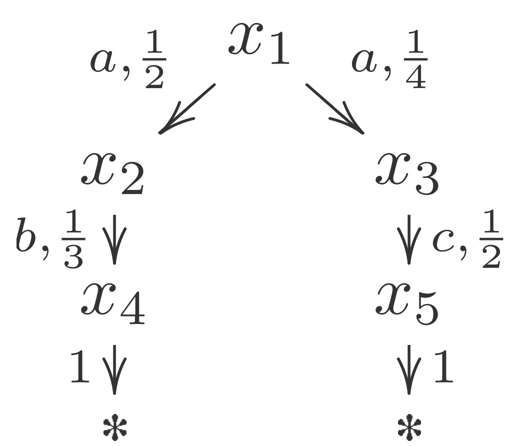

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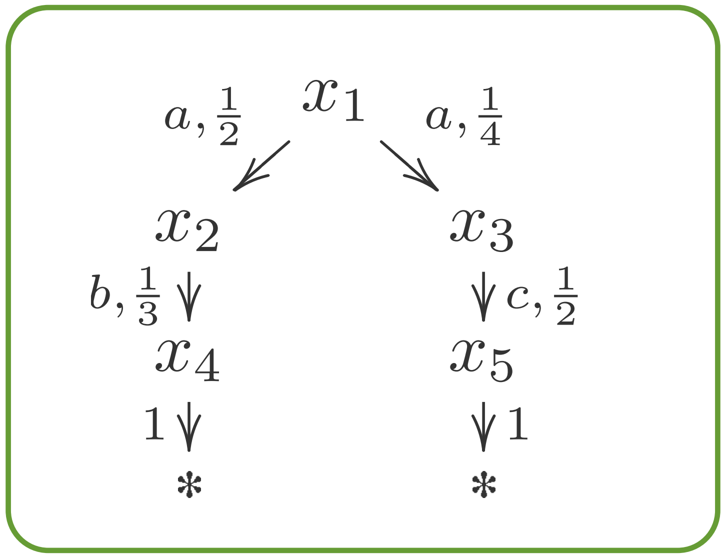
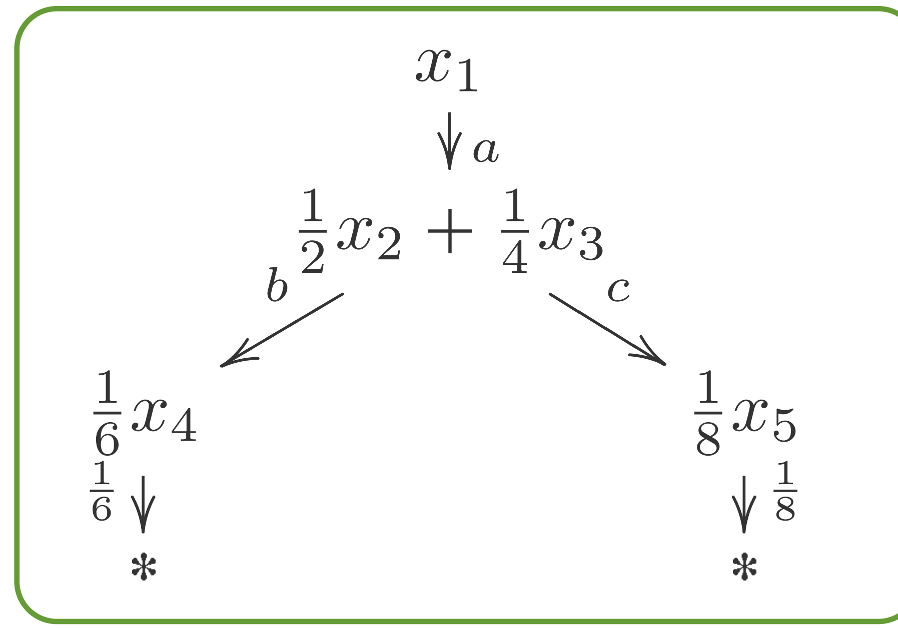


DFA

$[0,1] \times (-)^A$ states $\mathcal{D}(-)$



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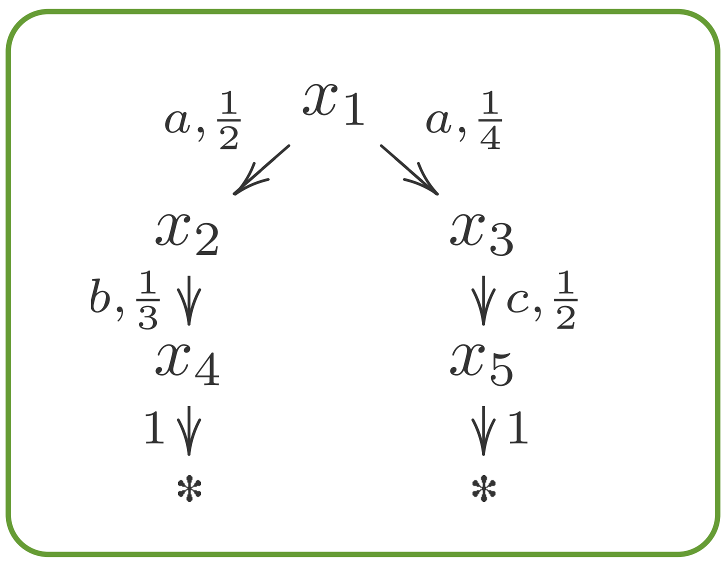

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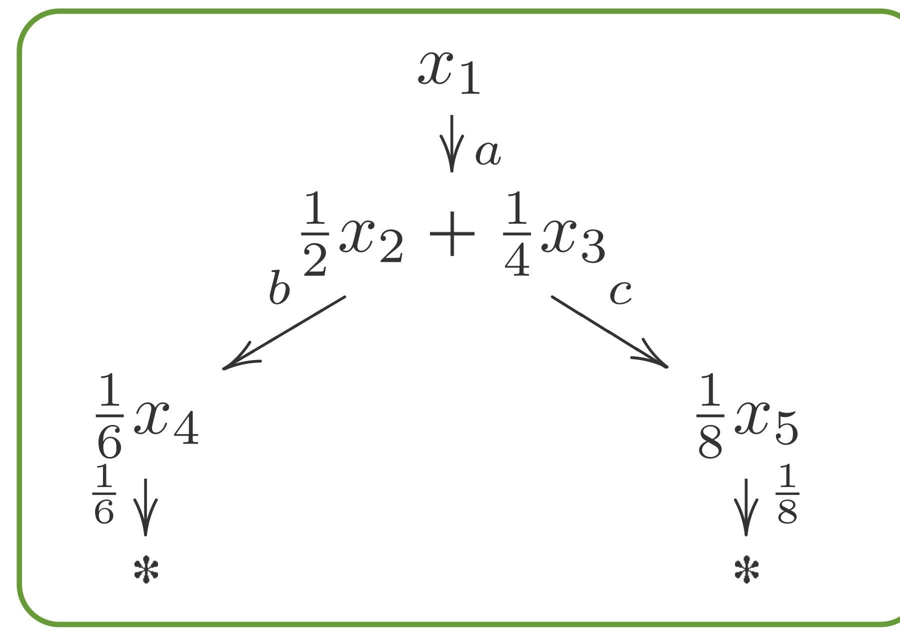

trace = bisimilarity after determinisation

Traces via determinisation

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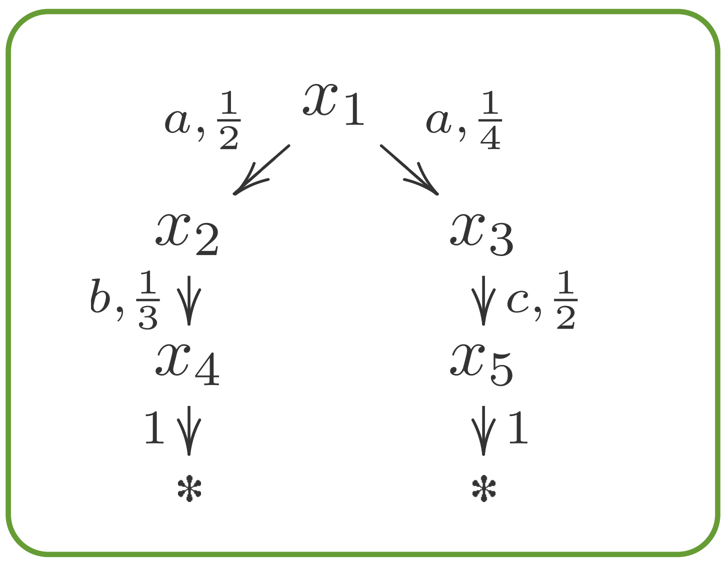

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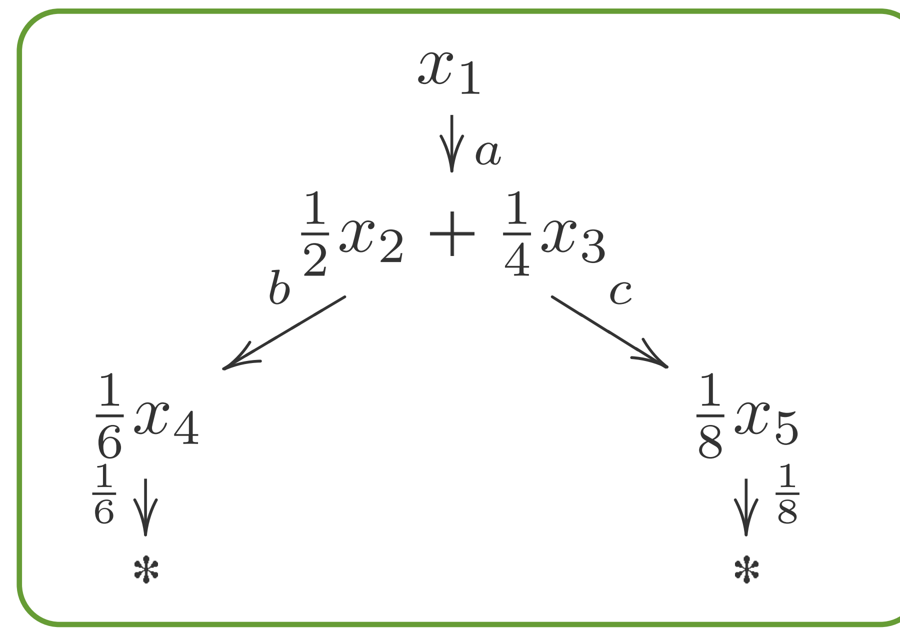

trace = bisimilarity after determinisation

Happens in $\mathcal{EM}(\mathcal{D})$

Traces via determinisation

Generative PTS

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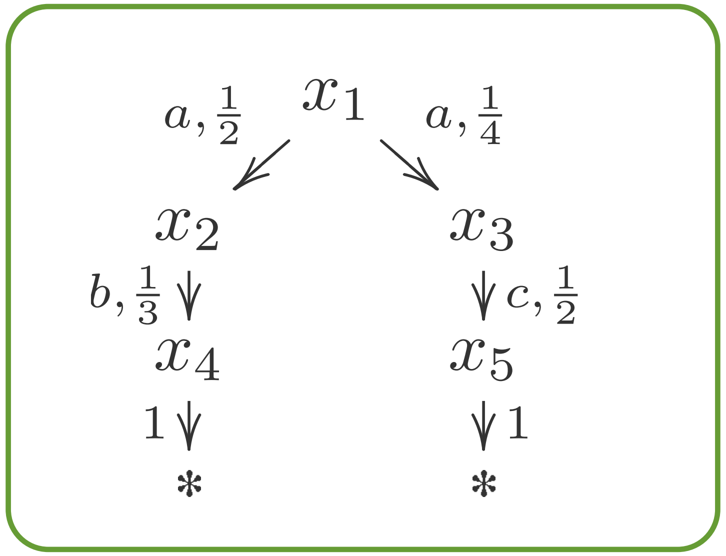

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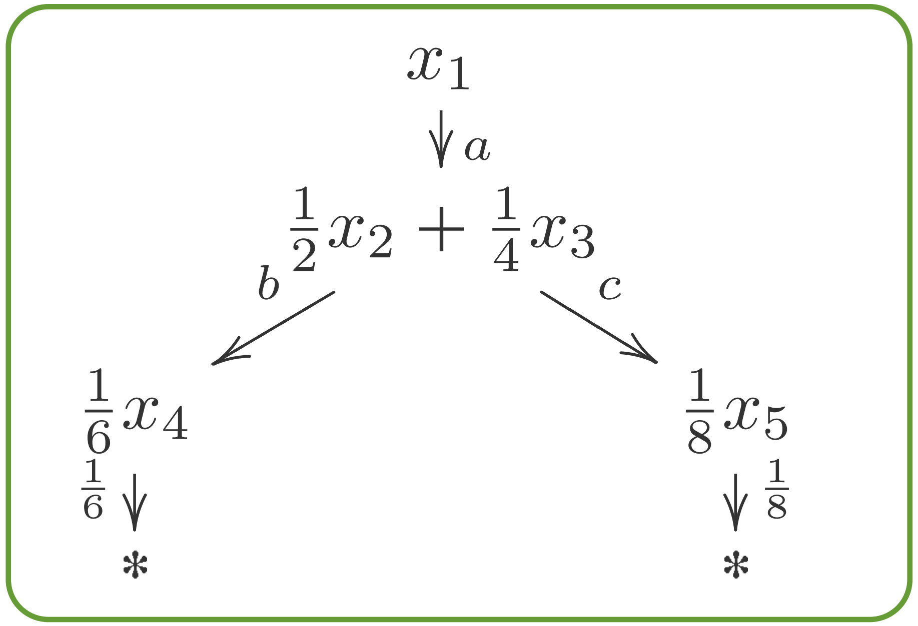


trace = bisimilarity after determinisation

Happens in $\mathcal{EM}(\mathcal{D})$

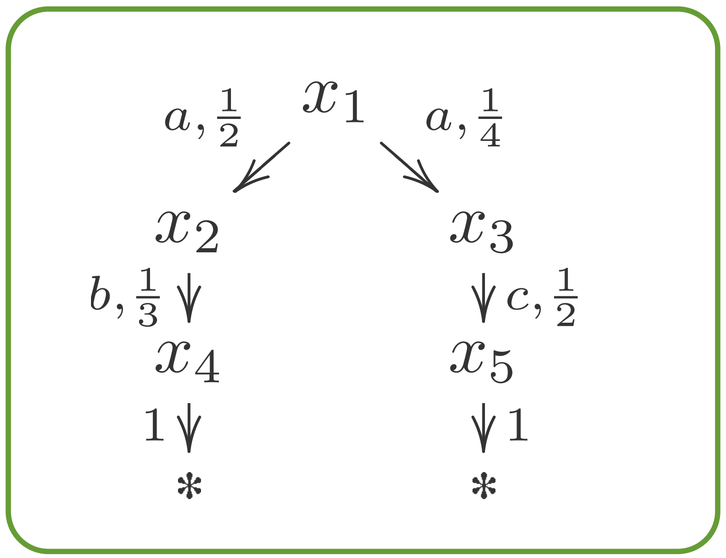
(positive) convex algebras

Traces via determinisation

we recently generalised this to PA too

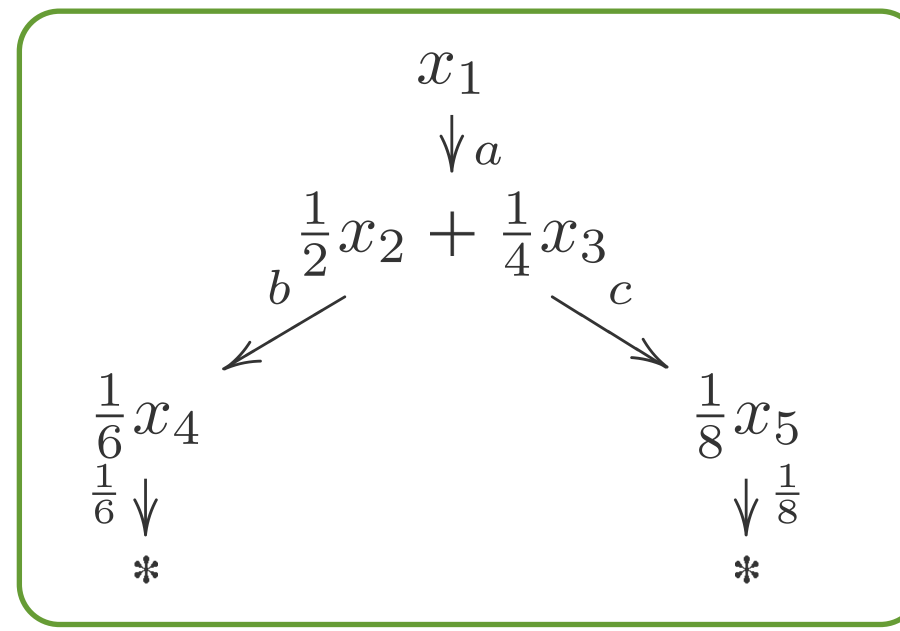
Generative PTS

$\mathcal{D}(1 + A \times (-))$



DFA

$[0,1] \times (-)^A$ states $\mathcal{D}(-)$



trace = bisimilarity after determinisation

Happens in $\mathcal{EM}(\mathcal{D})$

(positive) convex algebras

Trace axioms for generative PTS

Axioms for bisimilarity

+

$$p \cdot a \cdot (p_1 \mathsf{E}_1 \oplus p_2 \mathsf{E}_2) \quad \equiv \quad p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \qquad (D)$$

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soundness and completeness !?

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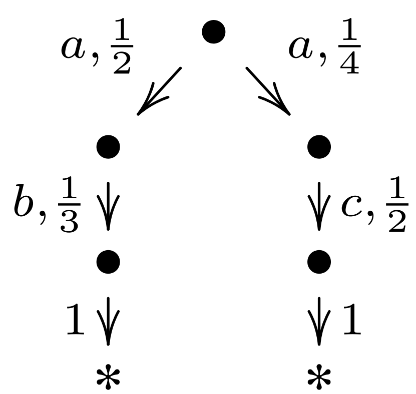

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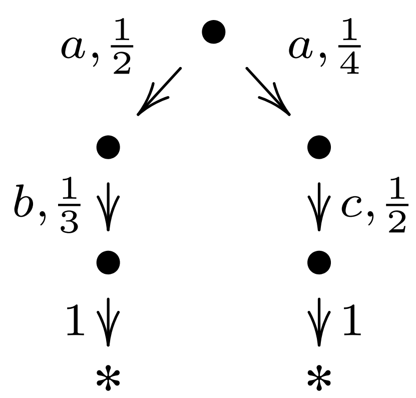

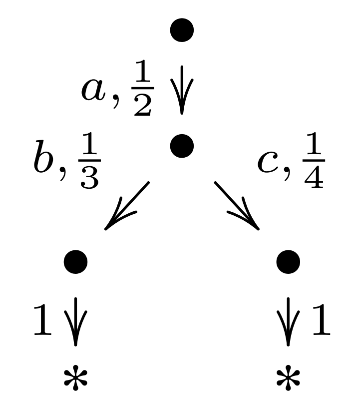

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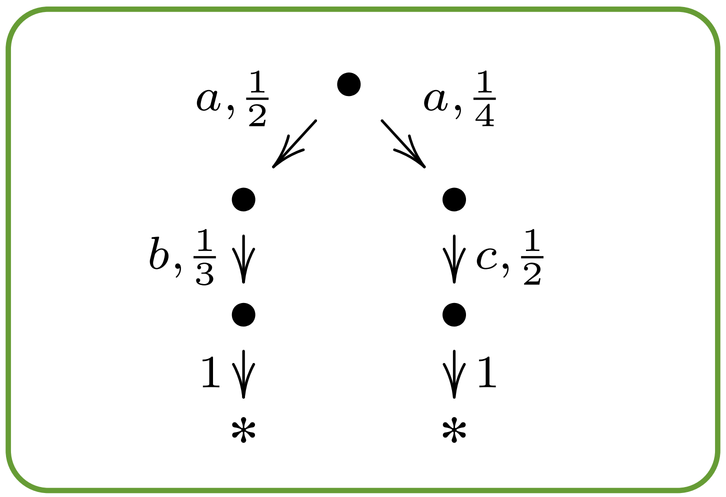

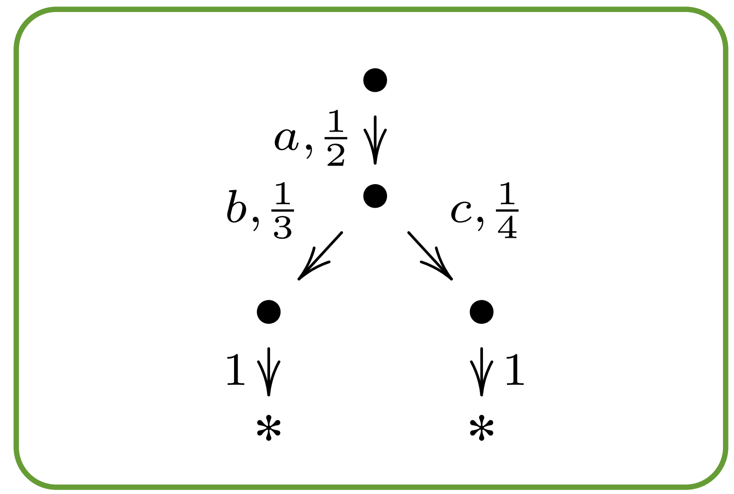

$$\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) \overset{(Cong)}{\equiv} \left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{2}\cdot a\cdot\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

$$\overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

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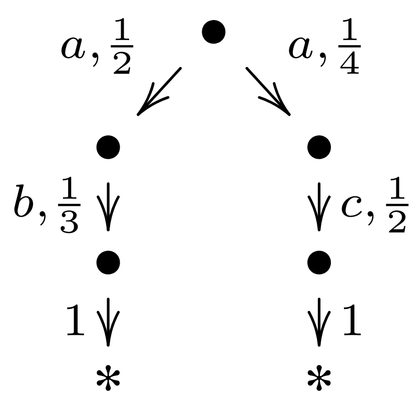

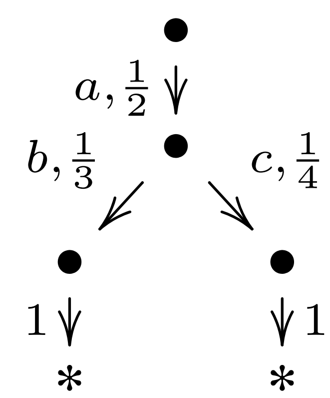

$$\frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot * \overset{(D)}{\equiv} \frac{1}{2} \cdot a \cdot \frac{1}{4} \cdot c \cdot 1 \cdot *$$

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The quest for completeness

Inspired lots of new research:

- A. S., H. Woracek Congruences of convex algebras JPAA'15

- S. Milius Proper functors CALCO'17

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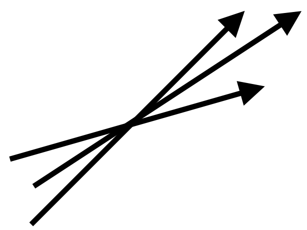
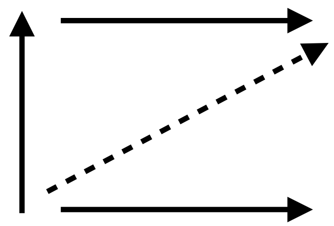
our axiomatisation would be proven complete if only one particular functor $\hat{G}$ on $\mathcal{EM}(\mathcal{D})$ were proper

if f.p. = f.g. and ★ then completeness

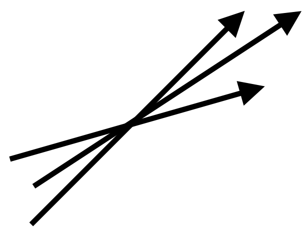
does not hold

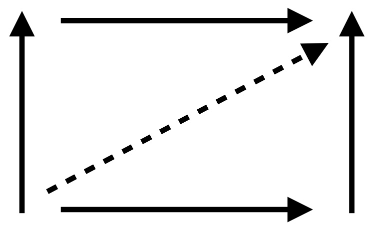




Part II

Proper convex functors





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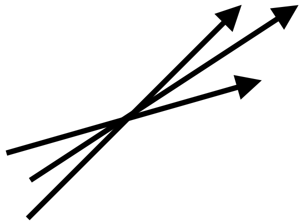
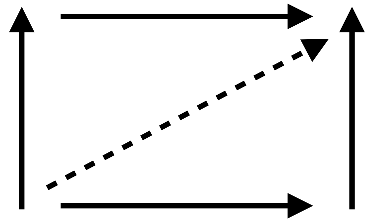
Part II

Proper convex functors

the trace axioms can be proven complete!





Part II

Proper convex functors

the trace axioms can be proven complete!

very new nontrivial results !
joint work with

Harald Woracek

Proper functors

$\mathcal{EM}(\mathcal{T})$

beh.equivalence

A functor F on an algebraic category is proper, if

- for any two F-coalgebras with free f.g. carriers TX ⟶FTX and TY ⟶ FTY
- for any two points x in TX, y in TY with η(x) ~ η(y)

there is a zigzag of F-coalgebras with free f.g. carriers that relates x and y

extends the notion of a proper semiring of Ésik and Maletti

a semiring S is proper iff S x $(-)^A$ is proper

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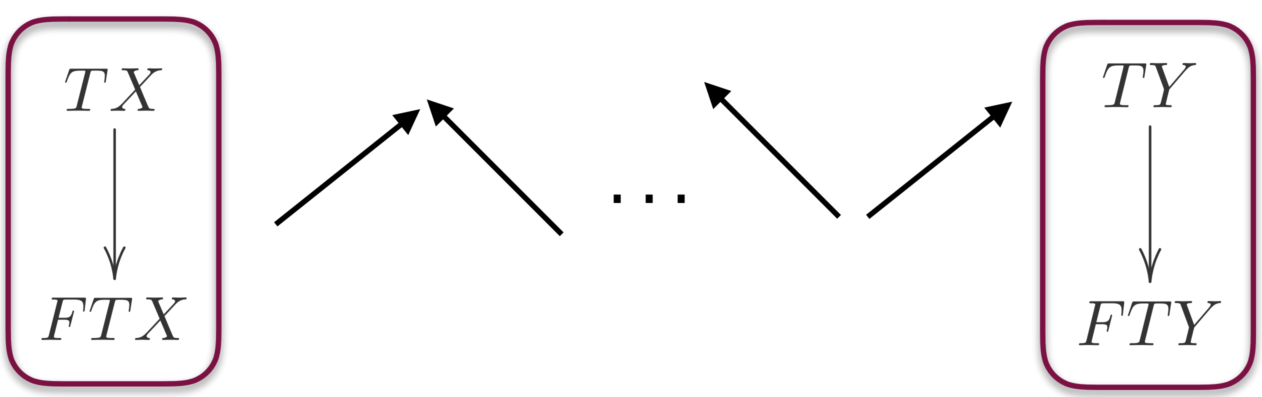

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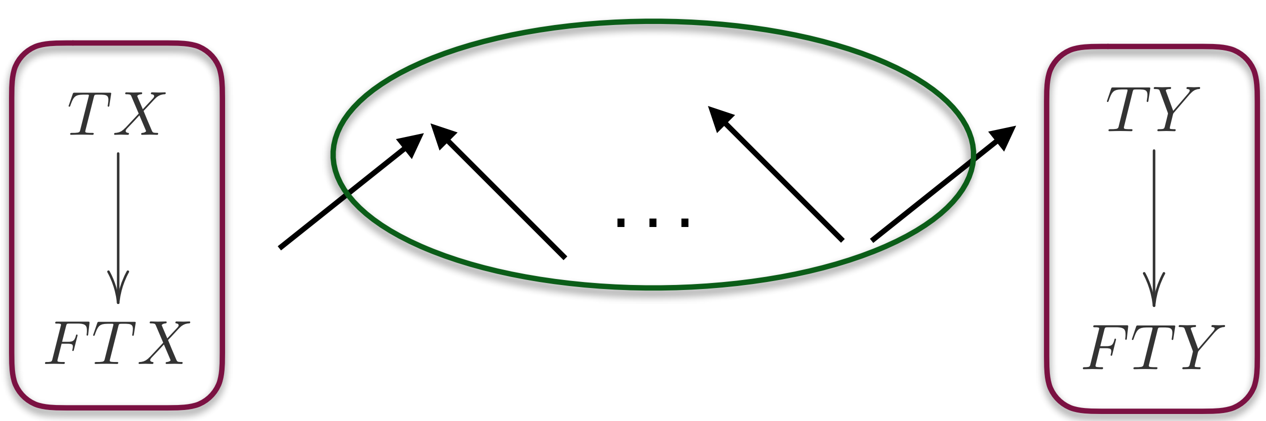

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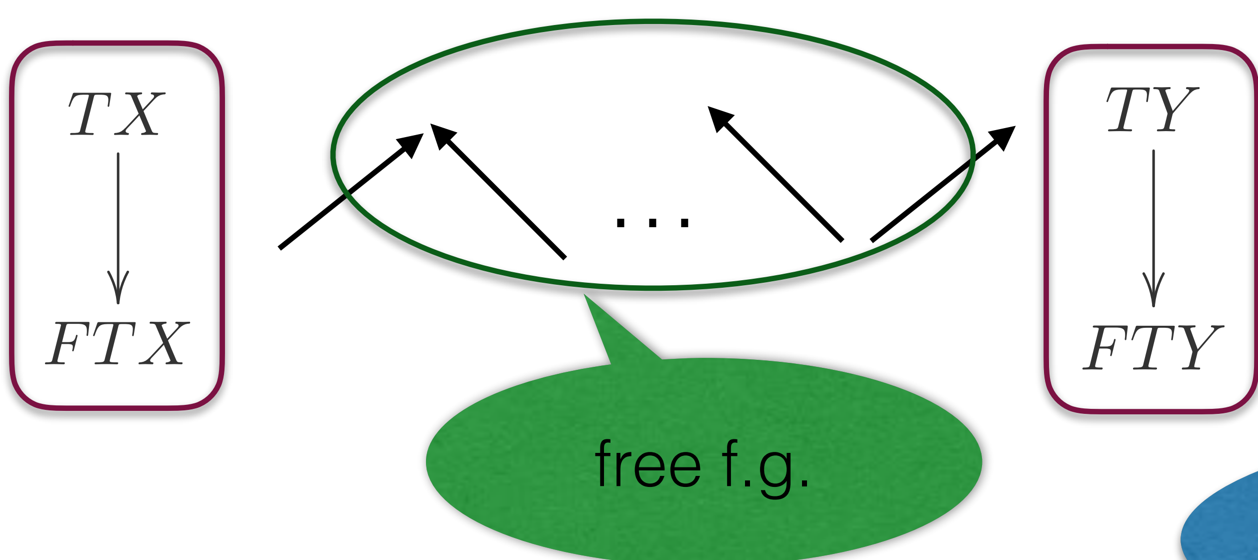

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Proper functors

Known

- any Noetherian semiring is proper, hence $\mathbb{Z}$, $\mathbb{R}$ are proper
- $\mathbb{N}$ is proper

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- If Ĝ on PCA is proper

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we have just proven all these proper

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Our results prove

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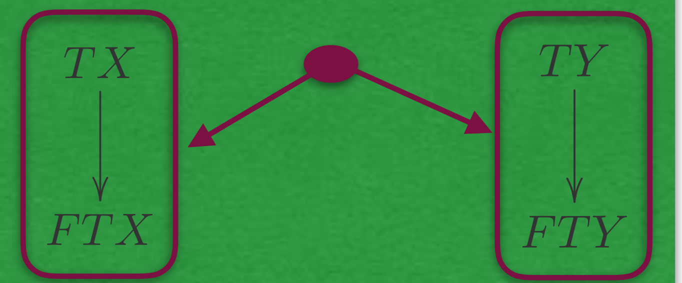
in all cases one span suffices



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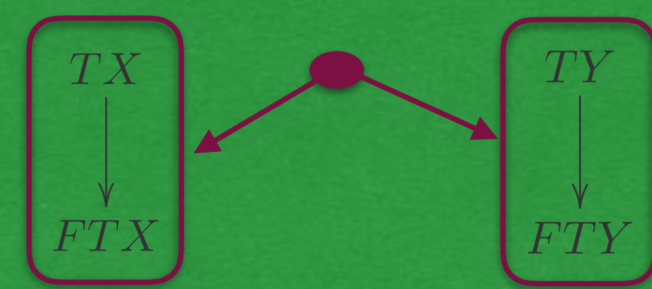

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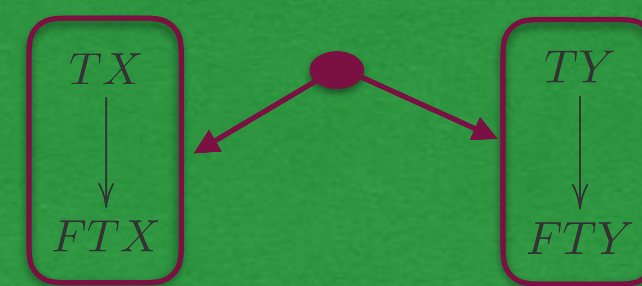

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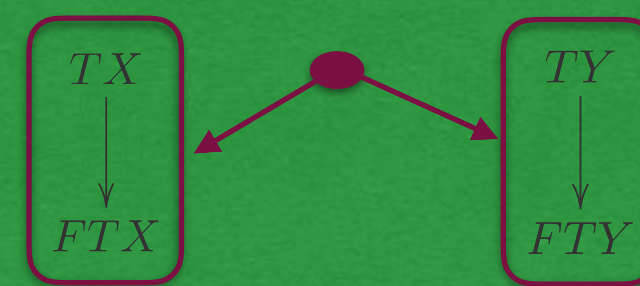

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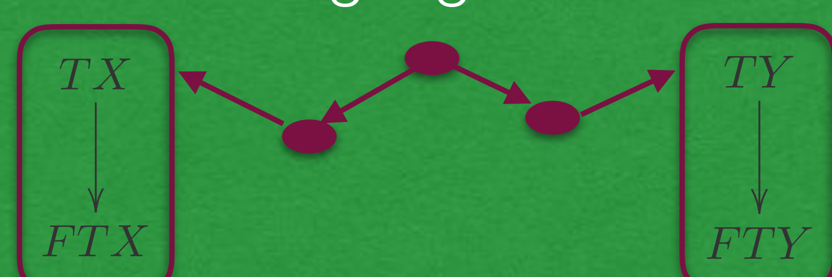

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Thank You !