Least and Alternating Fixed-Point Specifications in Coalgebraic System Modeling

Ichiro Hasuo

National Institute of Informatics
Tokyo, Japan

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Collaborators

- Hasuo, Shimizu & Cirstea,
  Lattice-theoretic progress measures and coalgebraic model checking.
  POPL 2016
- Urabe, Shimizu & Hasuo,
  Coalgebraic Trace Semantics for Büchi and Parity Automata.
  CONCUR 2016
- Urabe, Hara & Hasuo,
  Categorical Liveness Checking by Corecursive Algebras.
  LICS 2017
- Urabe & Hasuo,
  Fair Simulation for Nondeterministic and Probabilistic Büchi Automata: a Coalgebraic Perspective.
  LMCS 2017, to appear

- Many slides today are by Natsuki

Natsuki Urabe

Shunsuke Shimizu

Masaki Hara

Corina Cirstea

Fixed Points in Comp. Sci.

* ... we all love them :-)

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  * lfp = liveness $\mathbf{F}p = \mu u.\ p \vee \mathbf{X}u$
    (reachability, termination, ...)
  * gfp = safety $\mathbf{G}p = \nu u.\ p \wedge \mathbf{X}u$
  * in-between: recurrence $\mathbf{GF}p$, persistence $\mathbf{FG}p$, ...
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  * initial algebra = algebraic data type
  * final coalgebra
    = semantic domain of state-based dynamics

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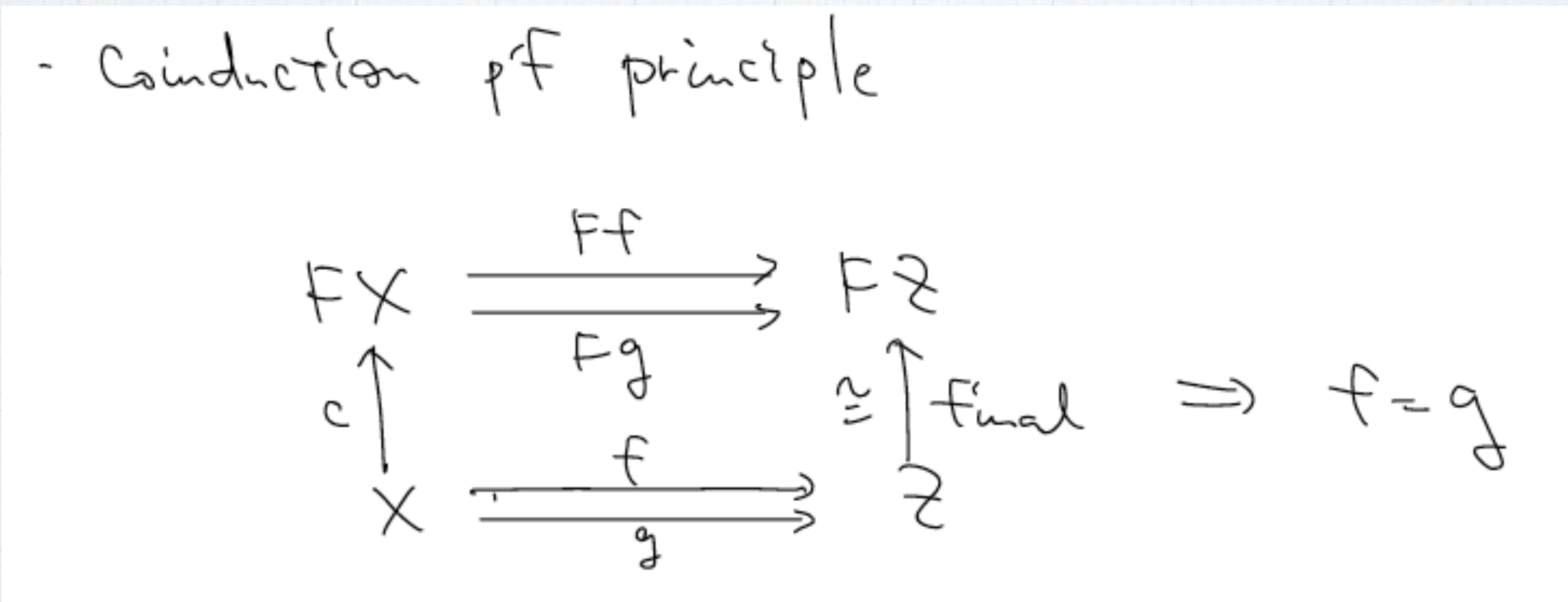
Reasoning Principles in the Theory of Coalgebras



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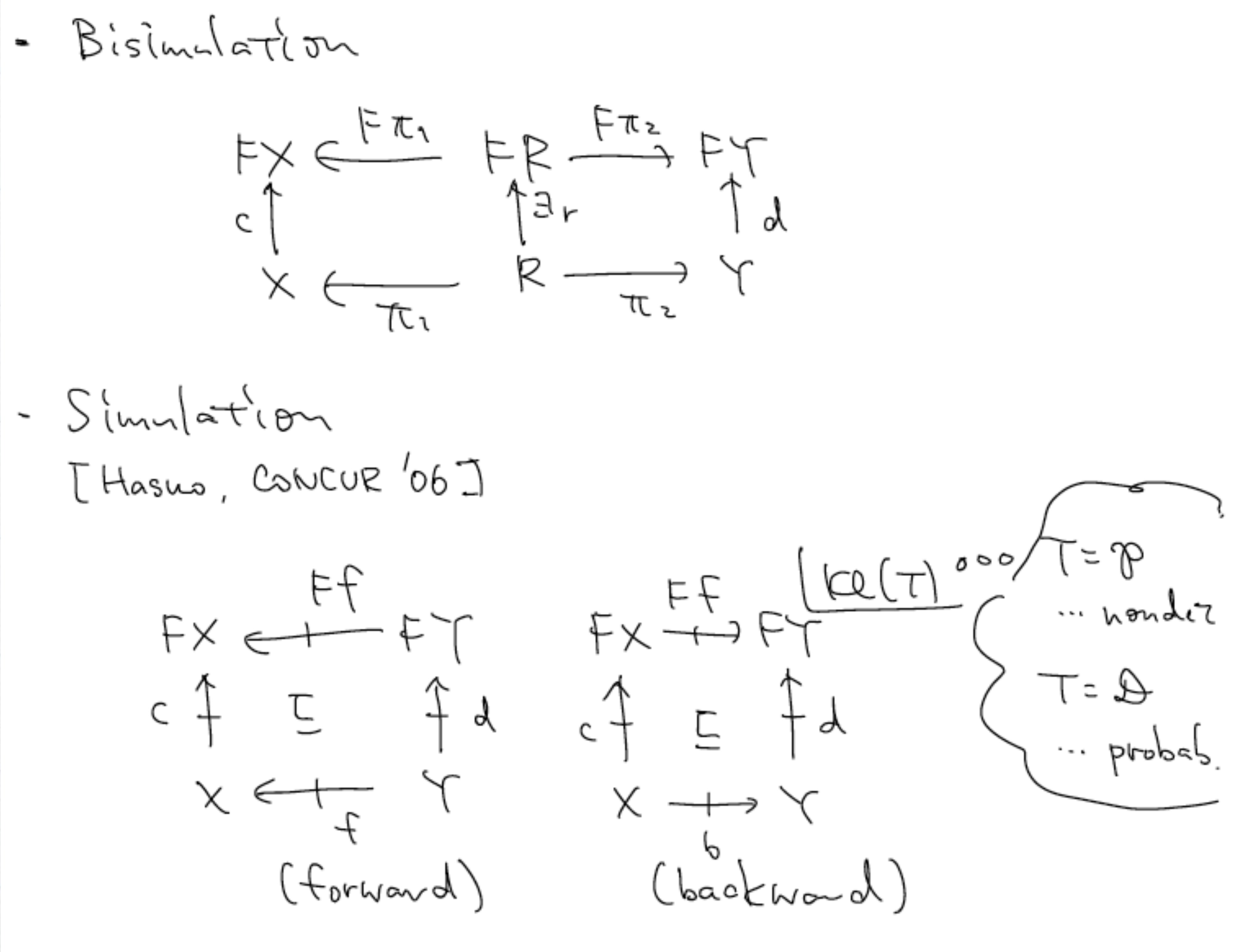


Reasoning Principles in the Theory of Coalgebras

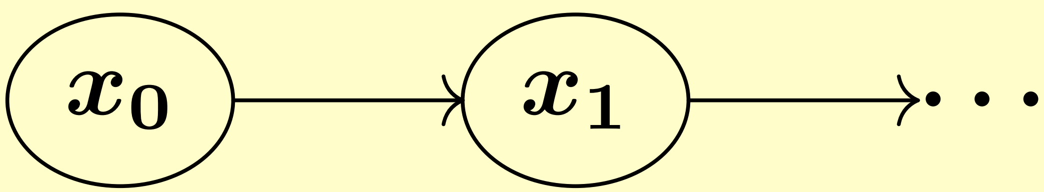
* Bisimulation, behavioral equivalence, simulation, ...

* These all come with greatest fixed point flavors... why?

Outline

* Least, greatest and alternating fixed points: a foundational view [POPL'16]
* Buechi and parity acceptance conditions in coalgebras [CONCUR'16]
* Categorical ranking functions by corecursive algebras [LICS'17]

Invariant vs. Ranking Function



A linear Kripke structure:
$\mathsf{succ}\colon X \to X, \quad [\![p]\!] \subseteq X$

* $\mathsf{G}p$ (everywhere p) is a gfp $\quad \nu u.\, p \wedge \mathsf{X} u$
  * the greatest solution of $u = p \wedge \mathsf{X}\, u$

* $\mathsf{F}p$ (eventually p) is an lfp $\quad \mu u.\, p \vee \mathsf{X} u$
  * the least solution of $u = p \vee \mathsf{X}\, u$

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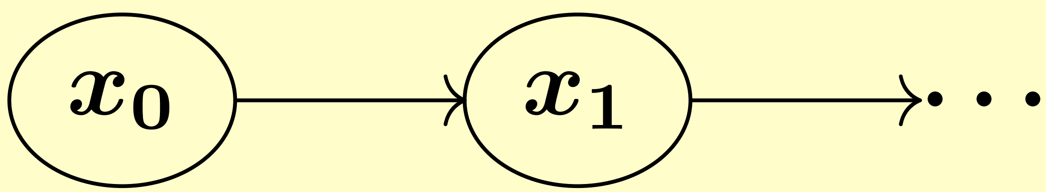

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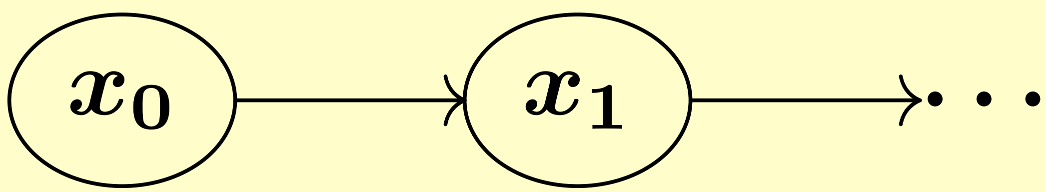

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Lem. (witnessing $\mathbf{G}\, p = \nu u.\, (p \wedge \mathbf{X}\, u)$)

$$\frac{I \subseteq [\![p]\!] \qquad x \in I \;\Rightarrow\; \mathsf{succ}(x) \in I}{I \subseteq [\![\mathbf{G}\, p]\!]}$$

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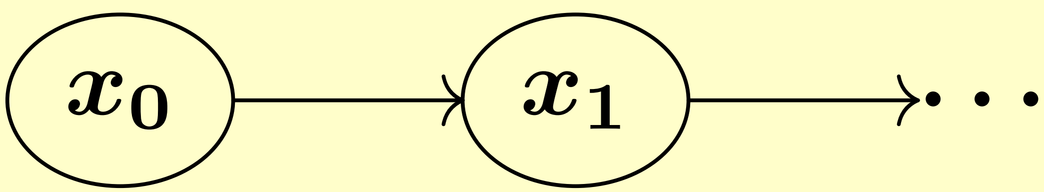

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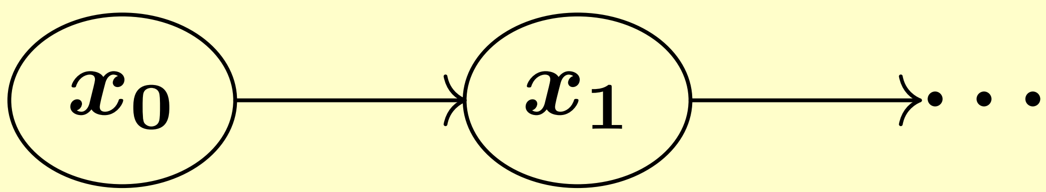

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$$\mathbf{rk}(x) = n \;\wedge\; x \notin [\![p]\!] \implies \mathbf{rk}(\mathsf{succ}(x)) \leq n-1 \;.$$

Then $\mathbf{rk}(x) \neq \spadesuit$ implies $x \in [\![\mathsf{F}\,p]\!]$.

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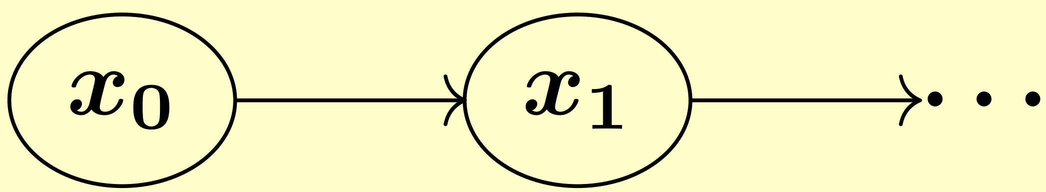

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* How come the difference?
  ➜ Let us take a **foundational** view...

Lattice-Theoretic Foundation

L: complete lattice, $f : L \to L$ monotone

Thm. (Knaster-Tarski)

- $\mu f = \min\{l \in L \mid f(l) \sqsubseteq l\}$
- $\nu f = \max\{l \in L \mid l \sqsubseteq f(l)\}$

Thm. (Cousot-Cousot)

$\bot \sqsubseteq f(\bot) \sqsubseteq \cdots \sqsubseteq f^{\omega}(\bot) \sqsubseteq \cdots$
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$\top \sqsupseteq f(\top) \sqsupseteq \cdots \sqsupseteq f^{\omega}(\top) \sqsupseteq \cdots$
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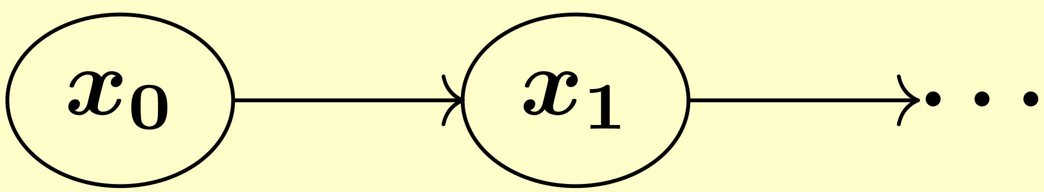
Sound approx. from below

Proof Methods for Unnested Fixed Points



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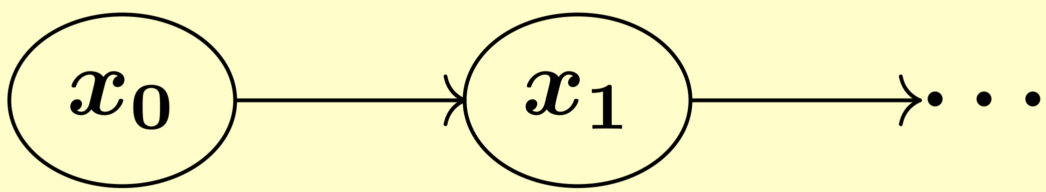
Lem. (invariants witness gfp's)
Let $f\colon 2^X \to 2^X$ be monotone, and $I \in 2^X$.

$$\frac{I \subseteq f(I)}{I \subseteq \nu u.\, f(u)}$$

Proof Methods for Unnested Fixed Points



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$U_n = \{x \mid \mathbf{rk}(x) \leq n\}$

Lem. (witnessing lfp's)
Let $f\colon 2^X \to 2^X$ be monotone.
If $U_0 \subseteq U_1 \subseteq \cdots \subseteq X$ satisfies

$$U_0 = \emptyset \quad \text{and} \quad U_{n+1} \subseteq f(U_n) \ ,$$

then $U_n \subseteq \mu u.\, f(u)$ for each n.

The Table

properties	witnessed by...
safety, gfp	invariants
liveness, lfp	ranking functions

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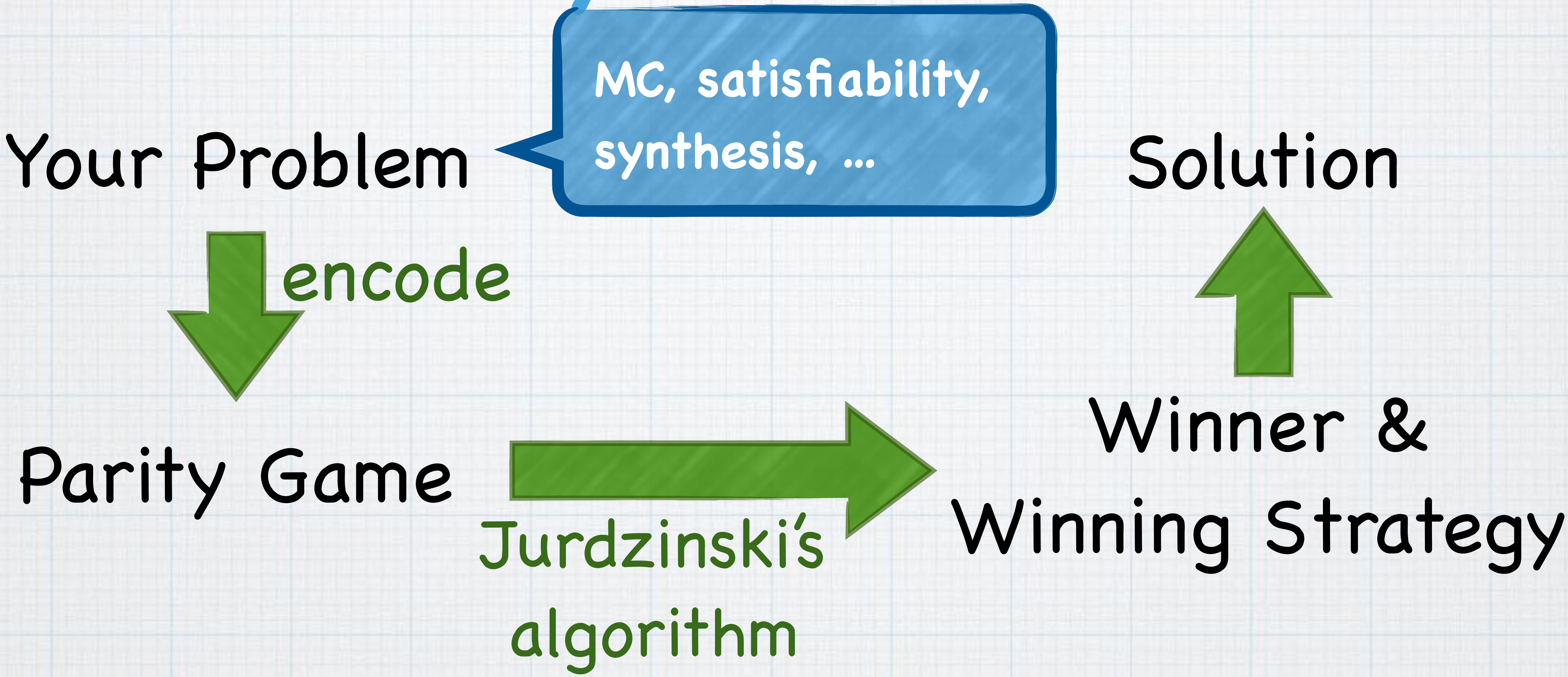
nested, alternating gfp's & lfp's	**winning strategies for a parity game** (if finitary)

The Parity-Game Workflow



The Parity-Game Workflow

MC, satisfiability, synthesis, ...

Your Problem

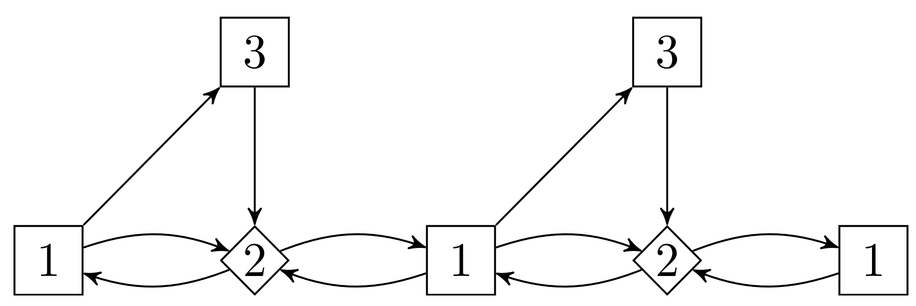
encode

Parity Game

Jurdzinski's algorithm

Winner & Winning Strategy

Solution

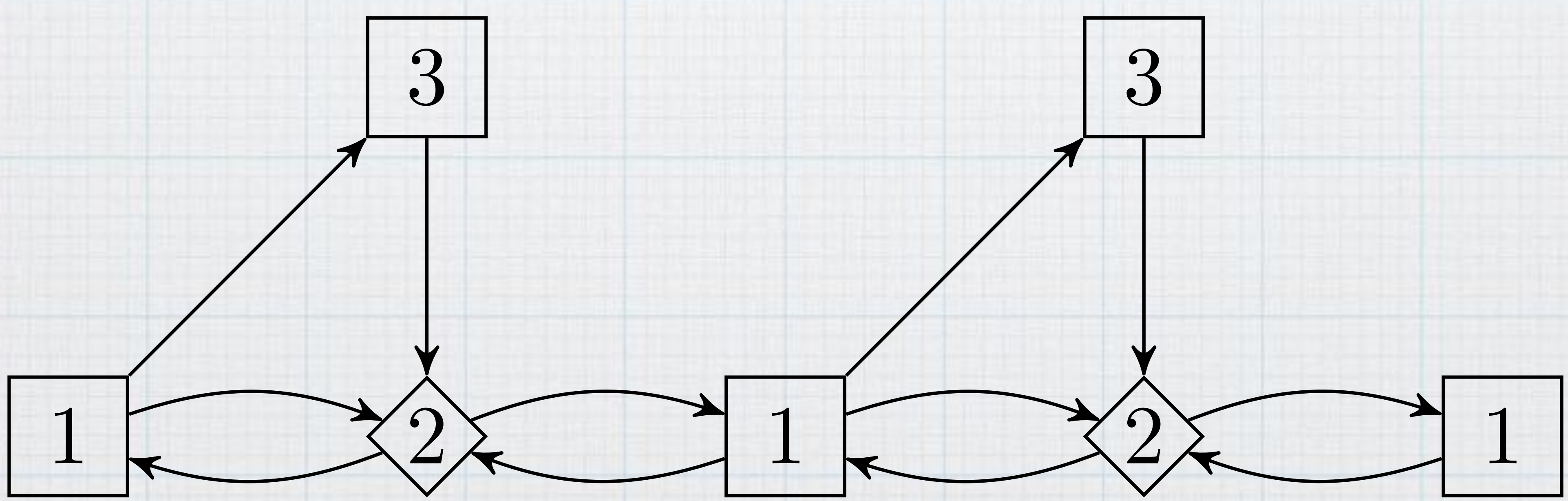


- In parity games:
  - alt. branching ($\forall$ vs $\exists$, $\wedge$ vs $\vee$)
  - parity acceptance cond.
    → alt. betw. μ and ν

Jurdzinski's Progress Measure for Parity Games: Intuitions

◇: your position
□: opponent's
goal: "visit bigger even"

You Win



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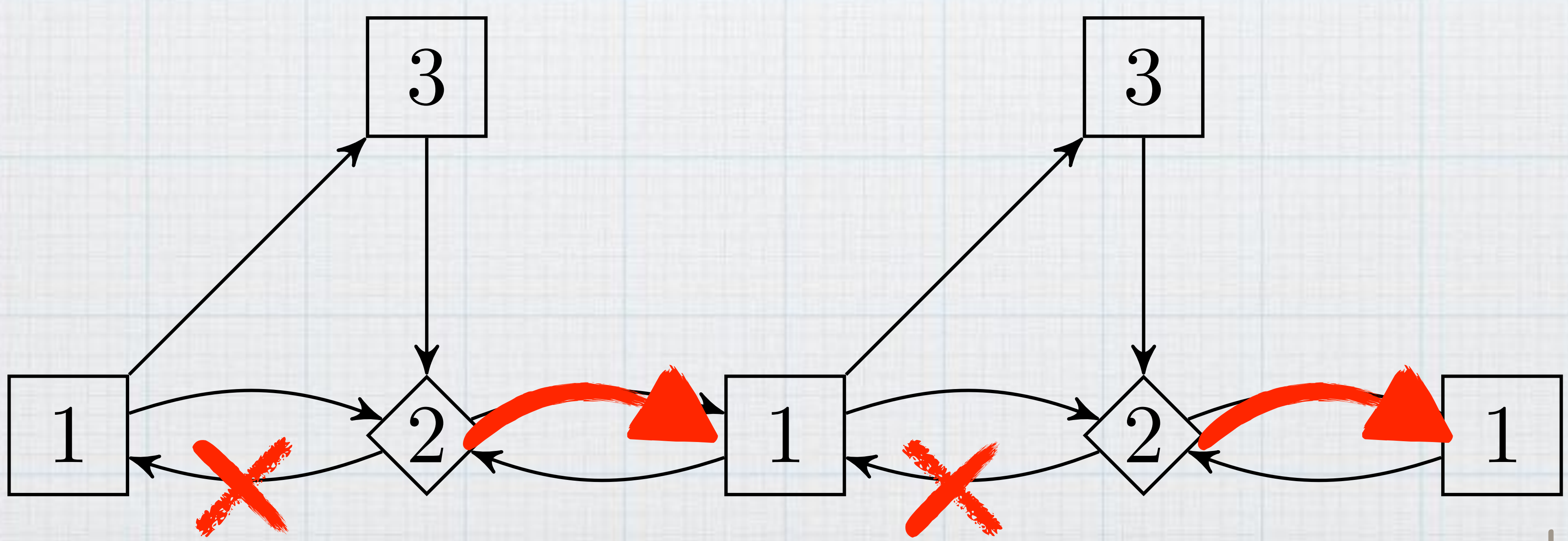

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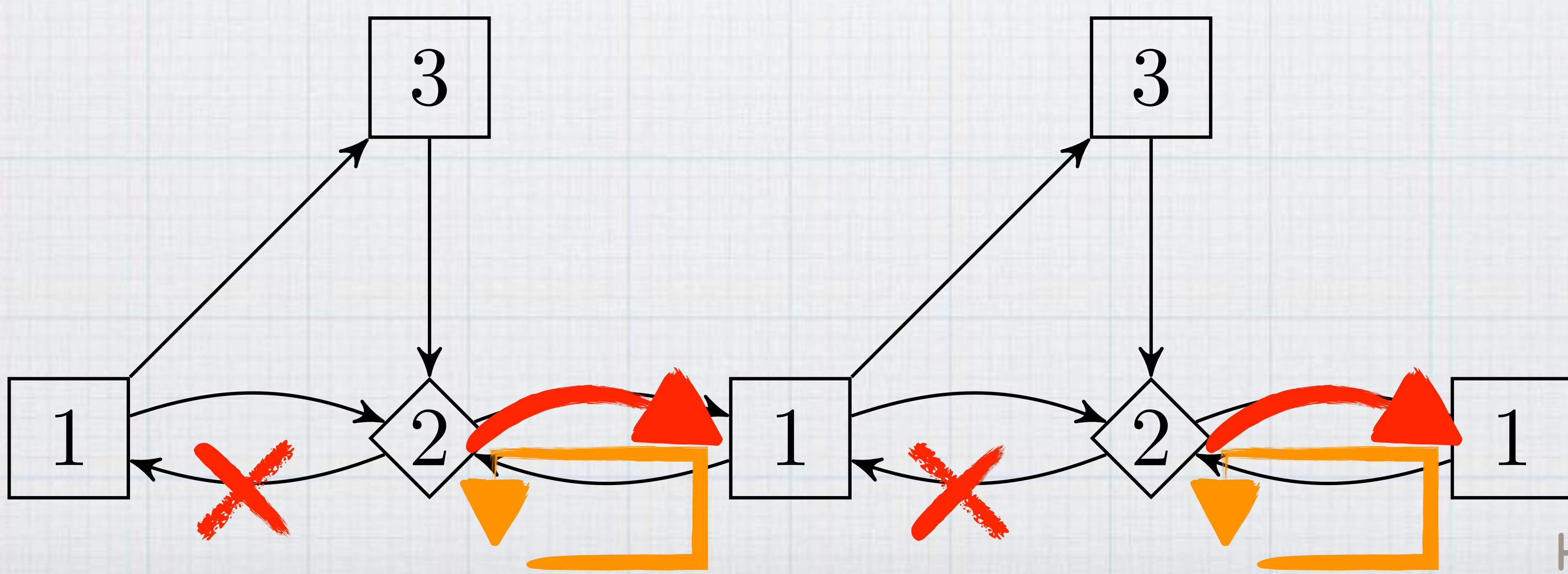

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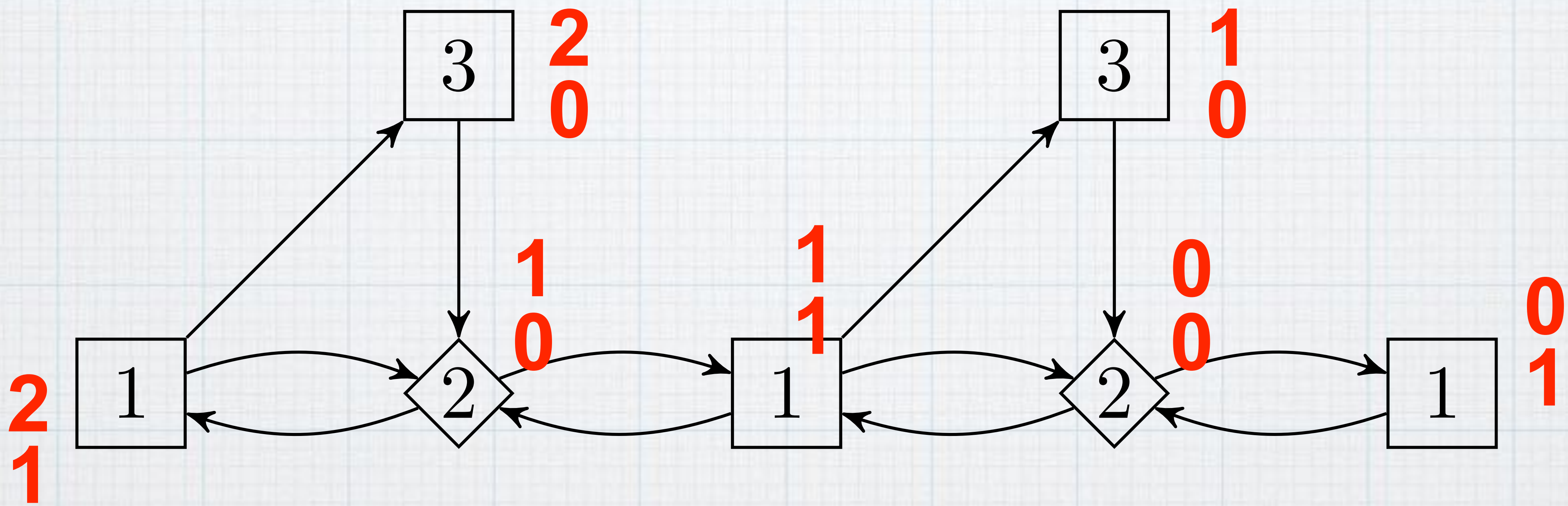

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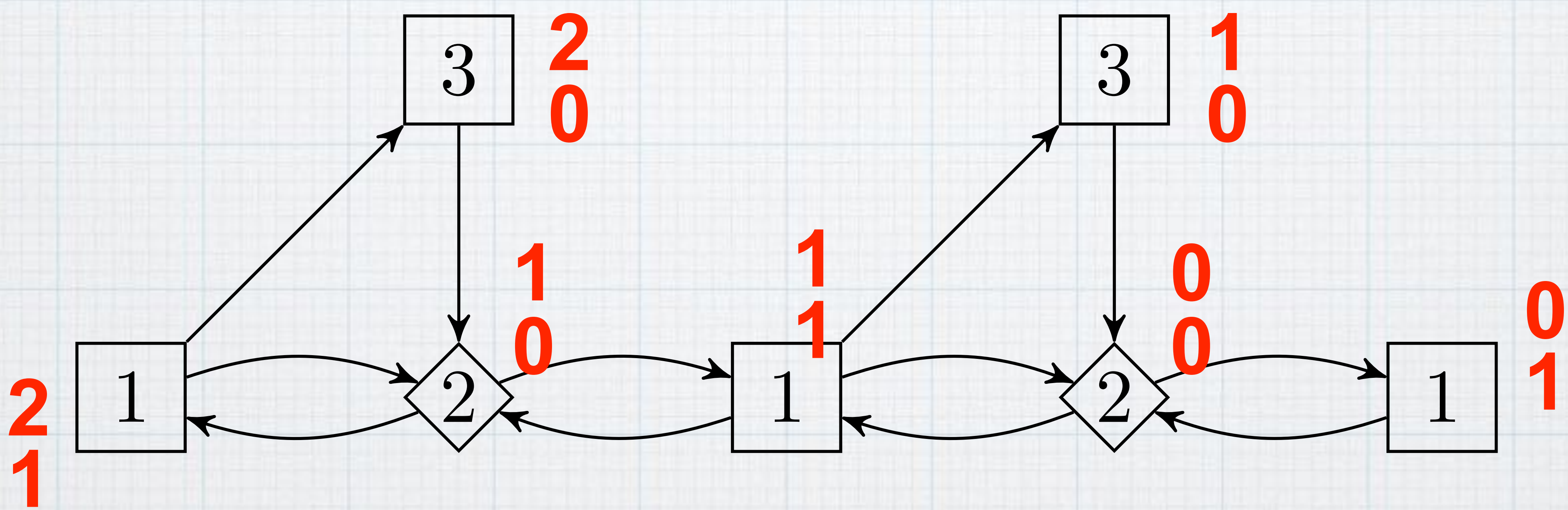

n_3 — how many 3's will be visited

n_1 — how many 1's will be visited (before visiting 2, a bigger even)

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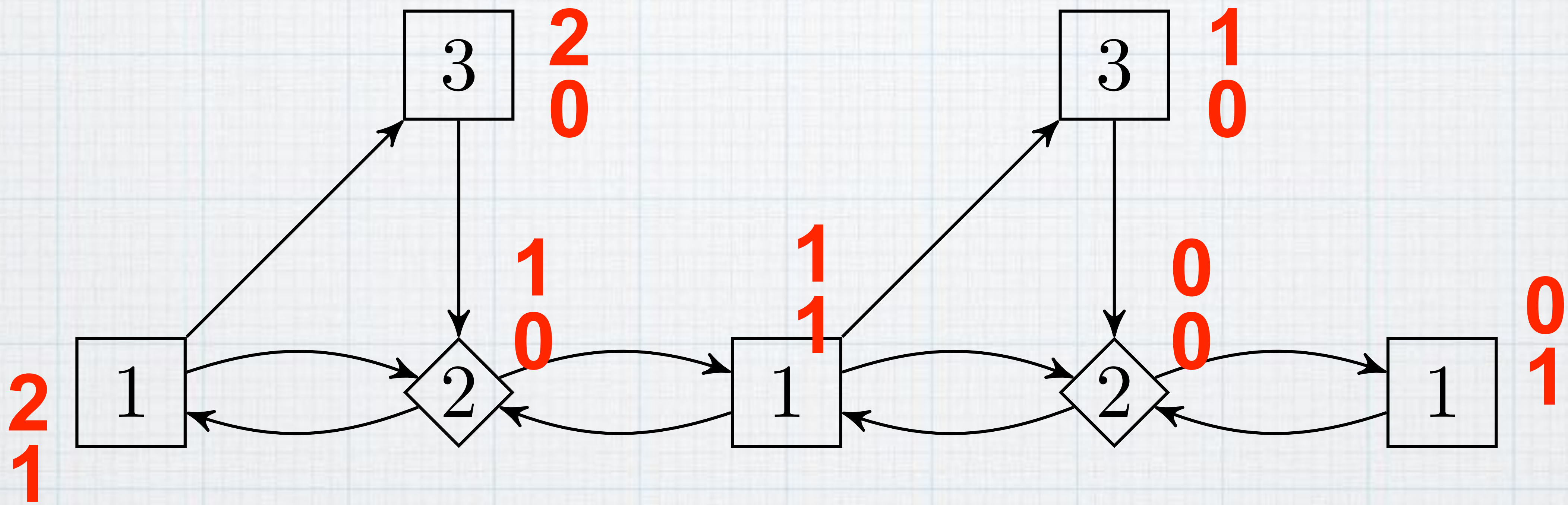

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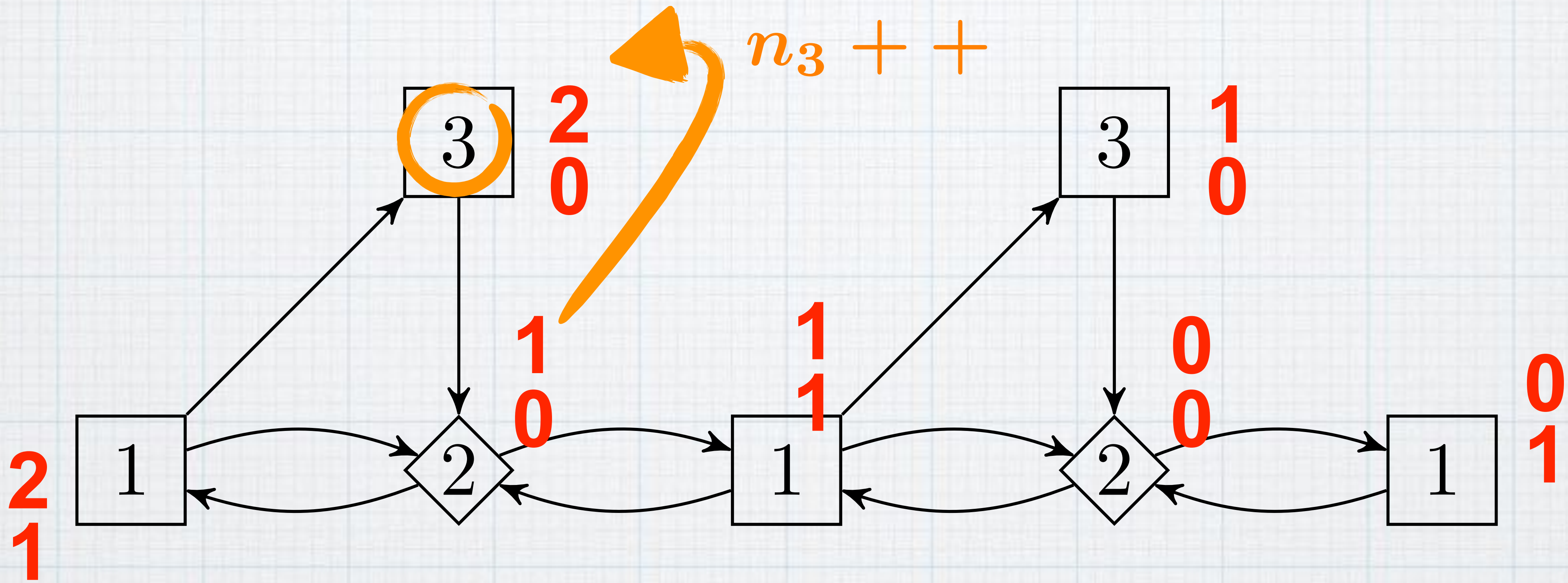

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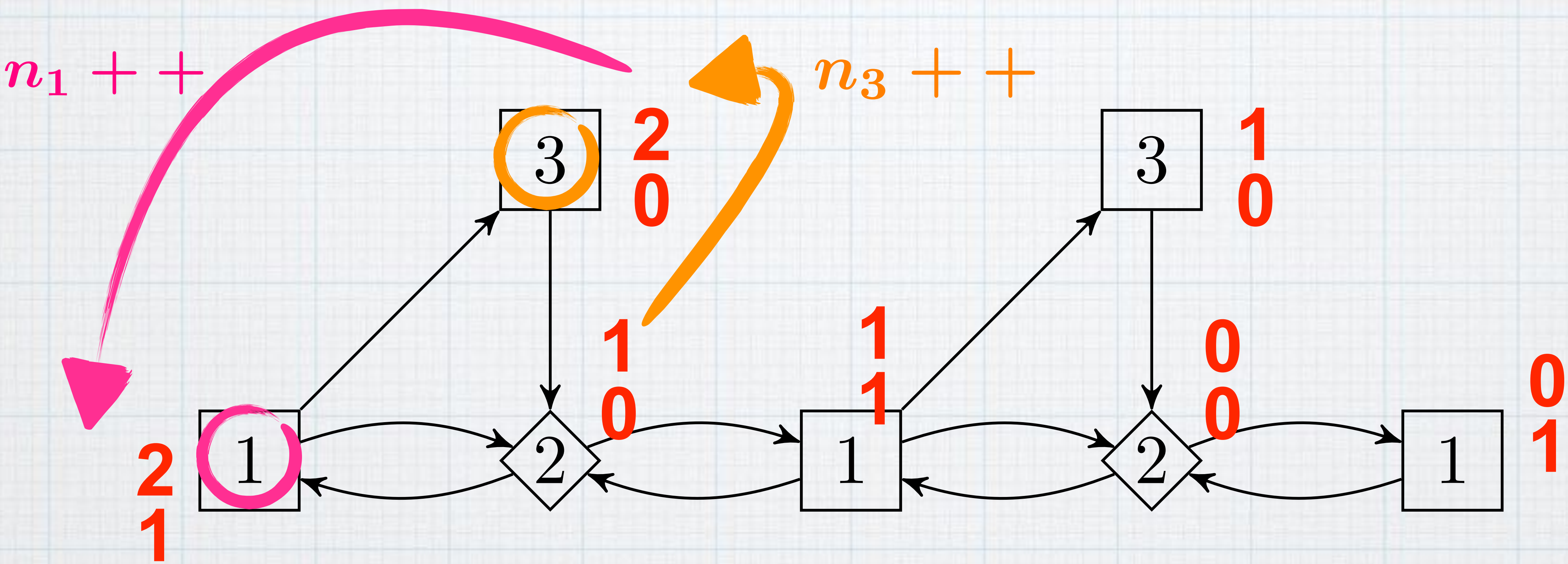

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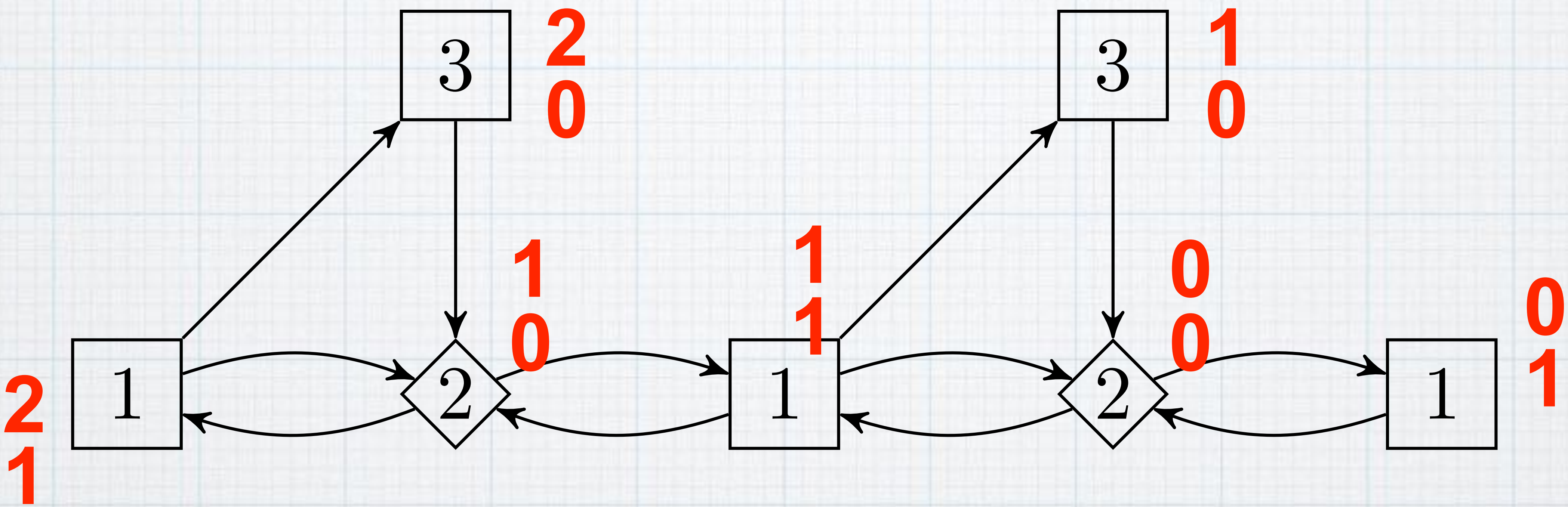

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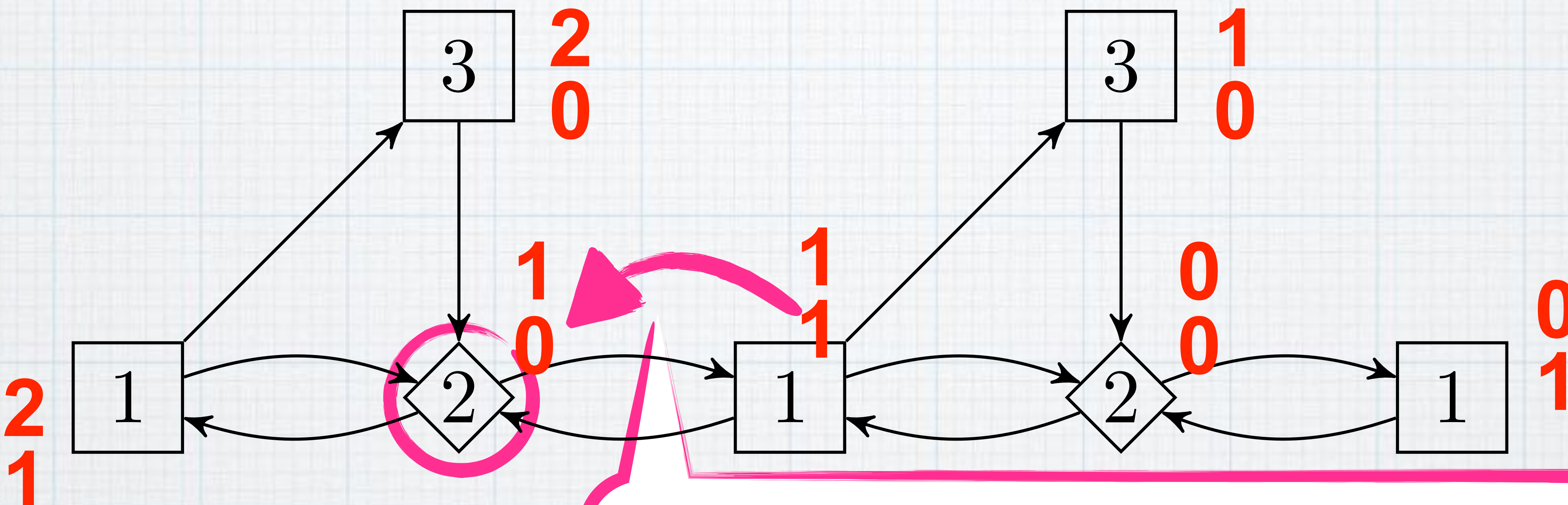

$n_1 := 0$

because visiting 2 **cancels out** visiting 1

n_3 — how many 3's will be visited

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Jurdzinski's Progress Measure

Definition

(Assuming priorities are 0, 1, …, 6)

* A **prioritized ordinal** is α_5 α_3 α_1 (each α_j is an ordinal)

Jurdzinski's Progress Measure

Definition

(Assuming priorities are 0, 1, ..., 6)

- A **prioritized ordinal** is $\begin{matrix}\alpha_5\\ \alpha_3\\ \alpha_1\end{matrix}$ (each α_j is an ordinal)

- for each $i = 0, 1, \ldots, 6,$
  the ***i*-th truncated lexicographic order**

$$\begin{matrix}\alpha_5\\ \alpha_3\\ \alpha_1\end{matrix} \preceq_i \begin{matrix}\beta_5\\ \beta_3\\ \beta_1\end{matrix}$$

is defined by

- the lexicographic order
- after truncating α_j, β_j for all $j < i$

- examples: $\begin{matrix}7\\ 142\\ 63\end{matrix} \preceq_1 \begin{matrix}8\\ 0\\ 0\end{matrix}$ $\quad\begin{matrix}2\\ 142\\ 63\end{matrix} \preceq_4 \begin{matrix}2\\ 0\\ 0\end{matrix}$

truncated

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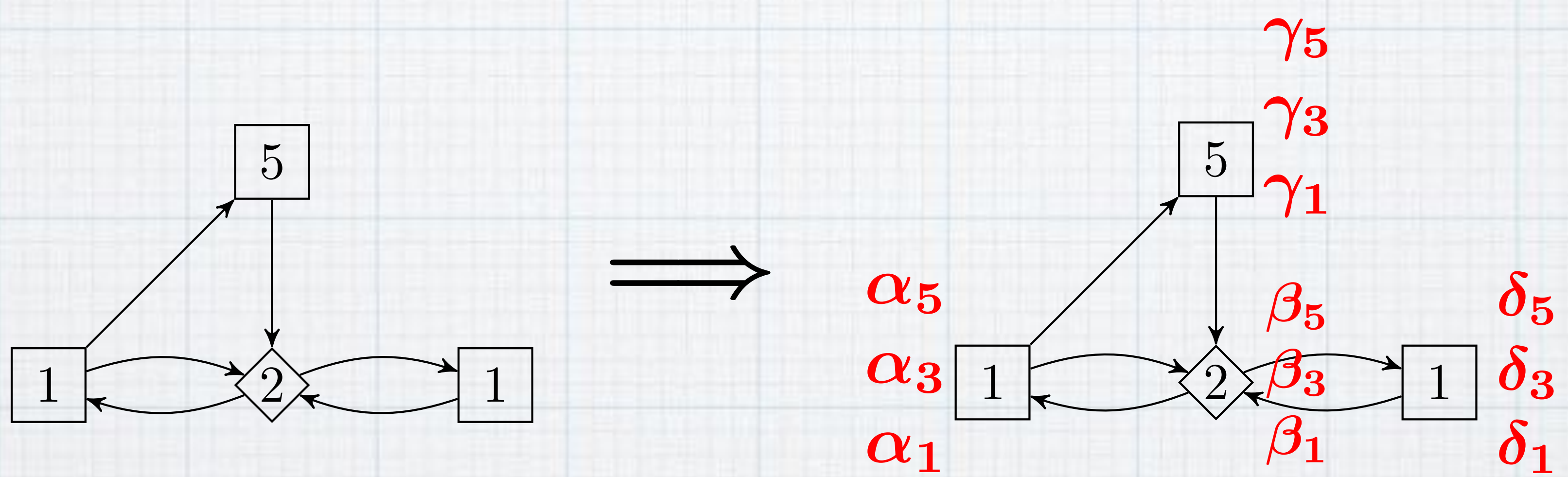

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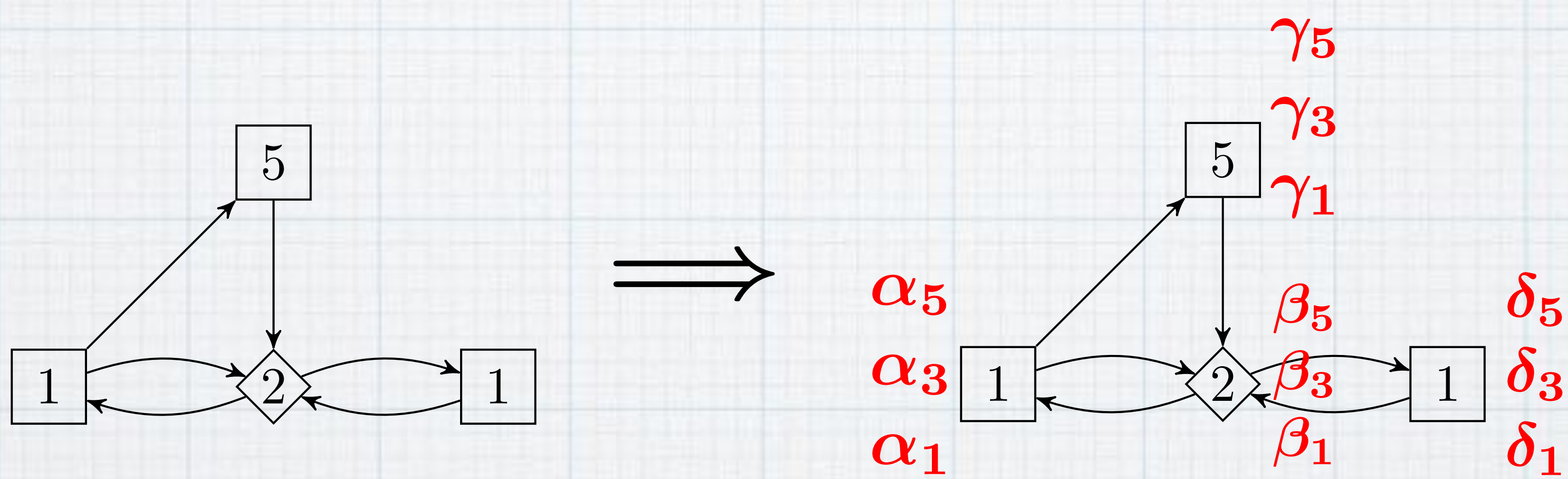

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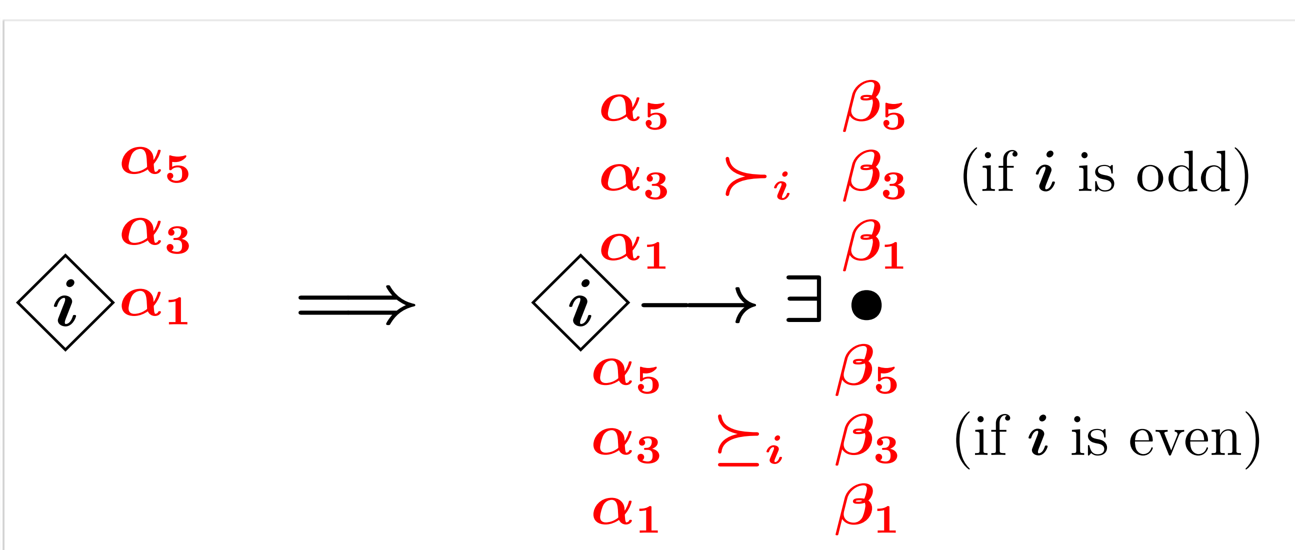

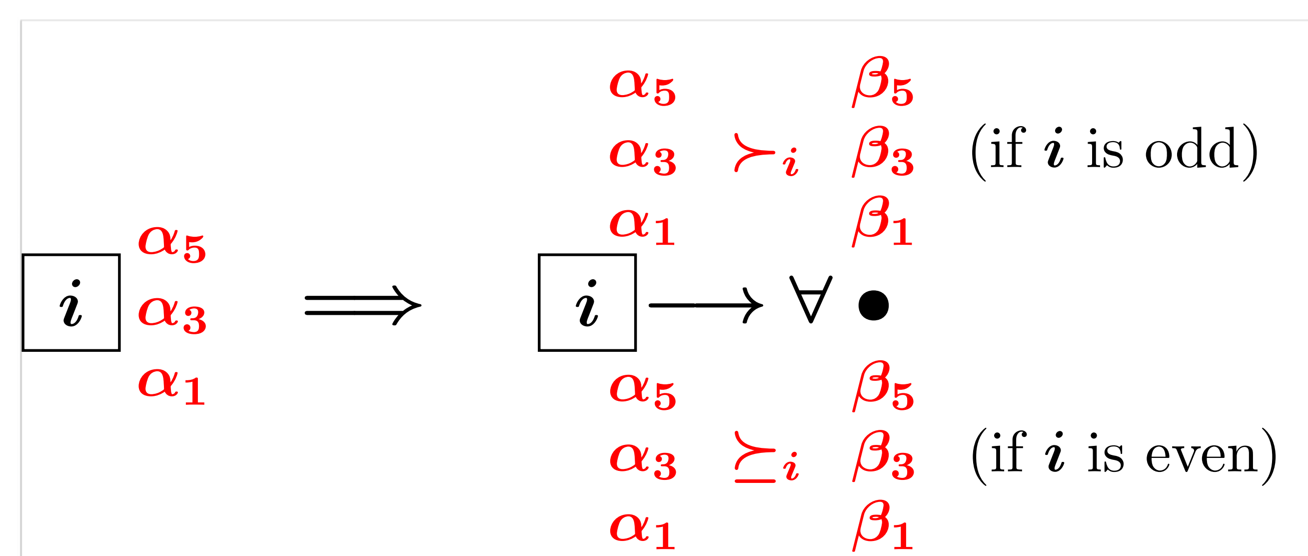

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liveness, lfp	ranking functions
nested, alternating gfp's & lfp's	**progress measure for a parity game** [Jurdzinski] **lattice-theoretic progress measure** **(Our first main contrib.)**

finite, algorithmic

The Table

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Knaster-Tarski

Cousot-Cousot

finite, algorithmic

infinite, symbolic, logical

Syntax: Equational Systems

[Arnold & Niwinski '01], [Cleaveland, Klein & Steffen, CAV'92], ...

<u>Def.</u> An *equational system* over a complete lattice $\boldsymbol{L}$ is

$$\begin{aligned} \boldsymbol{u_1} &=_{\eta_1} \boldsymbol{f_1(u_1, \ldots, u_m)}, \\ &\vdots \\ \boldsymbol{u_m} &=_{\eta_m} \boldsymbol{f_m(u_1, \ldots, u_m)} \end{aligned}$$

where

- $\boldsymbol{f_1, \ldots, f_m \colon L^m \to L}$ are monotone, and
- $\boldsymbol{\eta_1, \ldots, \eta_m \in \{\mu, \nu\}}$.

Syntax: Equational Systems

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$$\nu u_2.\, f_2(\, \mu u_1.\, f_1(u_1, u_2),\ u_2\,)$$

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solved first

$$u_1 =_{\mu} f_1(u_1, u_2),$$
$$u_2 =_{\nu} f_2(u_1, u_2)$$

$$\nu u_2.\, f_2(\,\mu u_1.\, f_1(u_1, u_2),\, u_2\,)$$

The order matters!

In [Hasuo, Shimizu & Cirstea, POPL'16]

* A lattice-theoretic generalization of Jurdzinski's progress measure,
* in the spirit of

Lem. (invariants witness gfp's)
Let $f\colon 2^X \to 2^X$ be monotone, and $I \in 2^X$.

$$\frac{I \subseteq f(I)}{I \subseteq \nu u.\, f(u)}$$

Lem. (witnessing lfp's)
Let $f\colon 2^X \to 2^X$ be monotone.
If $U_0 \subseteq U_1 \subseteq \cdots \subseteq X$ satisfies

$$U_0 = \emptyset \quad \text{and} \quad U_{n+1} \subseteq f(U_n) \;,$$

then $U_n \subseteq \mu u.\, f(u)$ for each n.

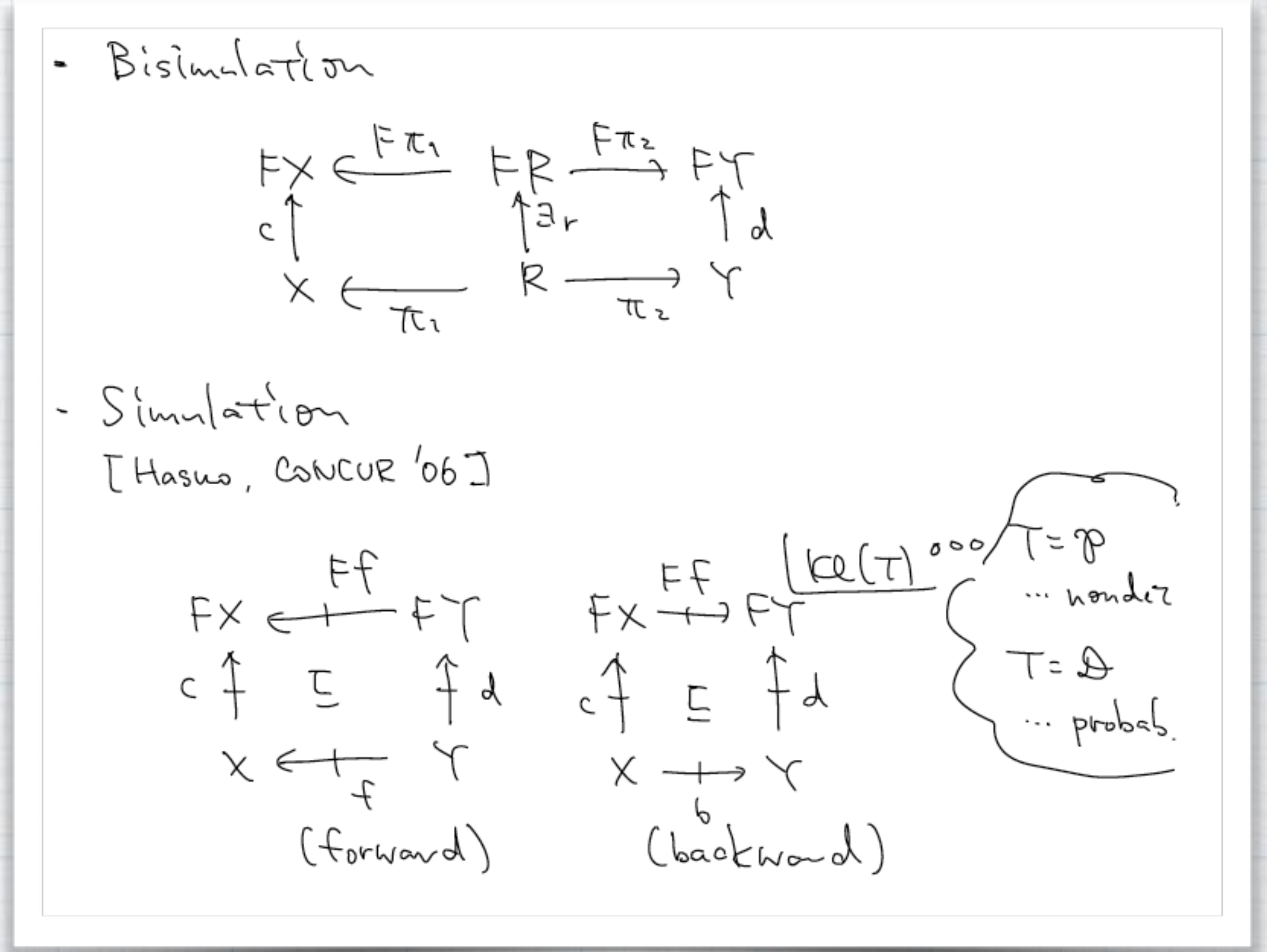
Fixed Points in Coalgebras Other Than Greatest

* Relevant to the grand challenges in the theory of coalgebras

  * **Weak (bi)simulation**: ignore finitely many τ-transitions

    (~ recurrence, GF(imitate))

  * **Buechi & parity acceptance**: recurrence & much more



Reasoning Principles in the Theory of Coalgebras

* ... how to model ranking function-like constructs categorically?
* Two answers:
  * Buechi & parity acceptance by equational systems [CONCUR'16, LMCS'17]
  * Categorical ranking functions by corecursive algebras [LICS'17]
* Underlying: lattice-theoretic progress measures as witnesses for alternating fixed points [POPL'16]

Outline

* Least, greatest and alternating fixed points: a foundational view [POPL’16]
* Buechi and parity acceptance conditions in coalgebras [CONCUR’16]
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Outline

- Least, greatest and alternating fixed points: a foundational view [POPL'16]
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Final Coalgebra in Kleisli Category

[Power & Turi, CTCS '99], [Hasuo, Jacobs & Sokolova, LMCS '07]

in $\mathbf{Sets}$

Nondeterministic Automaton

$$\mathcal{A} = (X, \Sigma, \delta, \mathsf{Acc})$$

where $\delta \subseteq X \times \Sigma \times X$

$$\begin{array}{c} FX \\ \uparrow c \\ X \end{array}$$

where $F = \mathcal{P}(\{\checkmark\} + \Sigma \times (\_))$

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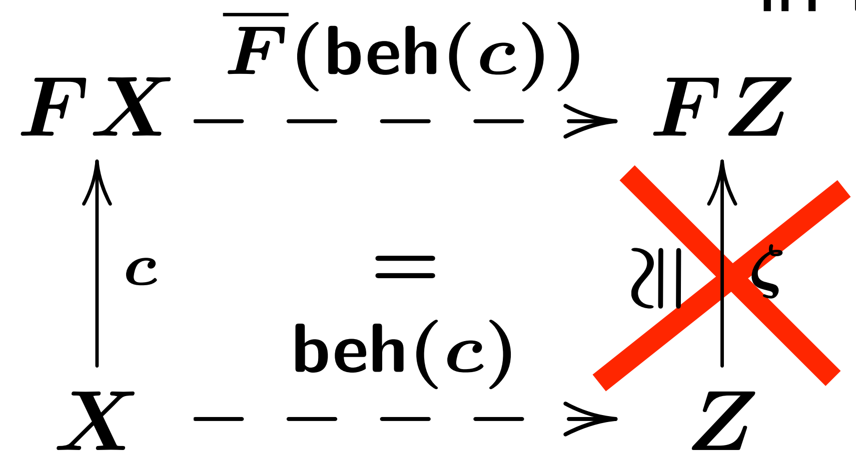

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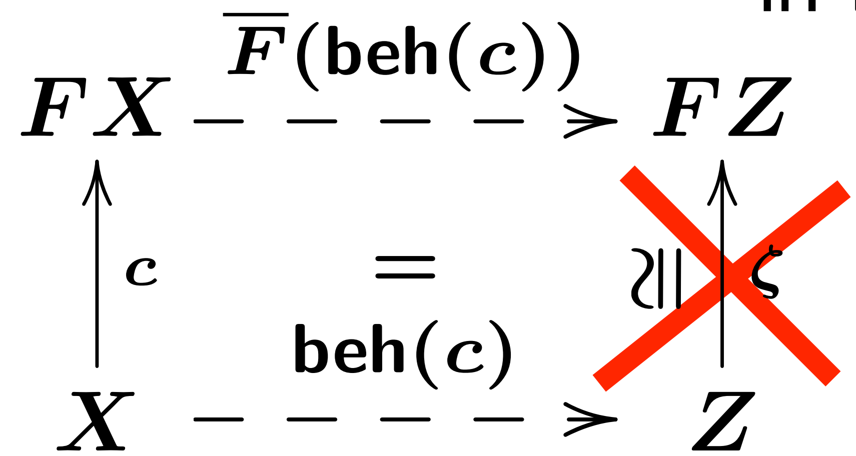

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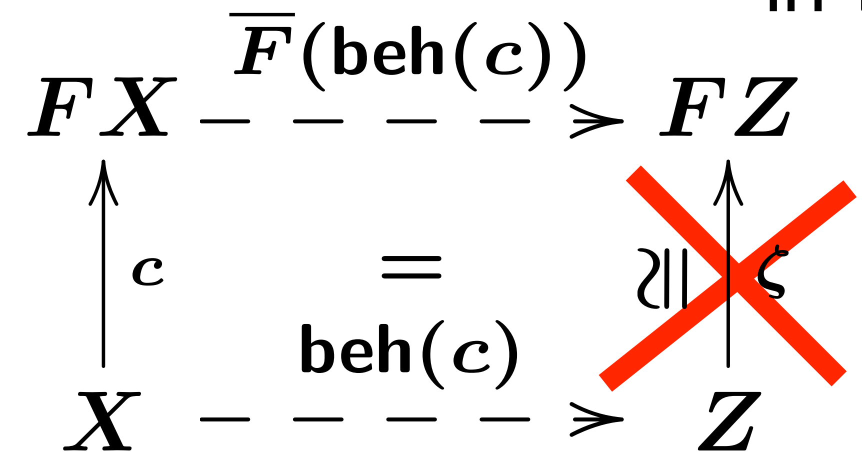

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$$\frac{f : X \nrightarrow Y \ \text{ in } \ \mathcal{K}\ell(\mathcal{P})}{f : X \to \mathcal{P}Y \ \text{ in } \ \mathbf{Sets}}$$

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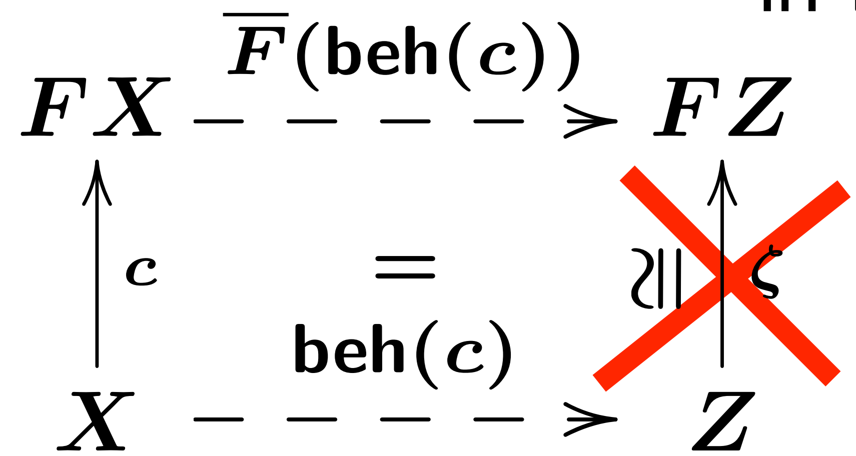

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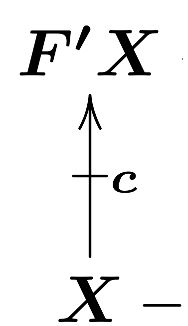
$\mathcal{K}\ell(\mathcal{P})$



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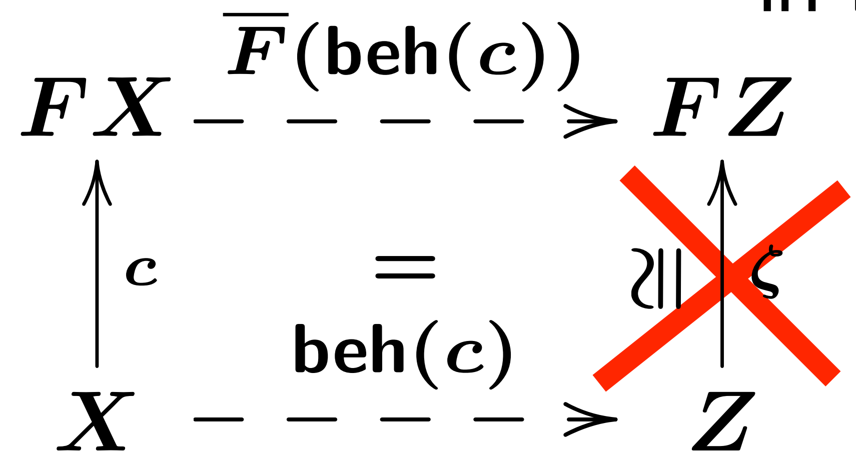

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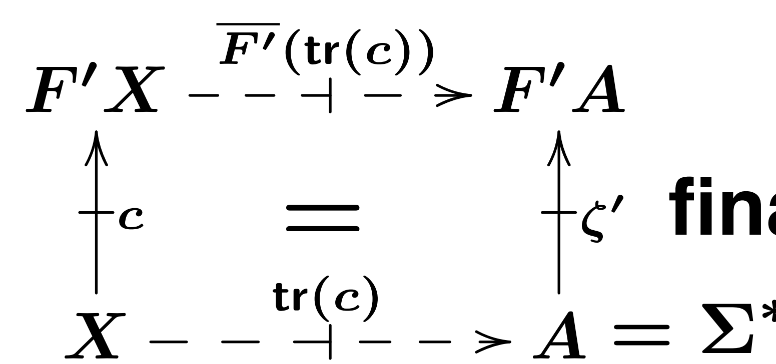

final

where $F' := \{\checkmark\} + \Sigma \times (\_)$

Recap: Kleisli Approach to Coalgebraic Linear-Time Semantics

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* **[Hasuo, Jacobs, Sokolova, LMCS '07]**

Recap: Kleisli Approach to Coalgebraic Linear-Time Semantics

* [Hasuo, Jacobs, Sokolova, LMCS ’07]
* Assume a monad T on Sets is “cpo-enriched”
  * (Kl(T) is Cpo_bot-enriched, composition is left-strict, ...)

Recap: Kleisli Approach to Coalgebraic Linear-Time Semantics

* [Hasuo, Jacobs, Sokolova, LMCS ’07]
* Assume a monad T on Sets is “cpo-enriched”
  * (Kl(T) is Cpo_bot-enriched, composition is left-strict, …)
* Then
  an initial F-algebra in Sets
  ➜ an initial F’-algebra in Kl(T)
  ➜ a final F’-coalgebra in Kl(T)
  * The last arrow is like [Smyth & Plotkin ’82]

Finite Trace Semantics

in $\mathcal{K}\ell(\mathcal{P})$

$$
\begin{array}{ccc}
F'X & \overset{\overline{F'}(\mathsf{tr}(c))}{- - \dashrightarrow} & F'A \\
\uparrow c & = & \cong\ \uparrow \zeta' \ \textbf{final} \\
X & \overset{\mathsf{tr}(c)}{- - \dashrightarrow} & A = \Sigma^*
\end{array}
$$

$$
\frac{\mathsf{tr}(c) : X \nrightarrow \Sigma^* \quad \text{in } \mathcal{K}\ell(\mathcal{P})}{\mathsf{tr}(c) : X \to \mathcal{P}\Sigma^* \quad \text{in } \mathbf{Sets}}
$$

Finite Trace Semantics

in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{array}{ccc} F'X & \xrightarrow{\overline{F'}(\mathrm{tr}(c))} & F'A \\ \uparrow c & = & \cong \uparrow \zeta' \ \textbf{final} \\ X & \xrightarrow{\mathrm{tr}(c)} & A = \Sigma^* \end{array}$$

$$\frac{\mathrm{tr}(c) : X \nrightarrow \Sigma^* \quad \text{in } \mathcal{K}\ell(\mathcal{P})}{\mathrm{tr}(c) : X \to \mathcal{P}\Sigma^* \quad \text{in Sets}}$$

Thm:

$\mathrm{tr}(c)$ characterizes **finite trace** $L(\mathcal{A})$

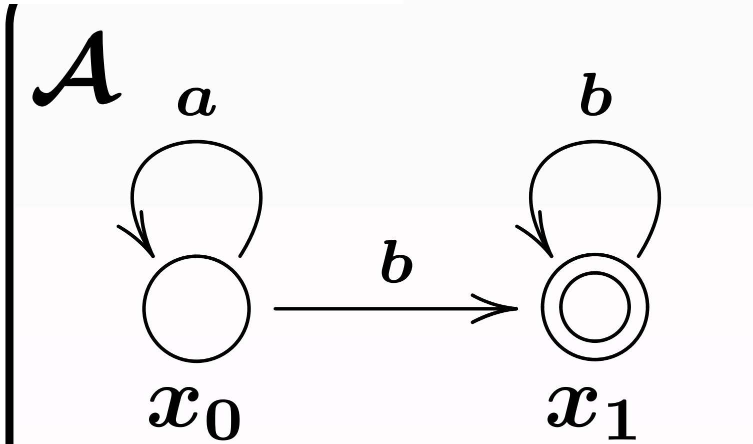
Def.

For $\mathcal{A} = (X, \Sigma, \delta, \mathsf{Acc})$,

finite trace semantics

$$L(\mathcal{A})(x) := \left\{ a_0 \ldots a_{n-1} \in \Sigma^* \;\middle|\; x = x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n \in \mathsf{Acc} \right\}$$

Example:



$L(\mathcal{A})(x_0) = a^* b b^*$

Extension to Various Systems

in $\mathcal{K}\ell(\mathcal{P})$

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final where

$$F' := \{\checkmark\} + \Sigma \times (\_)$$

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- $T = \mathcal{P}$

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- $F' = \{\checkmark\} + \Sigma \times (\_)$ ➡ $F' = \coprod_i \Sigma_i \times (\_)^i$

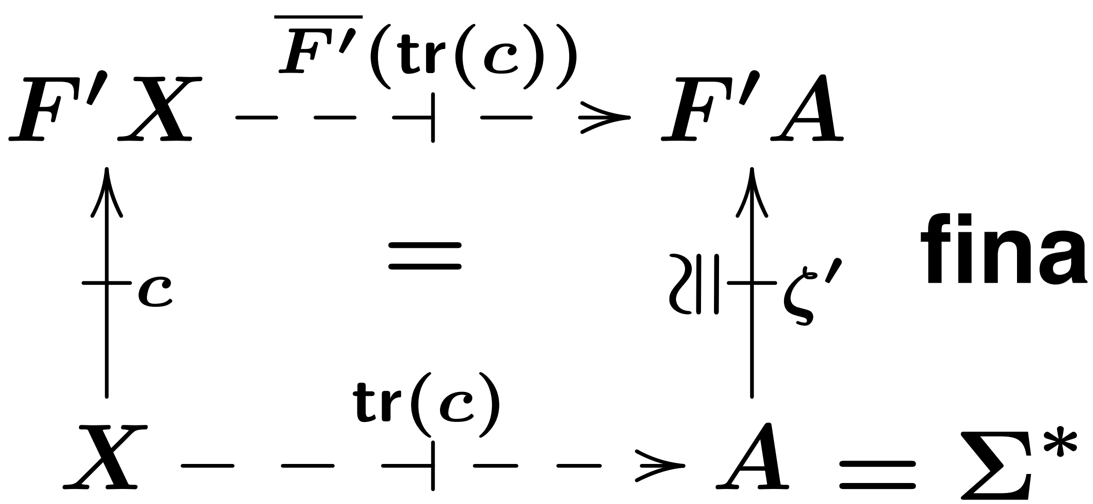
  (polynomial functor)

  - **Words** to **Trees**

- $T = \mathcal{P}$

Extension to Various Systems

in $\mathcal{K}\ell(\mathcal{P})$



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- $F' = \{\checkmark\} + \Sigma \times (\_)$ → $F' = \coprod_i \Sigma_i \times (\_)^i$ (polynomial functor)
  - **Words** to **Trees**
- $T = \mathcal{P}$ → $T = \mathcal{G}$ (the sub-Giry monad)
  - **Nondeterministic** to (generative) **Probabilistic**

Coalgebraic Finite Trace Semantics

Finite Trace

$$L(\mathcal{A})(x) := \left\{ a_0 \dots a_{n-1} \in \Sigma^* \;\middle|\; x = x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n \in \mathsf{Acc} \right\}$$

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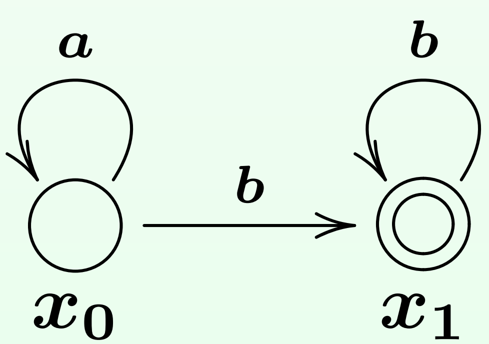


$$L(\mathcal{A}) = a^* b b^*$$

$$F = \{\checkmark\} + \Sigma \times (\_)$$

in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{array}{ccc} FX & \xrightarrow{\overline{F}(\mathsf{tr}(c))} & F\Sigma^* \\ \uparrow c & = & \cong \uparrow \zeta' \\ X & \xrightarrow{\mathsf{tr}(c)} & \Sigma^* \end{array}$$

final

unique

Thm:

$\mathsf{tr}(c)$ characterizes **finite trace** $L(\mathcal{A})$

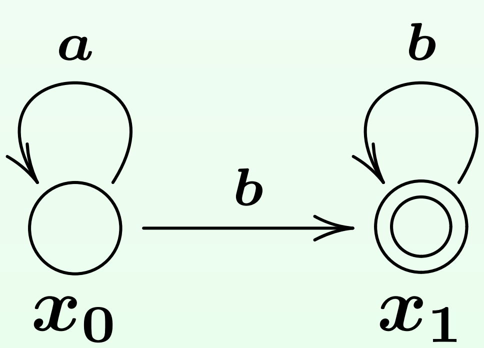
Coalgebraic Infinitary Trace Semantics

[Jacobs, '04]

Infinitary Trace

$$L^{\infty}(\mathcal{A})(x) := L(\mathcal{A})(x) \cup \left\{ a_0 a_1 \ldots \in \Sigma^{\omega} \;\middle|\; x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \ldots \right\}$$

Example:



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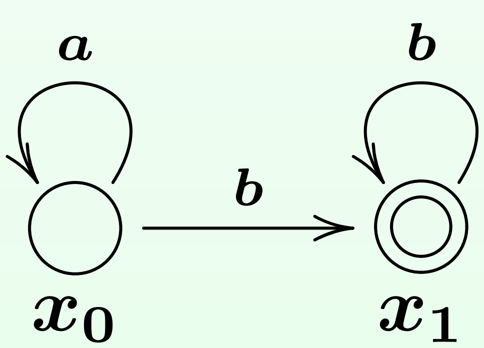

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$\mathcal{K}\ell(\mathcal{P})$

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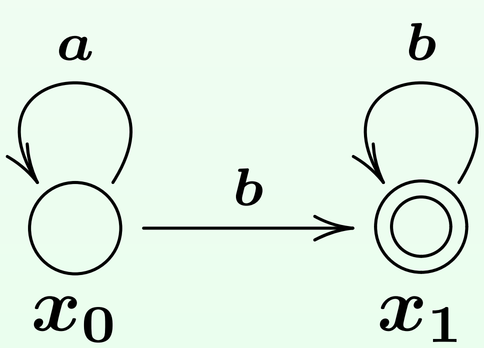

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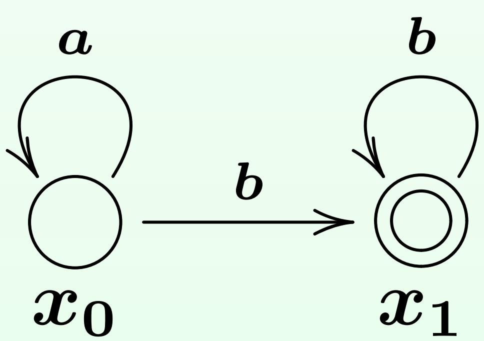

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$\mathcal{K}\ell(\mathcal{P})$

FX

$\uparrow c$

X

in Sets

$F\Sigma^{\infty}$

$\cong \uparrow \zeta'$ final

Σ^{∞}

$\|$

$\Sigma^* \cup \Sigma^{\omega}$

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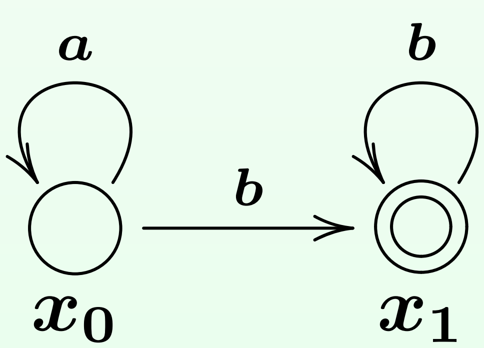

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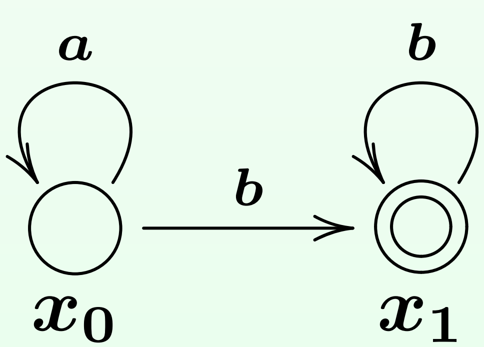

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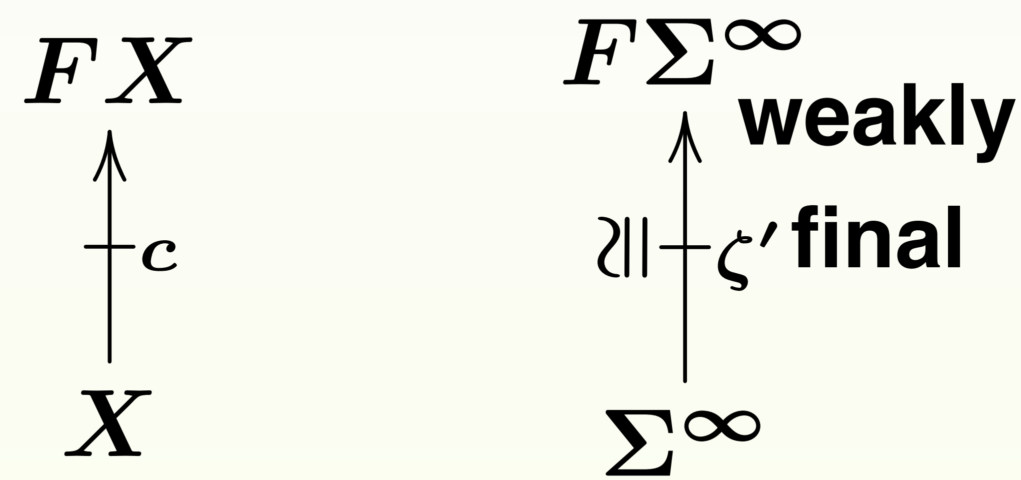

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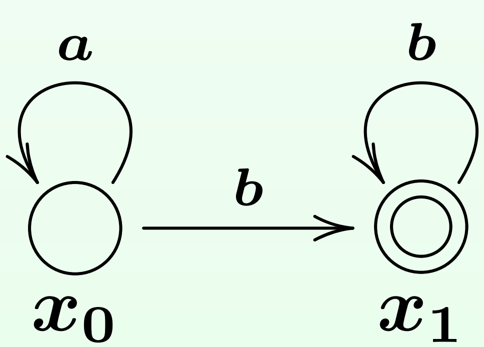

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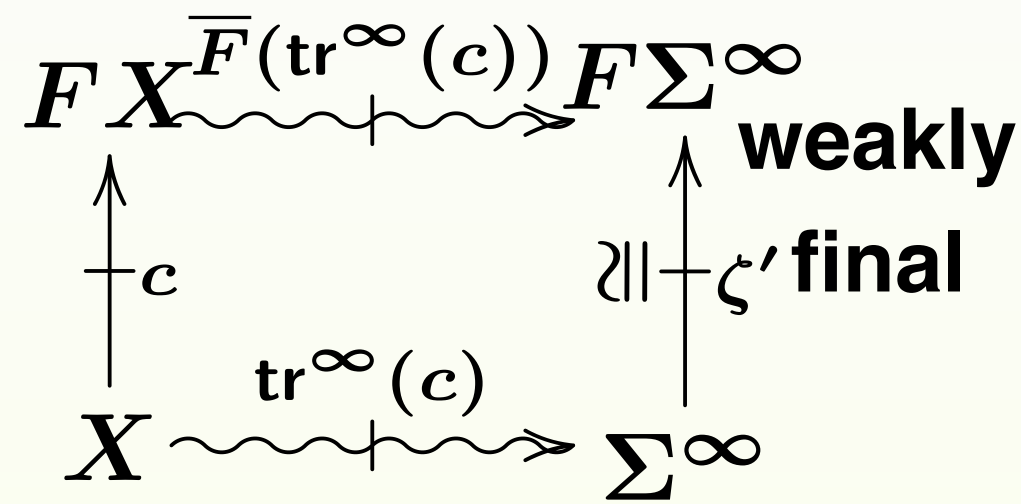

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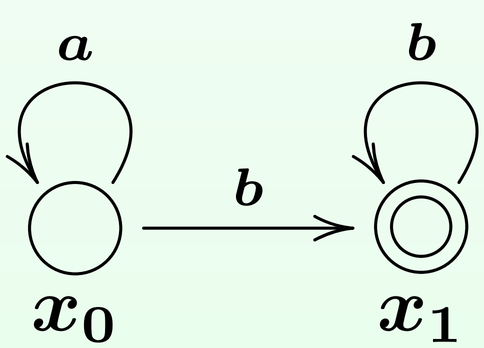

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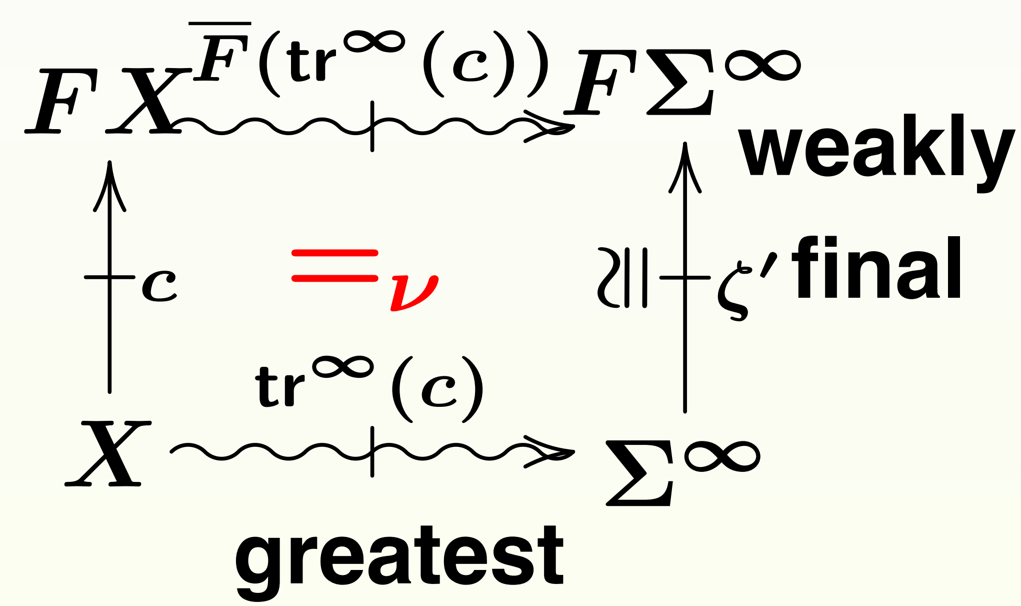


Coalgebraic Infinitary Trace Semantics

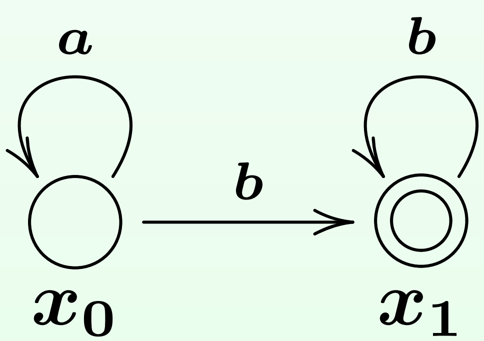
[Jacobs, '04]

Infinitary Trace

$$L^\infty(\mathcal{A})(x) := L(\mathcal{A})(x)$$

$$\cup \left\{ a_0 a_1 \ldots \in \Sigma^\omega \;\middle|\; x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \ldots \right\}$$

Example:



$$L^\infty(\mathcal{A})(x_0) = a^* b b^* + a^\omega + a^* b^\omega$$

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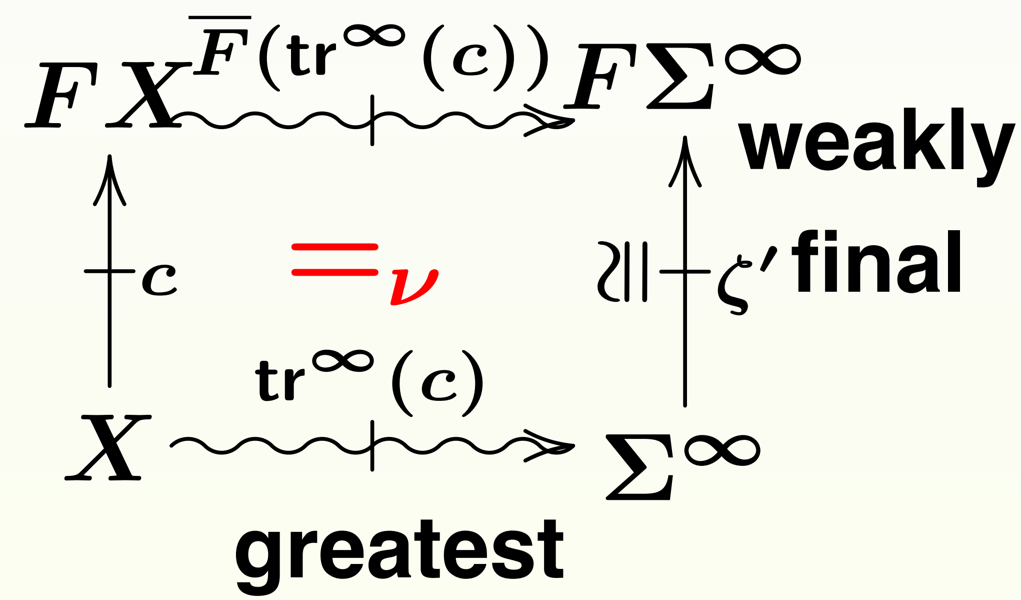

$$\frac{f : X \nrightarrow \Sigma^\infty \text{ in } \mathcal{K}\ell(\mathcal{P})}{f : X \to \mathcal{P}\Sigma^\infty \text{ in Sets}}$$

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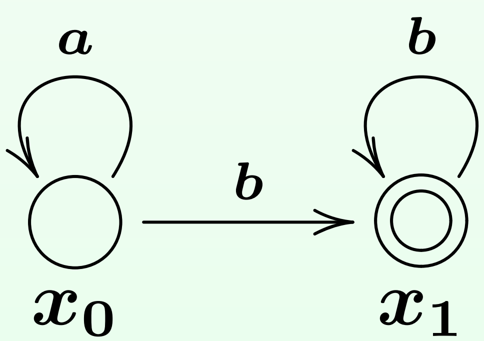

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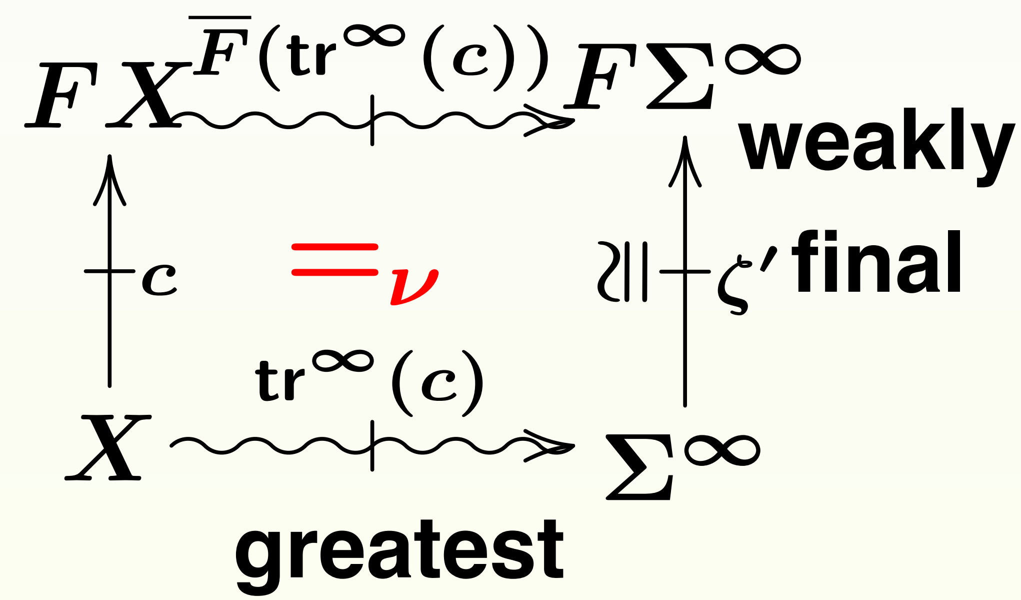

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Thm:

$\mathrm{tr}^\infty(c)$ characterizes **infinitary trace** $L^\infty(\mathcal{A})$

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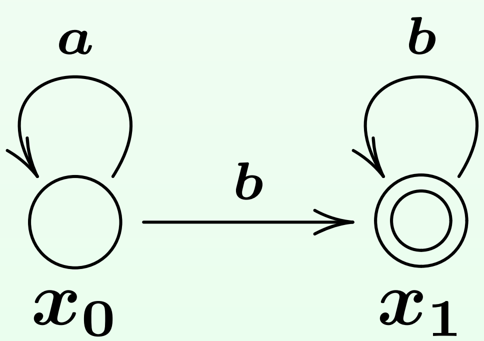

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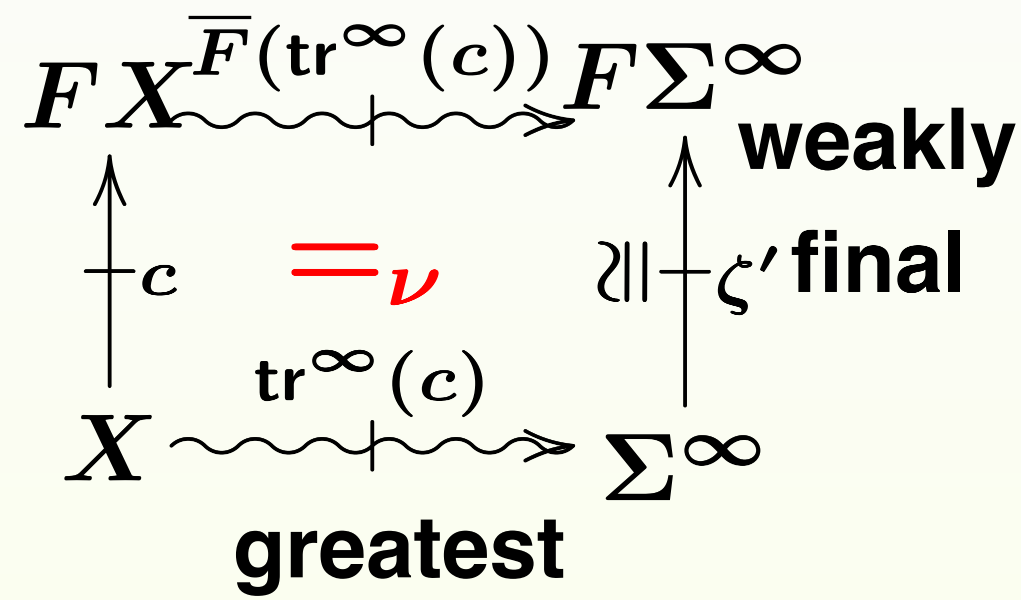

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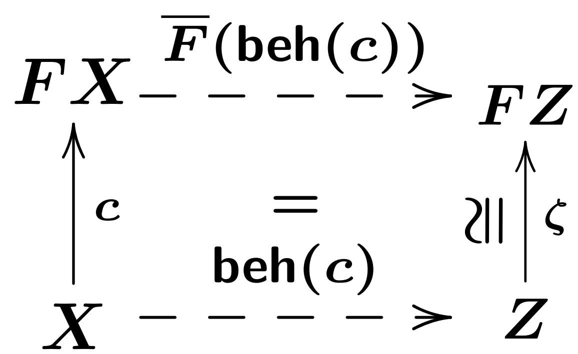
Thm:

$\mathrm{tr}^{\infty}(c)$ characterizes **infinitary trace** $L^{\infty}(\mathcal{A})$

Leave finality!

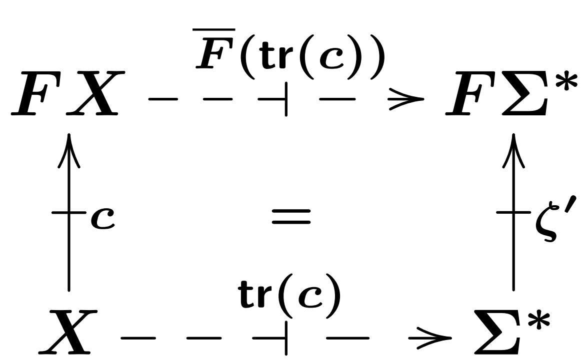
Summary

- Coalgebra is a model for **state-based dynamics**
- **Final coalgebra** captures the behavior

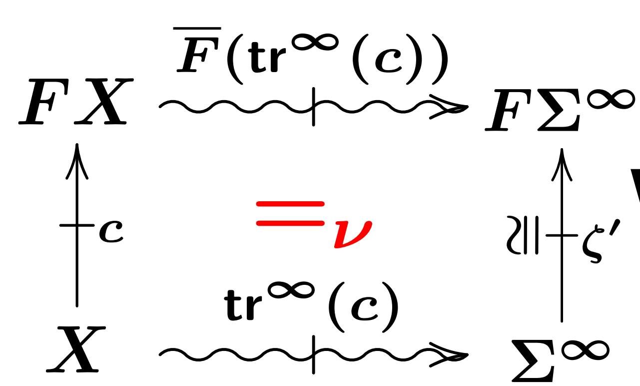


final in $\mathbf{Sets}$

- For nondet. & prob. automata,
  - the final coalgebra (coming from init. alg. in Sets) in the **Kleisli category** captures the finite trace semantics
  - a weakly final coalgebra (a final coalg. in Sets) in the **Kleisli category** captures the infinitary trace semantics



final



weakly final in $\mathcal{K}\ell(\mathcal{P})$

Büchi Automaton $\mathcal{A} = (X, \Sigma, \delta, \mathsf{Acc})$

Def.

X : state space Σ : alphabet

$\delta : X \to \mathcal{P}(\Sigma \times X)$: transition relation

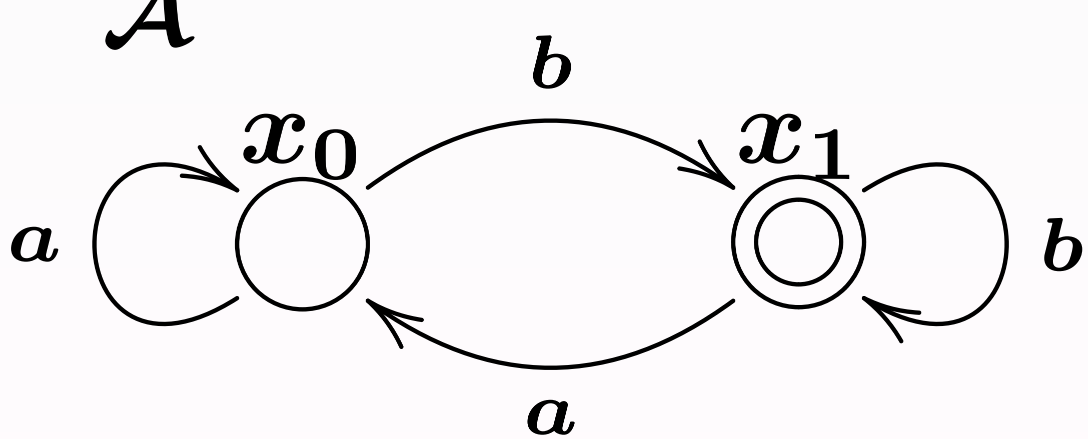
$\mathsf{Acc} \subseteq X$: accepting states

Büchi language $L^B(\mathcal{A}) : X \to \mathcal{P}(\Sigma^\omega)$

$$L^B(\mathcal{A})(x) := \left\{ a_0 a_1 \ldots \in \Sigma^\omega \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \\ \text{s.t. } x_k \in \mathsf{Acc} \text{ for inf. many } k\text{'s} \end{array} \right\}$$

Example:

$\mathcal{A}$



$$L^B(\mathcal{A})(x_0) = \left\{ w \;\middle|\; \begin{array}{l} w \text{ contains} \\ \text{infinitely many } b\text{'s} \end{array} \right\}$$

Parity Automaton $\mathcal{A} = (X, \Sigma, \delta, p)$

Def.

X : state space $\quad\Sigma$: alphabet

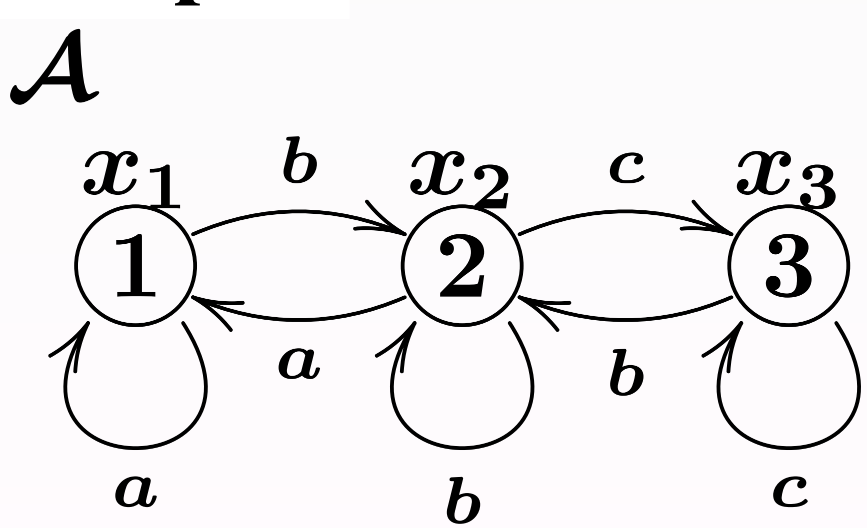
$\delta : X \to \mathcal{P}(\Sigma \times X)$: transition relation

$p : X \to \{1, \ldots, 2n\}$: priority function

parity language $L^p(\mathcal{A}) : X \to \mathcal{P}(\Sigma^\omega)$

$$L^p(\mathcal{A})(x) := \left\{ a_0 a_1 \ldots \in \Sigma^\omega \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \\ \text{s.t. } \limsup_{k\to\infty} p(x_k) \text{ is even} \end{array} \right\}$$

Example:



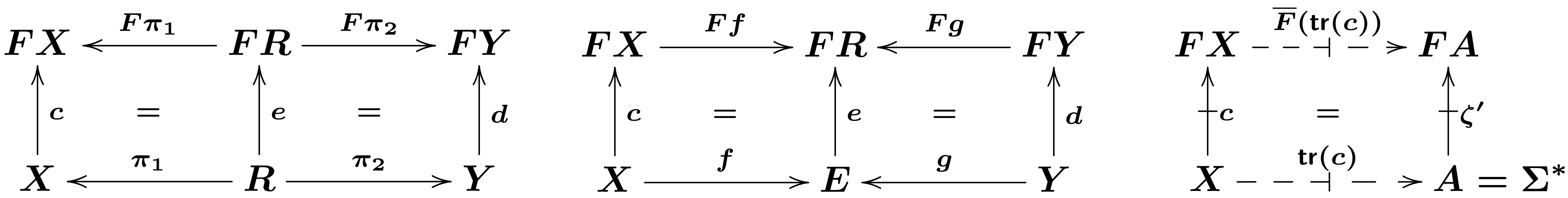
$L^p(\mathcal{A})(x_1)$

$$= \left\{ w \;\middle|\; \begin{array}{l} w \text{ contains} \\ \quad \text{infinitely many } b\text{'s, but} \\ \quad \text{only finitely many } c\text{'s} \end{array} \right\}$$

Difficulty

- Theory of coalgebra is centered around **homomorphisms**

  ≈ **stepwise correspondence**



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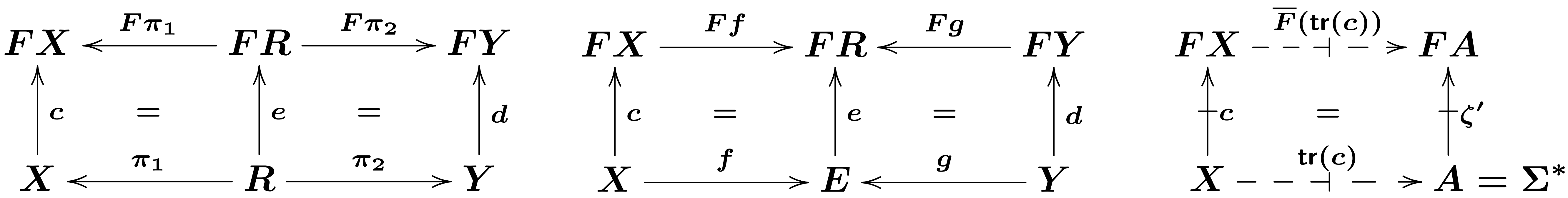

➡ **Local**

Difficulty

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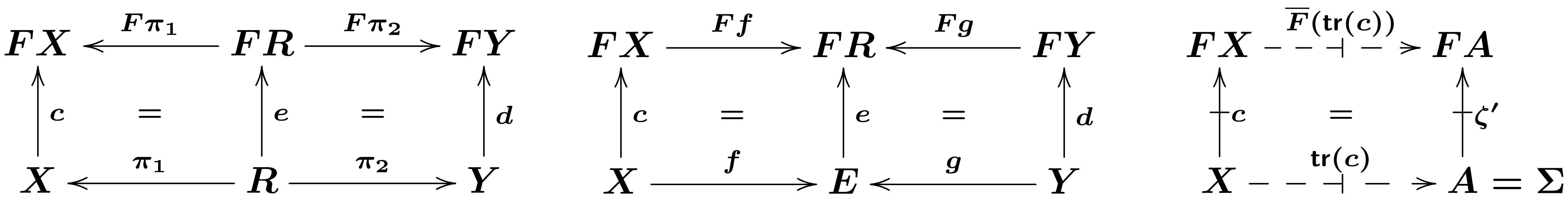

Local

- Büchi/parity acceptance condition considers **infinite** behaviors
  - “Visit **Acc** **infinitely** many times”
  - “Maximum **infinitely** visited priority is even”

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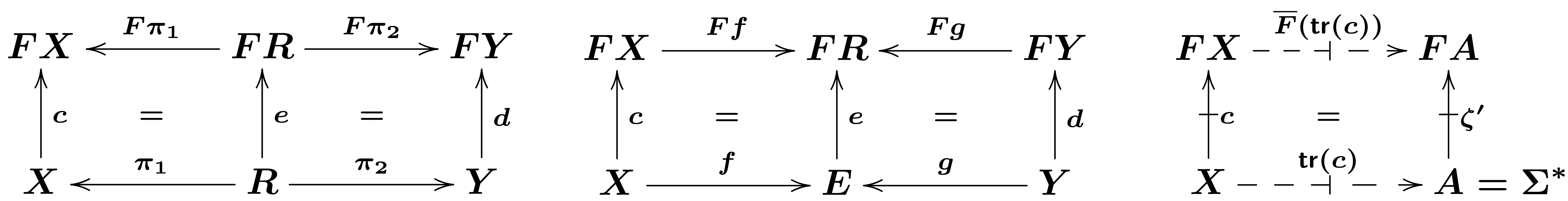

Local

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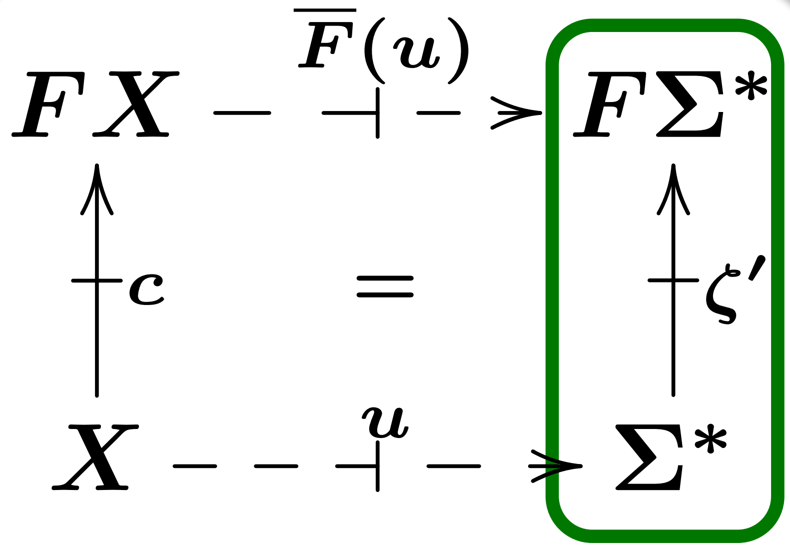
Nonlocal

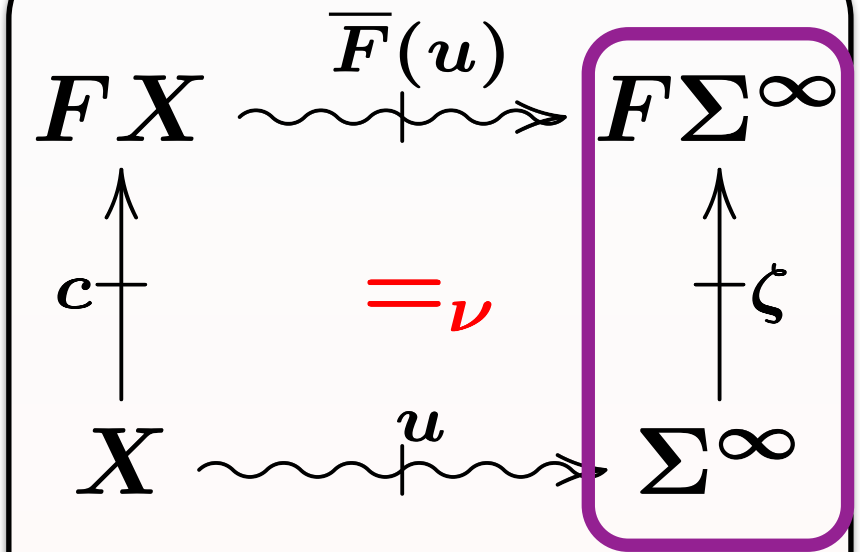
Least Homomorphism?



Unique Homomorphism

$$L(\mathcal{A}) \in \mathcal{P}(\Sigma^*)^X$$

Finite Trace

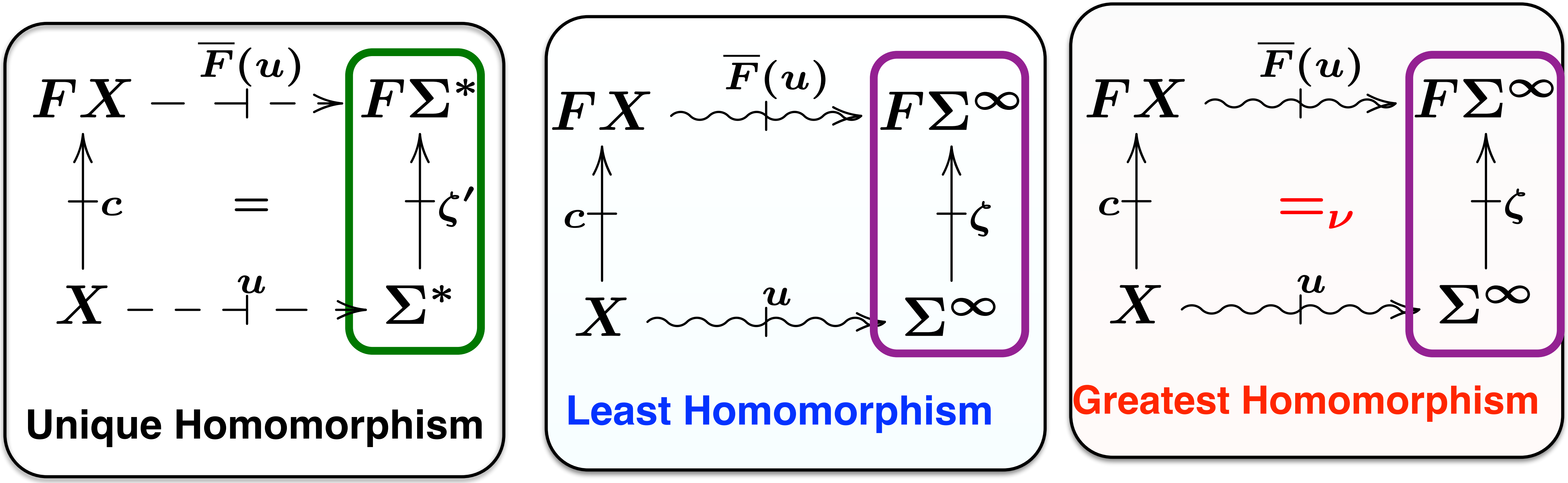


Greatest Homomorphism

$$L^\infty(\mathcal{A}) \in \mathcal{P}(\Sigma^\infty)^X$$

Infinitary Trace

Least Homomorphism?



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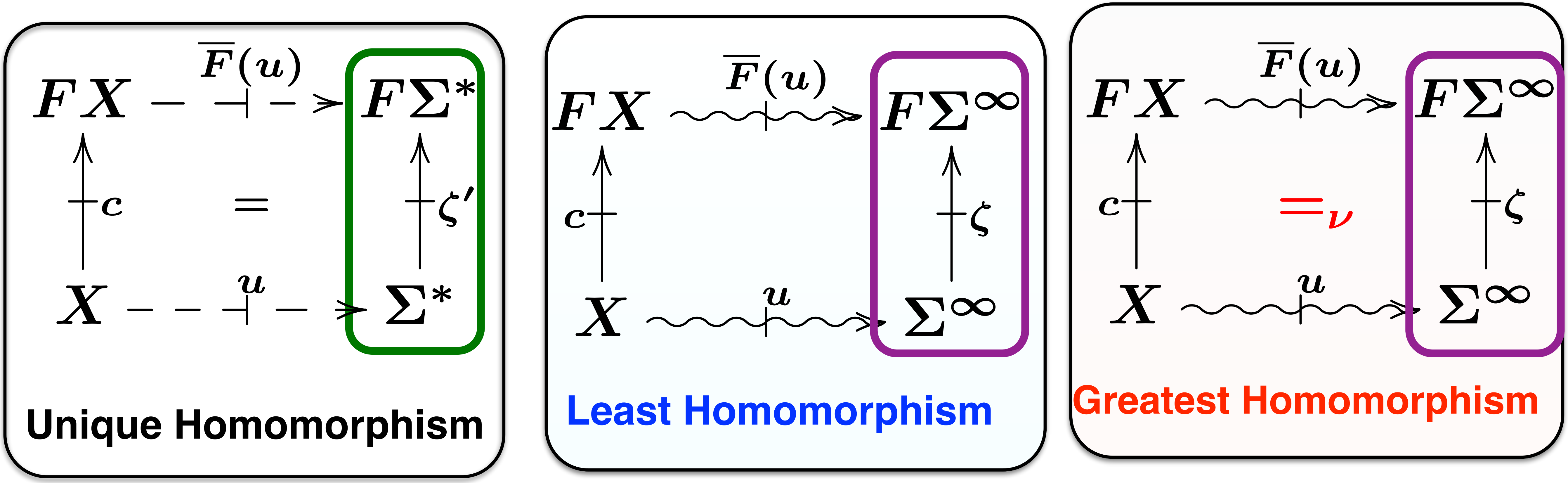

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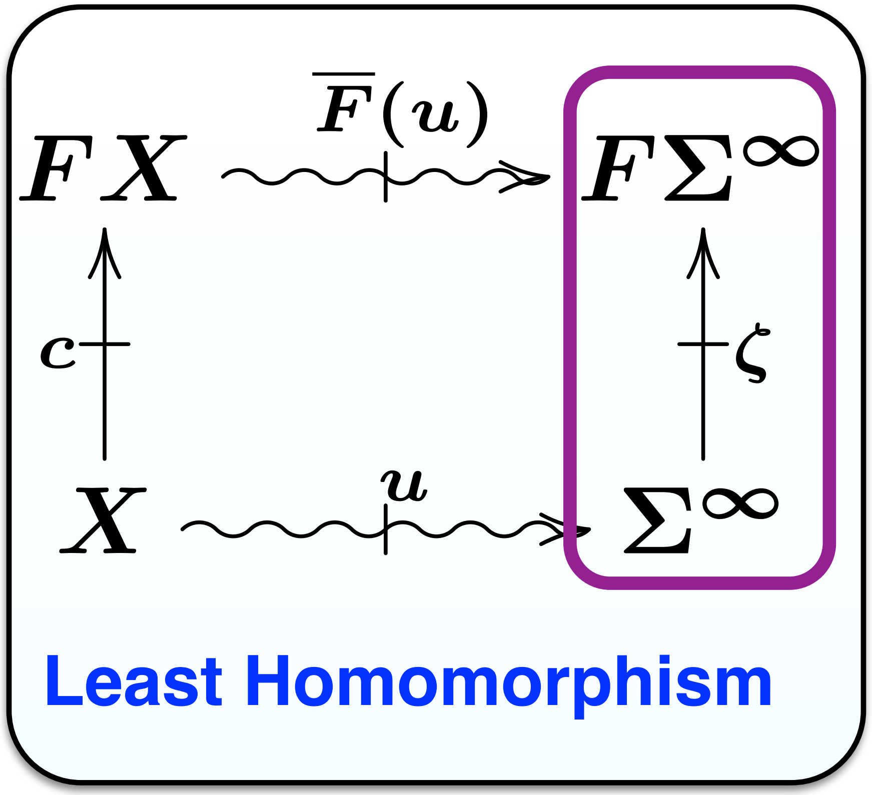
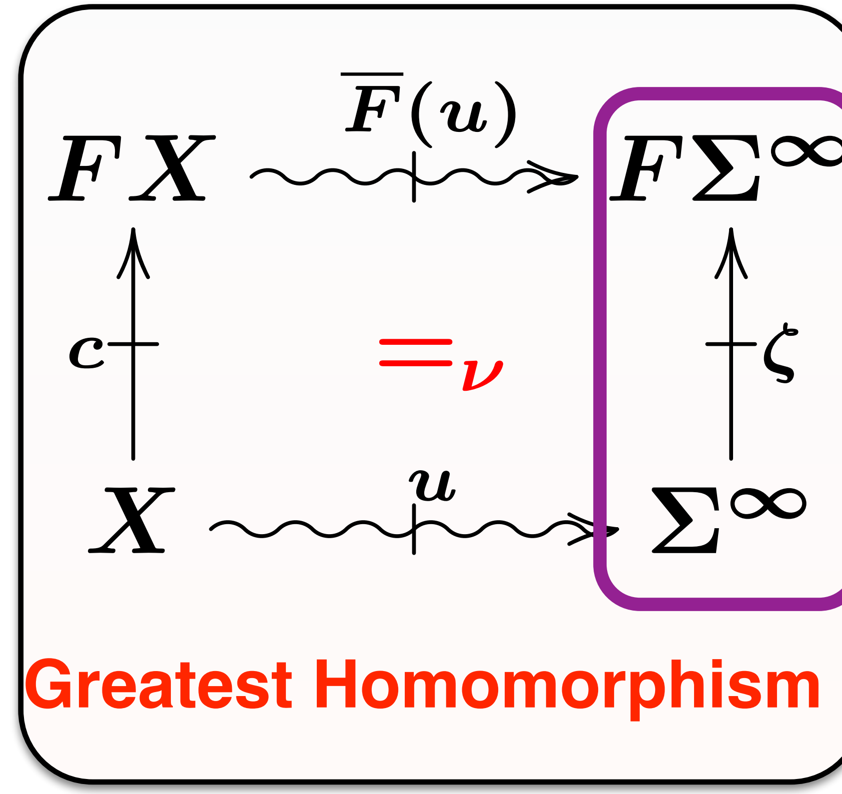
Infinitary Trace

Between the Least and Greatest





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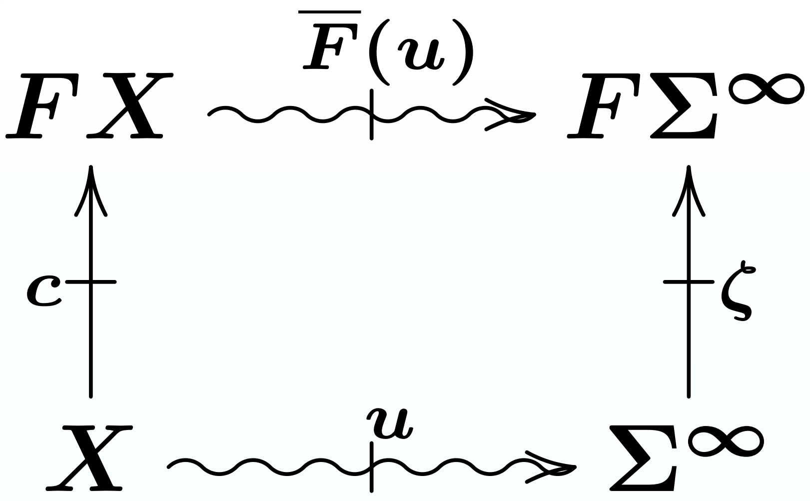
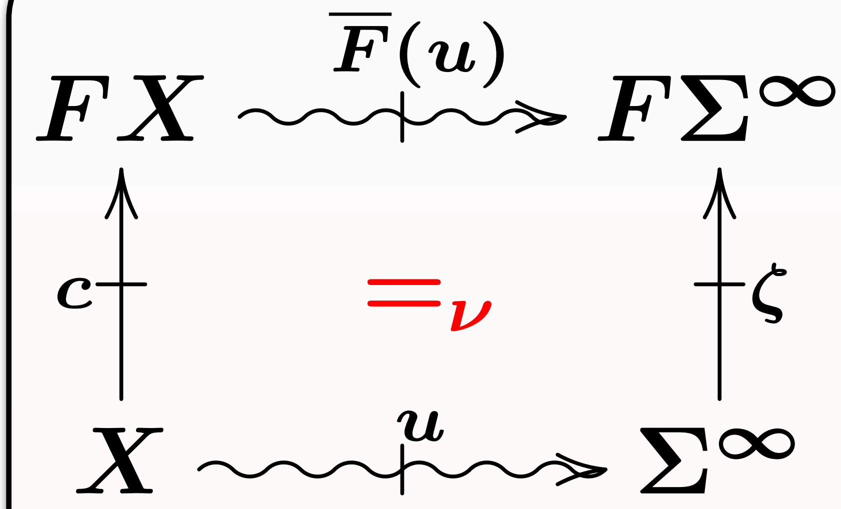


Least Homomorphism



Greatest Homomorphism

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Finite Trace

$$L^\infty(\mathcal{A}) \in \mathcal{P}(\Sigma^\infty)^X$$

Infinitary Trace

Between the Least and Greatest

$$\begin{array}{ccc} FX & \overset{\overline{F}(u)}{\rightsquigarrow} & F\Sigma^{\infty} \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \zeta} \\ X & \overset{u}{\rightsquigarrow} & \Sigma^{\infty} \end{array}$$

Least Homomorphism

$$\begin{array}{ccc} FX & \overset{\overline{F}(u)}{\rightsquigarrow} & F\Sigma^{\infty} \\ {\scriptstyle c}\uparrow & =_{\nu} & \uparrow{\scriptstyle \zeta} \\ X & \overset{u}{\rightsquigarrow} & \Sigma^{\infty} \end{array}$$

Greatest Homomorphism

⇩

$$L(\mathcal{A}) \in \mathcal{P}(\Sigma^{*})^{X}$$

Finite Trace

(**No** infinite word)

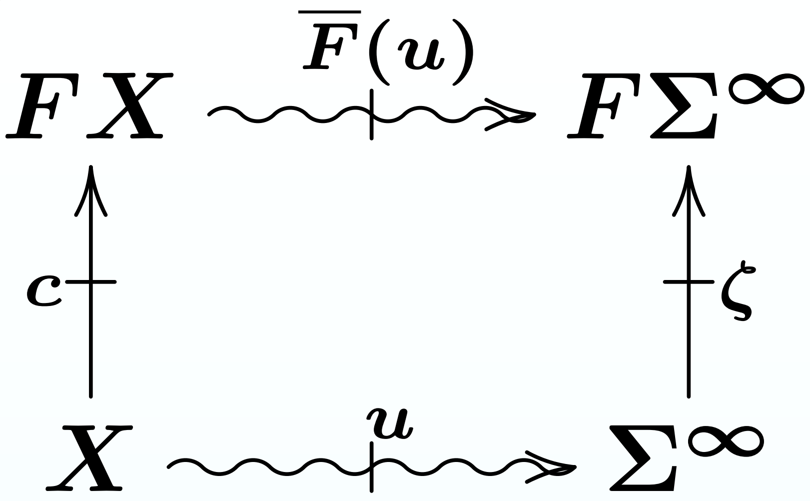
⇩

$$L^{\infty}(\mathcal{A}) \in \mathcal{P}(\Sigma^{\infty})^{X}$$

Infinitary Trace

(**All** infinite words)

Between the Least and Greatest



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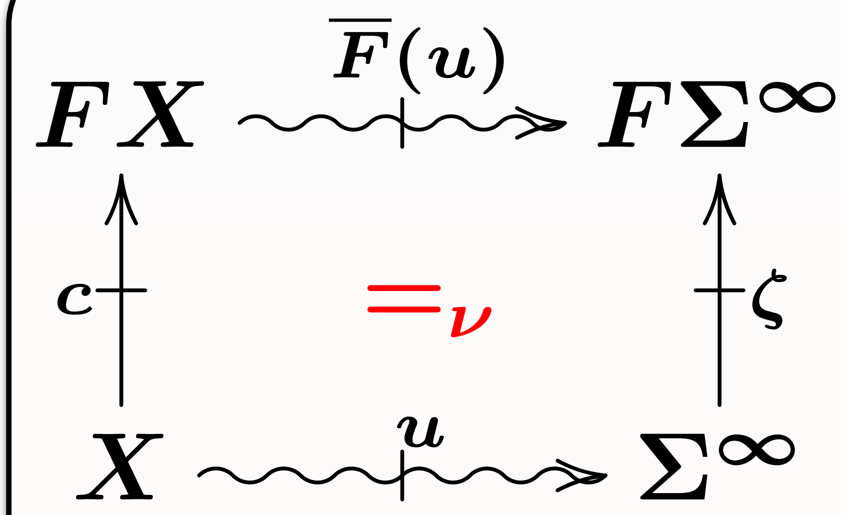

Greatest Homomorphism

$L(\mathcal{A}) \in \mathcal{P}(\Sigma^*)^X$	$L^p(\mathcal{A}) \in \mathcal{P}(\Sigma^\omega)^X$	$L^\infty(\mathcal{A}) \in \mathcal{P}(\Sigma^\infty)^X$
Finite Trace	**Parity Language**	**Infinitary Trace**
(**No** infinite word)	(**Accepted** infinite words)	(**All** infinite words)

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Intermediate

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Parity Language

(**Accepted** infinite words)

$$L^\infty(\mathcal{A}) \in \mathcal{P}(\Sigma^\infty)^X$$

Infinitary Trace

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Coalgebraic Modeling of Parity Automaton

parity automaton $\mathcal{A} = (X, \Sigma, \delta, p)$

$c : X \nrightarrow \Sigma \times X$ in $\mathcal{K}\ell(\mathcal{P})$ and

$$X = X_1 + \cdots + X_{2n}$$

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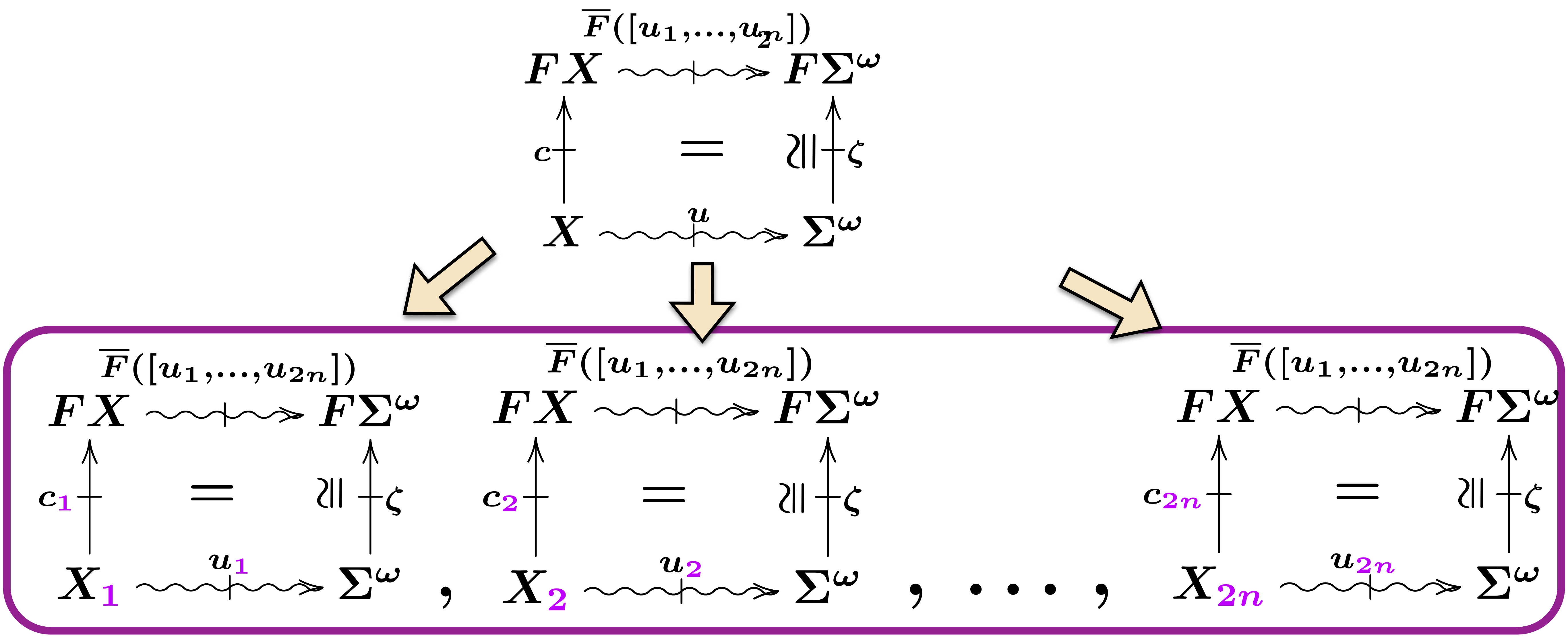

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Coalgebraic Modeling of Parity Automaton

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Solution of System of Diagrams

$$
\begin{array}{ccc} FX & \xrightarrow{\overline{F}([u_1,\dots,u_{2n}])} & F\Sigma^\omega \\ {\scriptstyle c_1}\uparrow & =_\mu & \cong\ \uparrow{\scriptstyle \zeta} \\ X_1 & \xrightarrow{u_1} & \Sigma^\omega \end{array}
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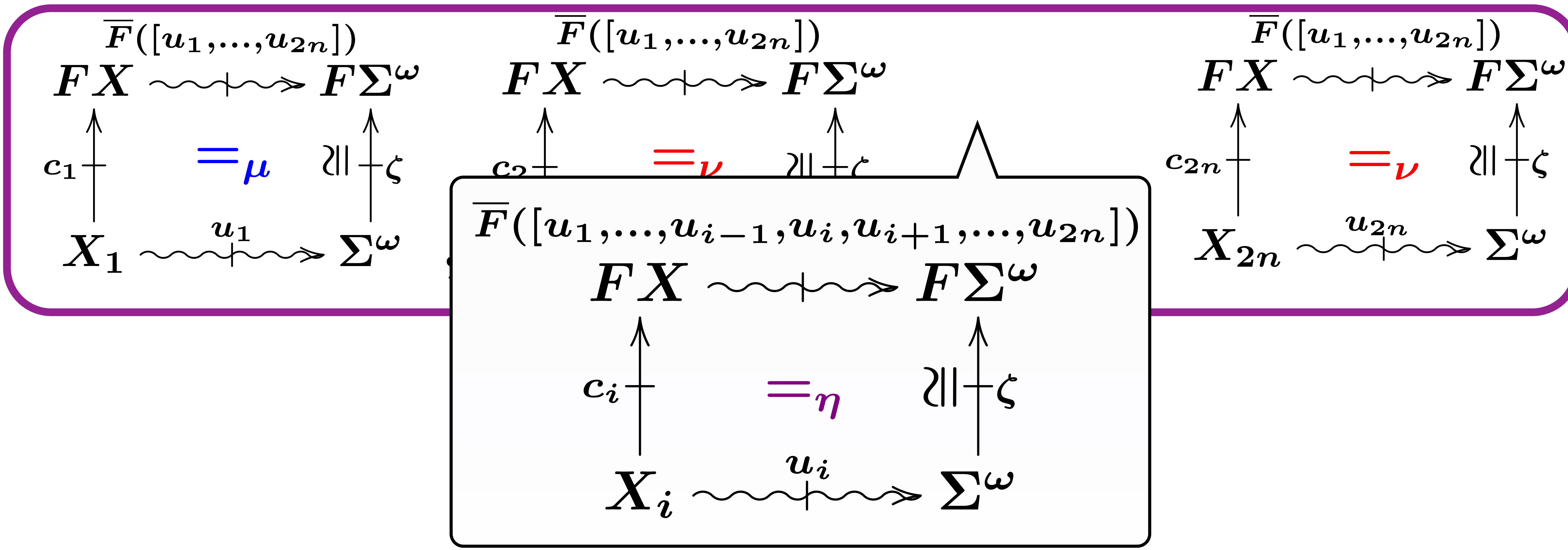
$$

Solution of System of Diagrams



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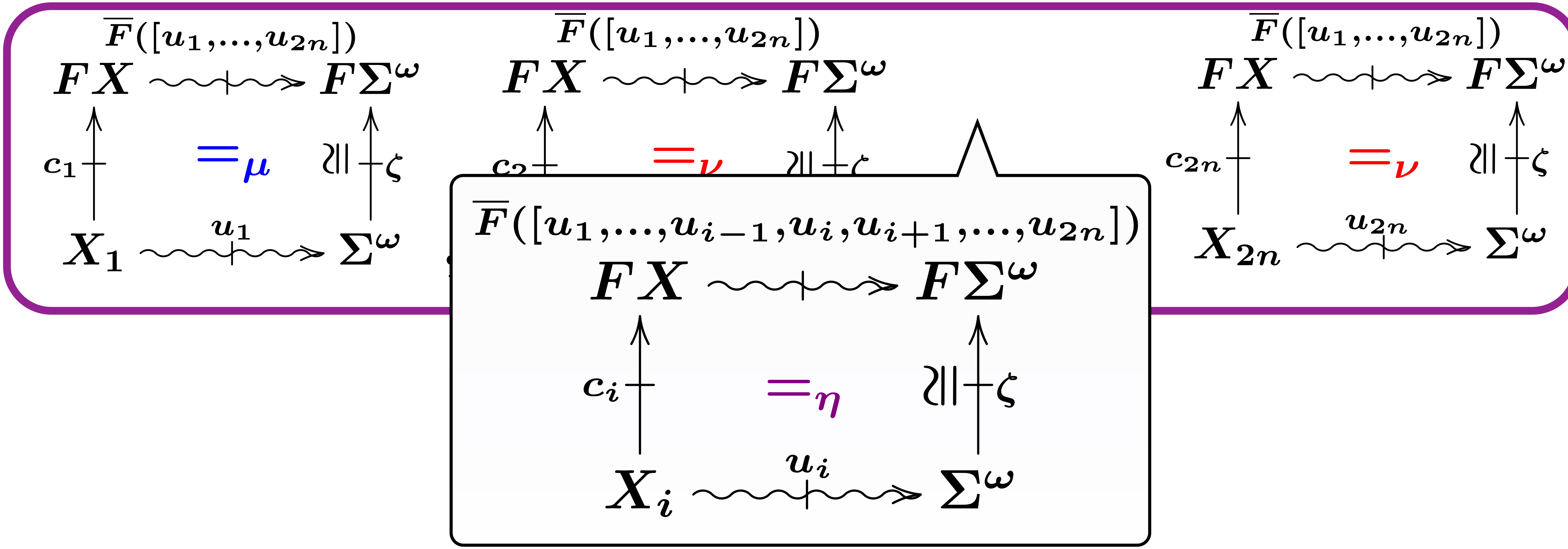
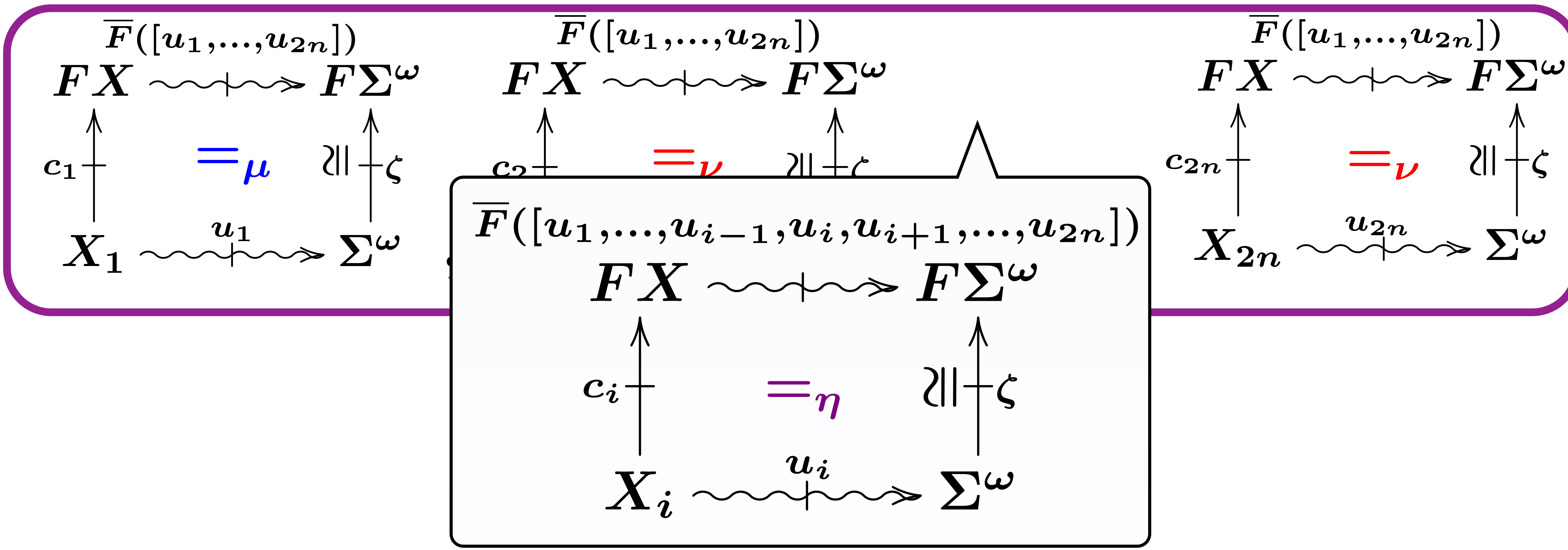


- We solve from the left to the right

Solution of System of Diagrams



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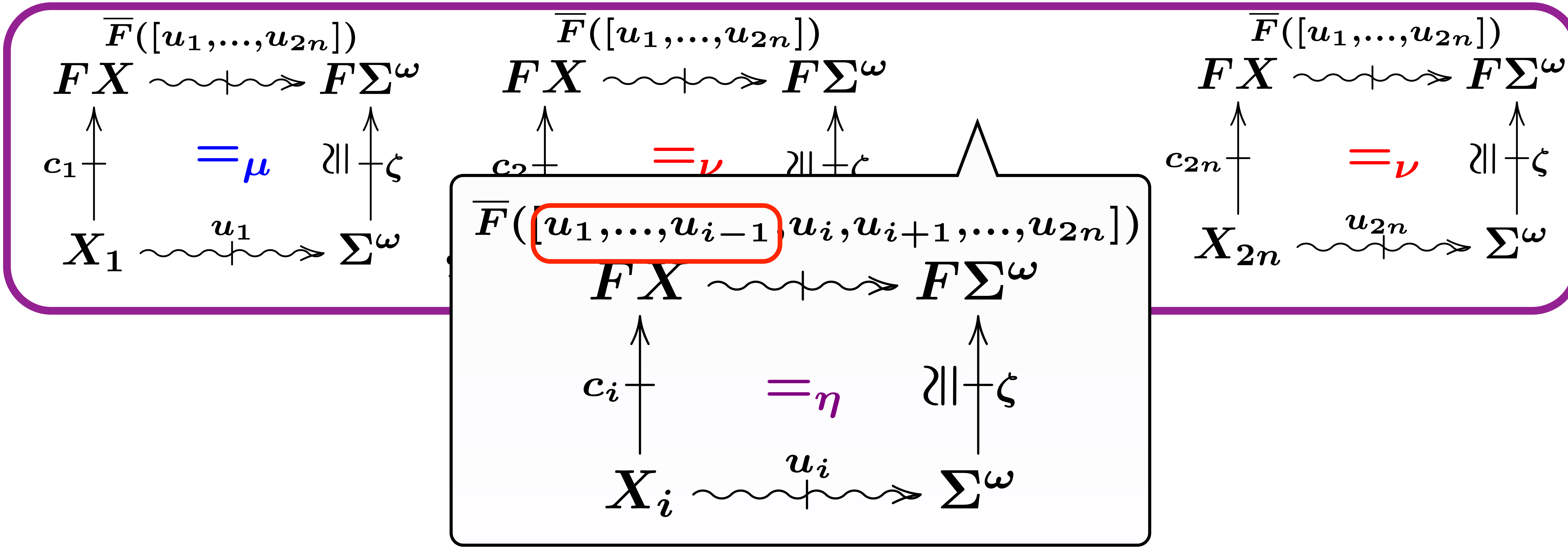
Solution of System of Diagrams



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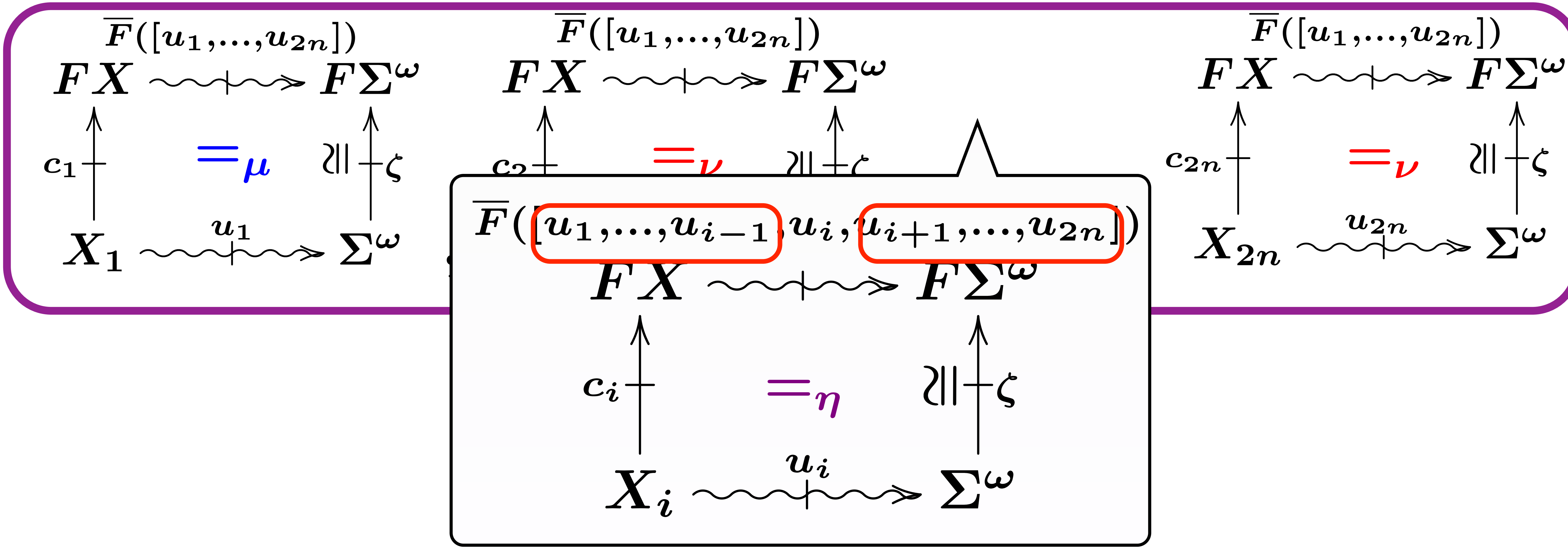
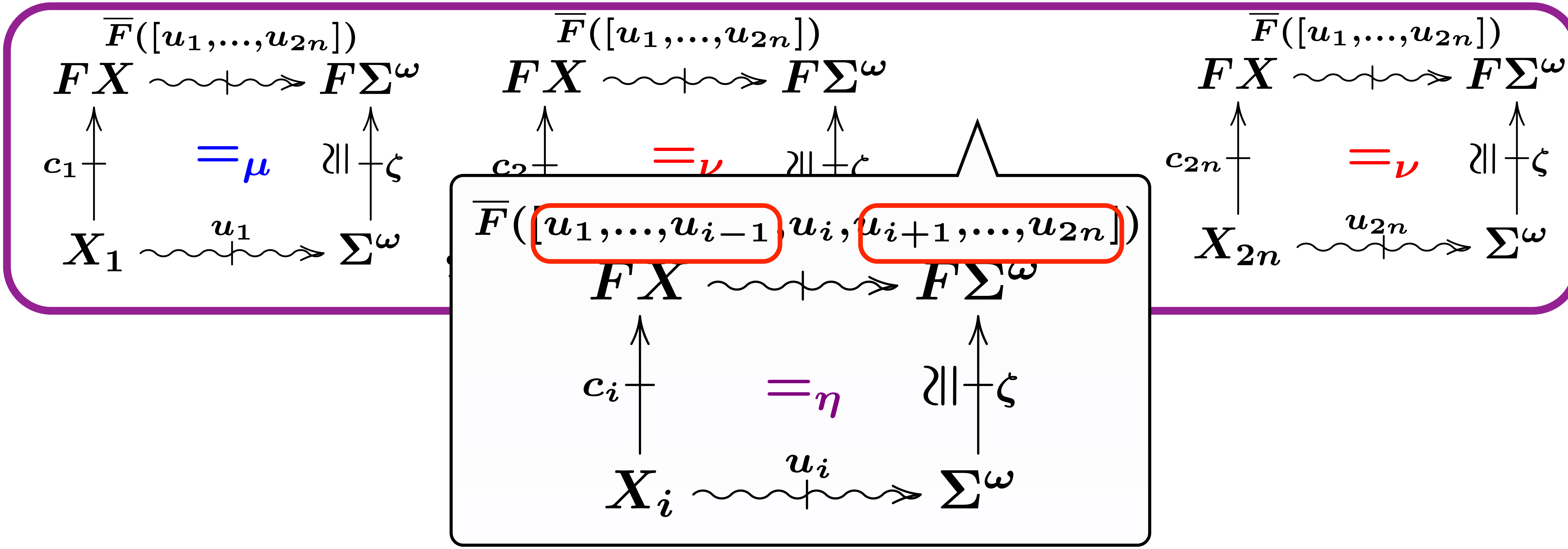
- To solve the i'th diagram,
  - substitute $u_1, \ldots, u_{i-1}$ by the solutions

Solution of System of Diagrams



- We solve from the left to the right
- To solve the i'th diagram,
  - substitute $u_1, \ldots, u_{i-1}$ by the solutions
  - regard $u_{i+1}, \ldots, u_{2n}$ as parameters

Solution of System of Diagrams



- We solve from the left to the right
- To solve the i'th diagram,
  - substitute $u_1, \dots, u_{i-1}$ by the solutions
  - regard $u_{i+1}, \dots, u_{2n}$ as parameters
- c.f. [Cleaveland et al., CAV '92], [Arnold & Niwinski, '01]

“Sanity-check Result”

Thm:

For a parity automaton $\mathcal{A} = (X, \Sigma, \delta, p)$, we define $c : X \nrightarrow \Sigma \times X$ in $\mathcal{K}\ell(\mathcal{P})$ and $X_1 + \cdots + X_{2n} = X$ by

$$c = \delta \quad \text{and} \quad X_i := p^{-1}(i) \qquad .$$

Let $u_1^{\mathrm{sol}}, \ldots, u_n^{\mathrm{sol}}$ be the solution of the following system.

$$\begin{array}{ccc} FX & \xrightarrow{\overline{F}([u_1,\ldots,u_{2n}])} & F\Sigma^\omega \\ {\scriptstyle c_1}\uparrow & =_\mu & \cong\uparrow{\scriptstyle \zeta} \\ X_1 & \xrightarrow{u_1} & \Sigma^\omega \end{array} \; , \; \begin{array}{ccc} FX & \xrightarrow{\overline{F}([u_1,\ldots,u_{2n}])} & F\Sigma^\omega \\ {\scriptstyle c_2}\uparrow & =_\nu & \cong\uparrow{\scriptstyle \zeta} \\ X_2 & \xrightarrow{u_2} & \Sigma^\omega \end{array} \; , \; \ldots \; , \; \begin{array}{ccc} FX & \xrightarrow{\overline{F}([u_1,\ldots,u_{2n}])} & F\Sigma^\omega \\ {\scriptstyle c_{2n}}\uparrow & =_\nu & \cong\uparrow{\scriptstyle \zeta} \\ X_{2n} & \xrightarrow{u_{2n}} & \Sigma^\omega \end{array}$$

Then we have:

$$\left[\, u_1^{\mathrm{sol}}, \ldots, u_{2n}^{\mathrm{sol}} \,\right] = L^p(\mathcal{A})$$

Function Φ_c

$$\Phi_c : \quad \{ f : X \nrightarrow \Sigma^\omega \} \;\to\; \{ f : X \nrightarrow \Sigma^\omega \}$$

$$\left(X \xrightarrow{\;f\;} \Sigma^\omega \right) \quad \longmapsto \quad \begin{array}{ccc} \overline{F}X & \xrightarrow{\;\overline{F}f\;} & \overline{F}\Sigma^\omega \\ \uparrow{\scriptstyle c} & & \cong \downarrow{\scriptstyle \zeta^{-1}} \\ X & & \Sigma^\omega \end{array}$$

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- f is a homomorphism $\Leftrightarrow$ f is a fixed point of Φ_c

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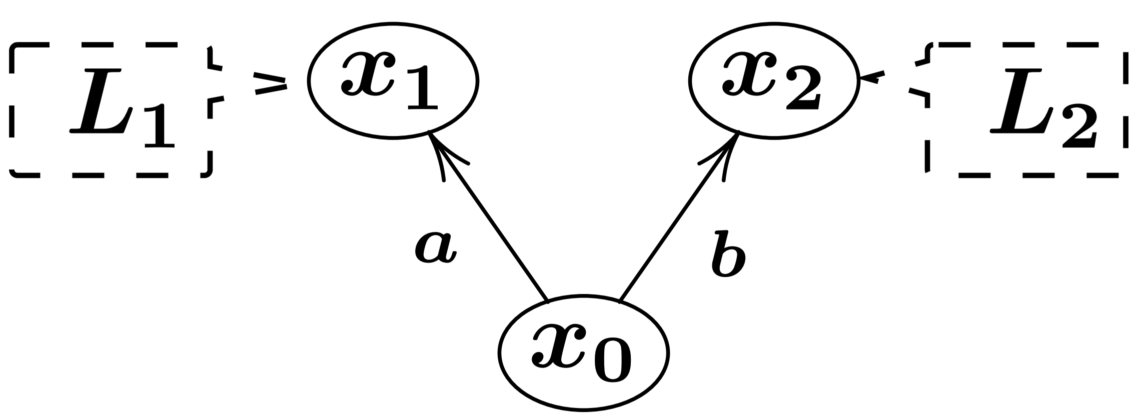

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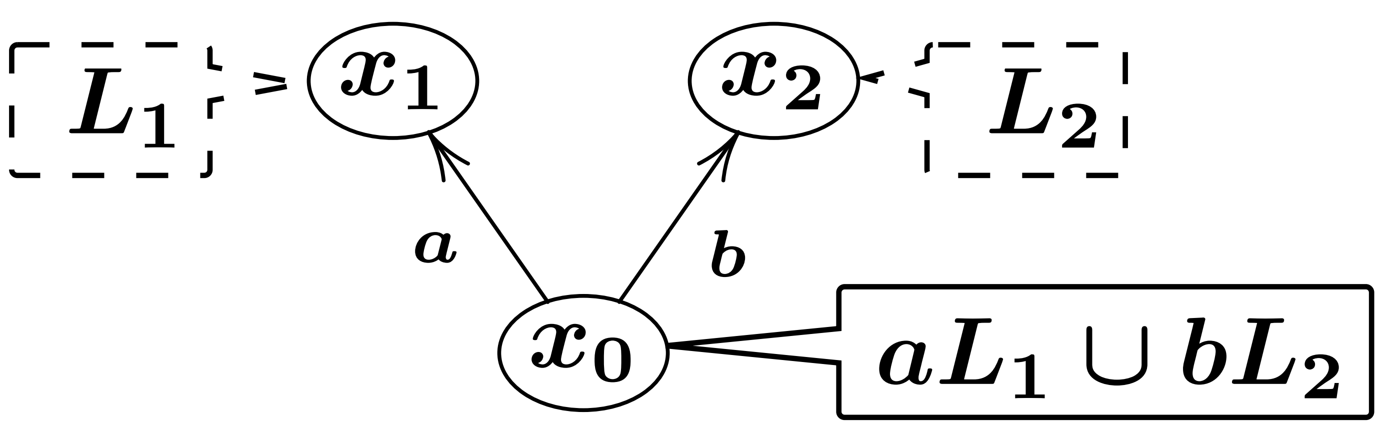

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$\Phi_c(f)$

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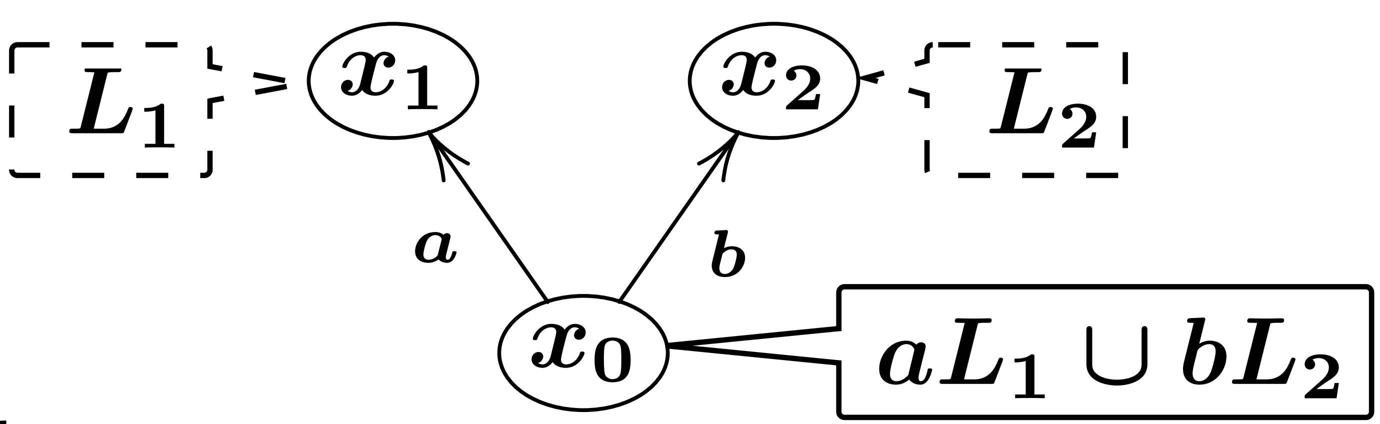

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$\Phi_c(f)$

- Φ_c is the one often denoted by $\Diamond_\delta : \mathcal{P}(\Sigma^\omega)^X \to \mathcal{P}(\Sigma^\omega)^X$

Fixed Point Semantics for Parity Automaton

$$
\Phi_c \quad
\begin{array}{ccc}
FX & \overset{\overline{F}([u_1,\dots,u_{2n}])}{\rightsquigarrow} & F\Sigma^\omega \\
{\scriptstyle c_1}\uparrow & =_\mu & \cong \uparrow {\scriptstyle \zeta} \\
X_1 & \overset{u_1}{\rightsquigarrow} & \Sigma^\omega
\end{array}
\;,\;
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X_2 & \overset{u_2}{\rightsquigarrow} & \Sigma^\omega
\end{array}
\;,\; \dots \;,\;
\begin{array}{ccc}
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X_{2n} & \overset{u_{2n}}{\rightsquigarrow} & \Sigma^\omega
\end{array}
$$

$$
\downarrow \qquad \overline{\overline{}}
$$

$$
\Diamond_\delta
$$

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$$\downarrow$$

$$\Diamond_\delta \begin{cases} u_1 =_\mu \Diamond_\delta([u_1, u_2, \ldots, u_{2n}]) \restriction_{X_1} & \in \mathcal{P}(\Sigma^\omega)^{X_1} \\ u_2 =_\nu \Diamond_\delta([u_1, u_2, \ldots, u_{2n}]) \restriction_{X_2} & \in \mathcal{P}(\Sigma^\omega)^{X_2} \\ \vdots & \\ u_{2n} =_\nu \Diamond_\delta([u_1, u_2, \ldots, u_{2n}]) \restriction_{X_{2n}} & \in \mathcal{P}(\Sigma^\omega)^{X_{2n}} \end{cases}$$

Equational System for Parity Automaton

Thm:

The solution of the following equational system characterizes **parity** language

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c.f.

$$\nu u_2.\Big(\mu u_1.\,(\Diamond_\delta u_1 \vee (F \wedge \Diamond_\delta u_2))\Big)$$ for Büchi

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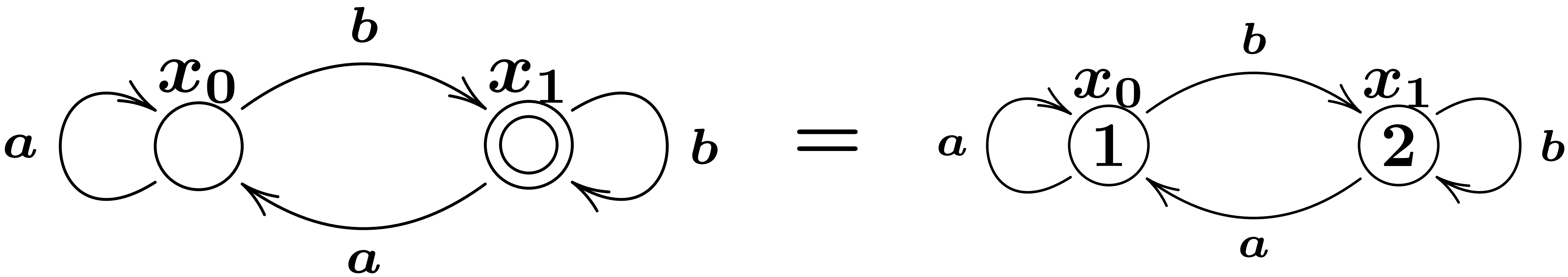
For Büchi Automata

- Büchi automaton is a **special case** of parity automaton



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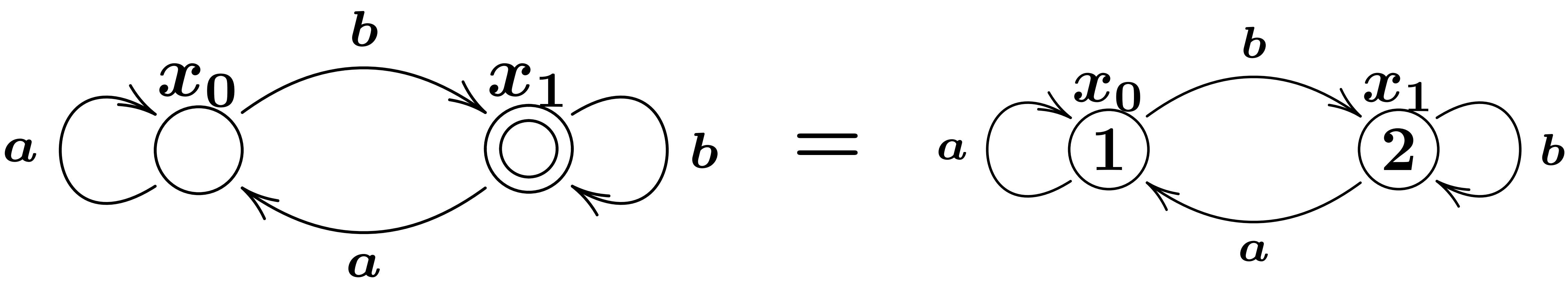
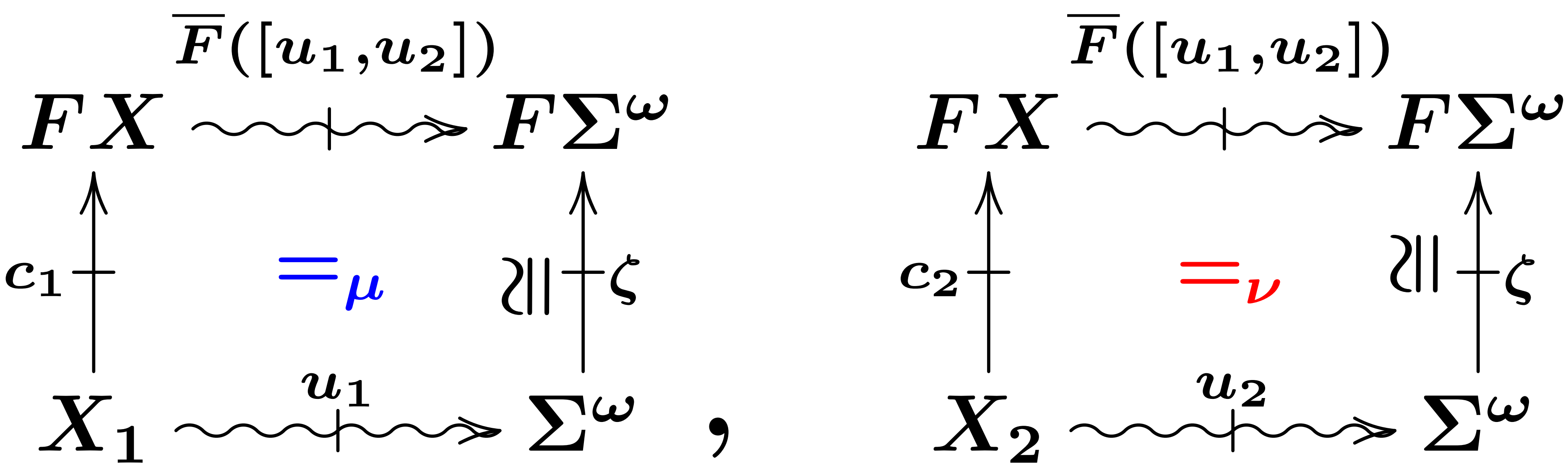
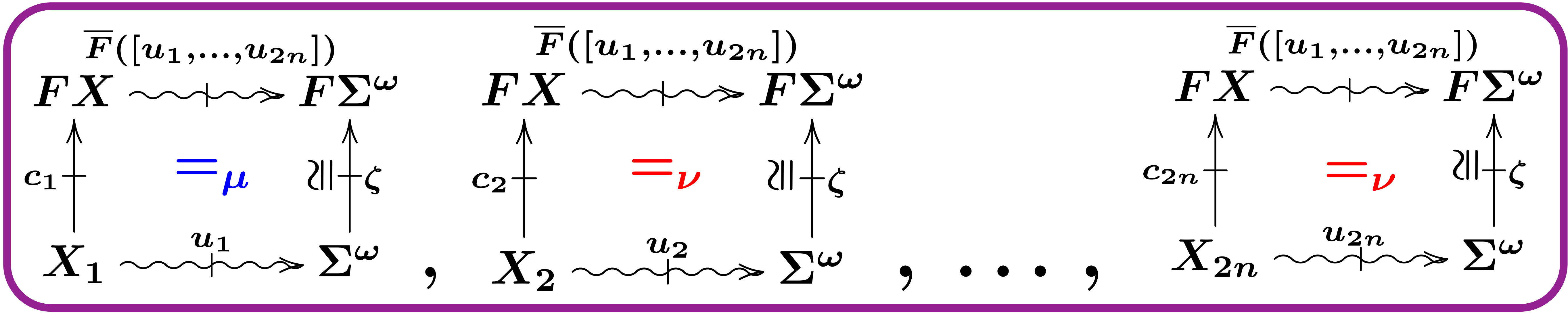


- Coalgebraic trace semantics is given by **two diagrams**



Extension to Various Systems

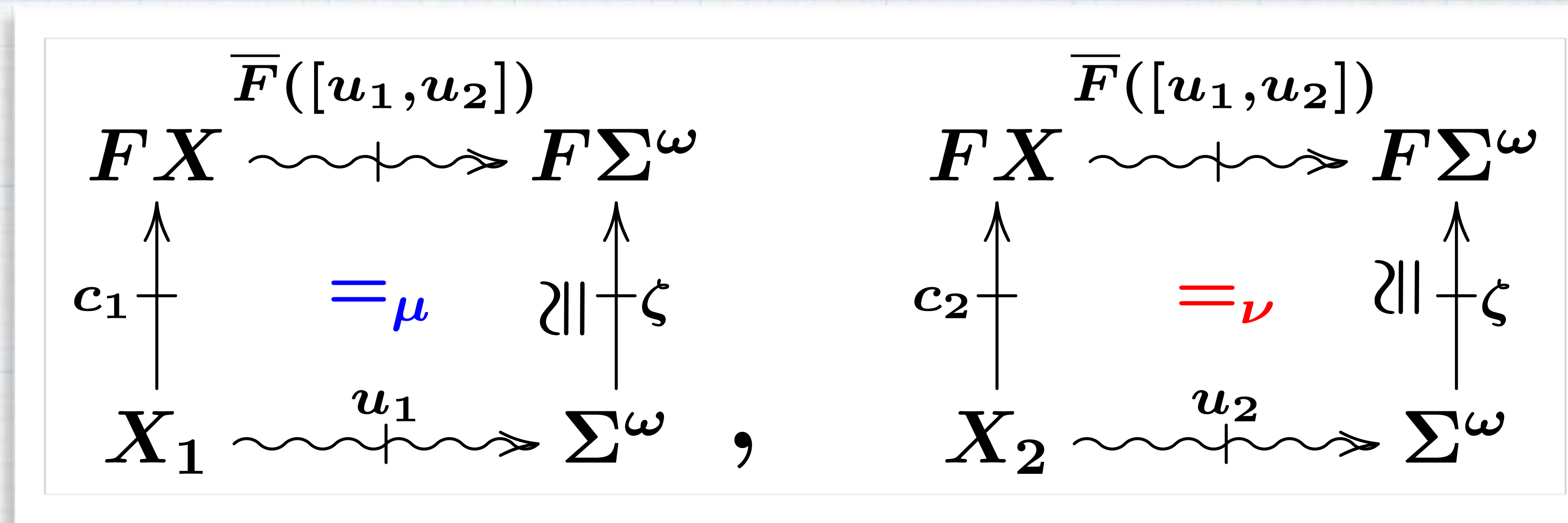


- $F = \Sigma \times (\_)$ ➡ $F = \coprod_i \Sigma_i \times (\_)^i$ (polynomial functor)
  - **Words** to **Trees**

- $T = \mathcal{P}$ ➡ $T = \mathcal{G}$ (the sub-Giry monad)
  - **Nondeterministic** to (generative) **Probabilistic**

Summary: Coalgebraic Modeling of Buechi and Parity Acceptance [CONCUR'16]

- Depart from "unique homomorphism"
  ➜ equational system



- So what? ➜ Coalgebraic fair simulations [LMCS'17]
- Based on the Kleisli approach (Kl(P), Kl(D), ...)
- Related work:
  - the Eilenberg-Moore approach
    [Silva, Bonchi, Bonsangue, Rot, Rutten, ...]
  - Buechi modeled in Sets^2 [Ciancia & Venema, CMCS'12]

Outline

* Least, greatest and alternating fixed points: a foundational view [POPL'16]
* Buechi and parity acceptance conditions in coalgebras [CONCUR'16]
* Categorical ranking functions by corecursive algebras [LICS'17]

Motivation

ranking function
&
soundness theorem

nondeterministic system

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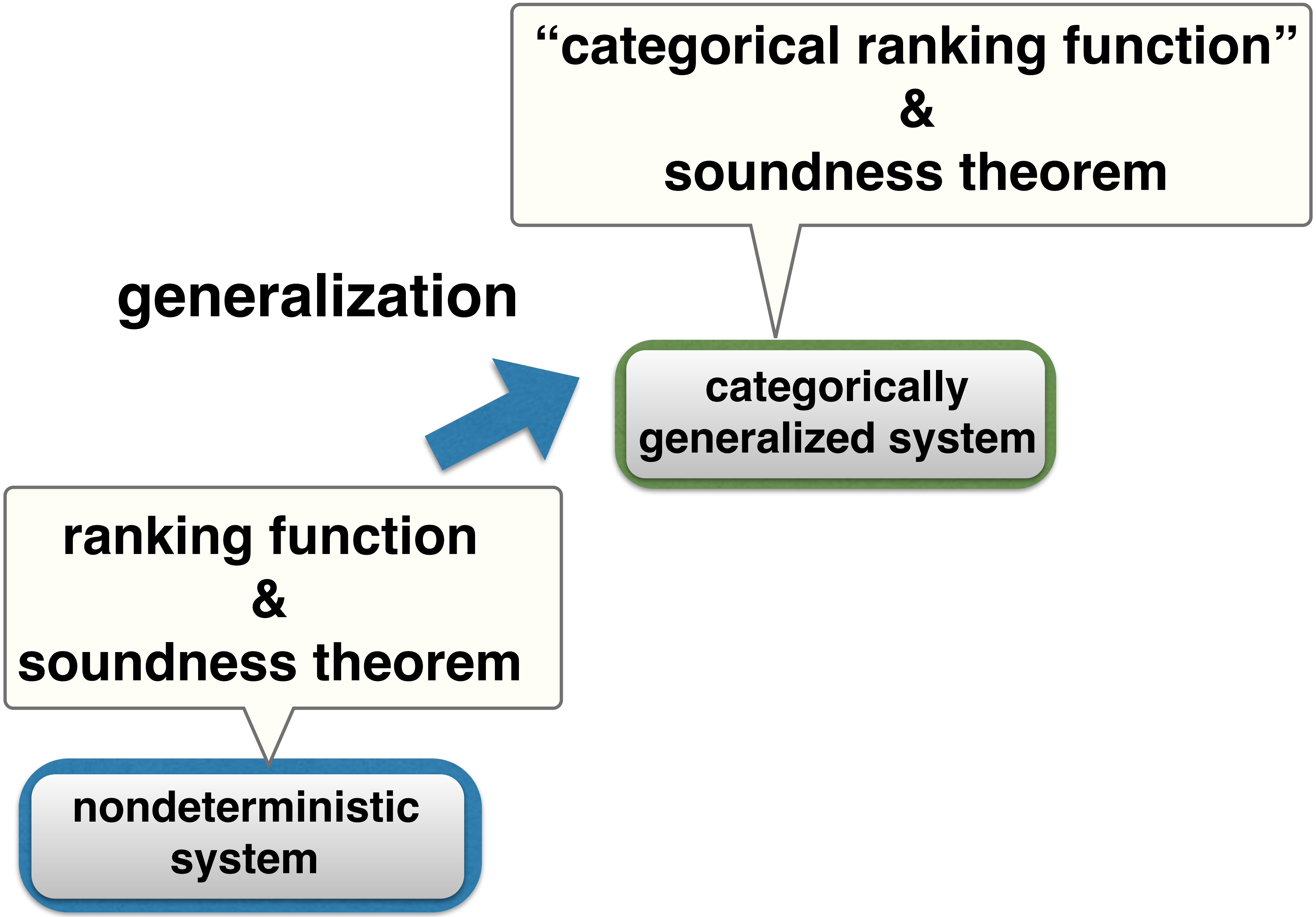

“categorical ranking function”
&
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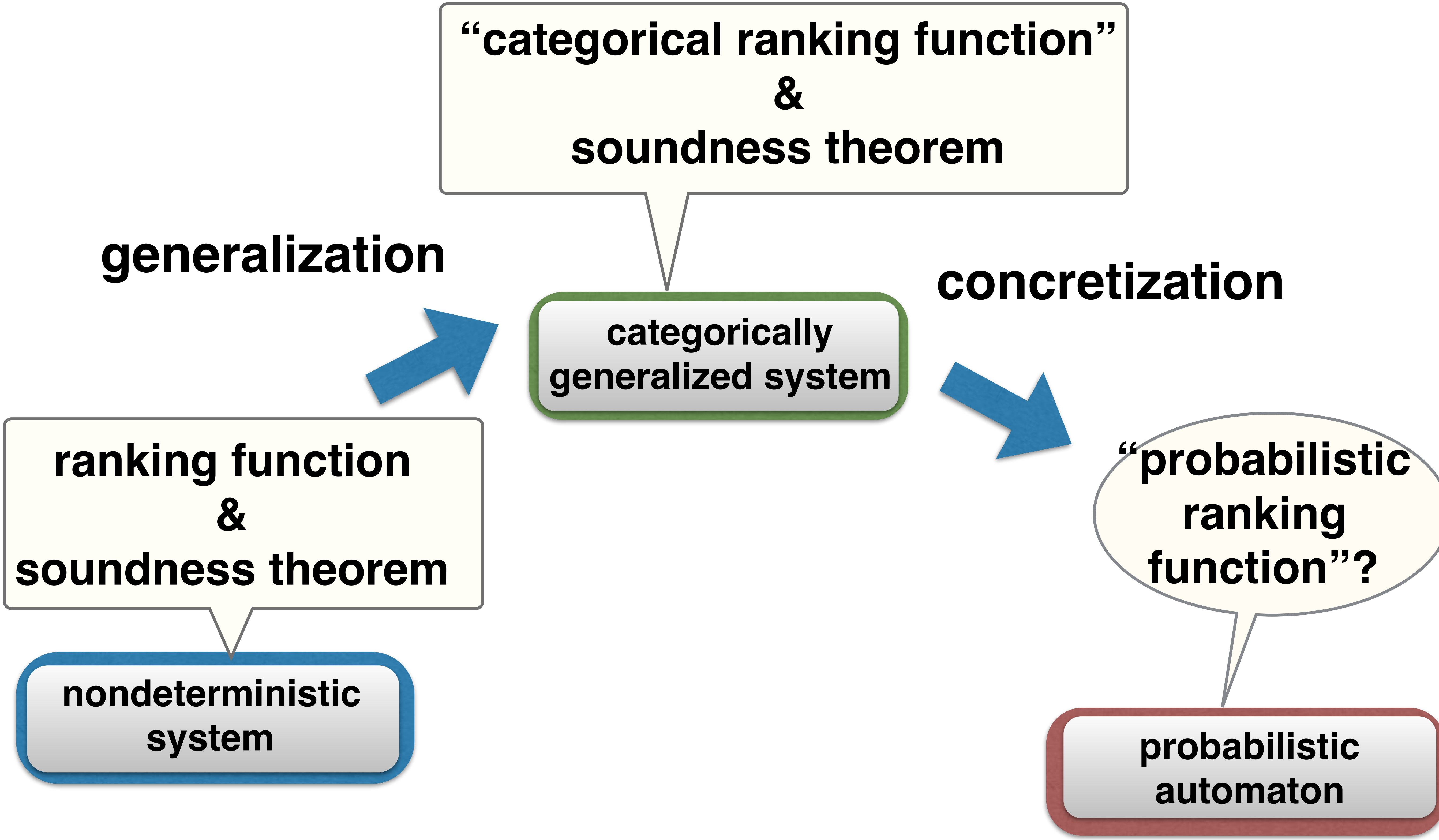
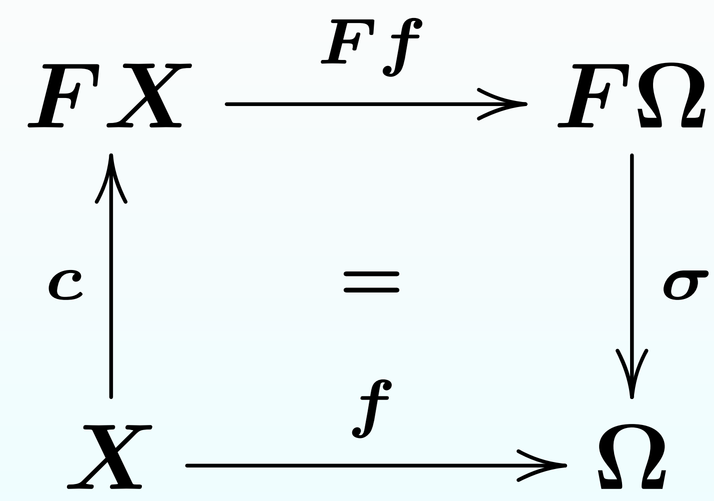


“categorical ranking function”
&
soundness theorem
generalization
concretization
categorically generalized system
ranking function
&
soundness theorem
“probabilistic ranking function”?
nondeterministic system
probabilistic automaton

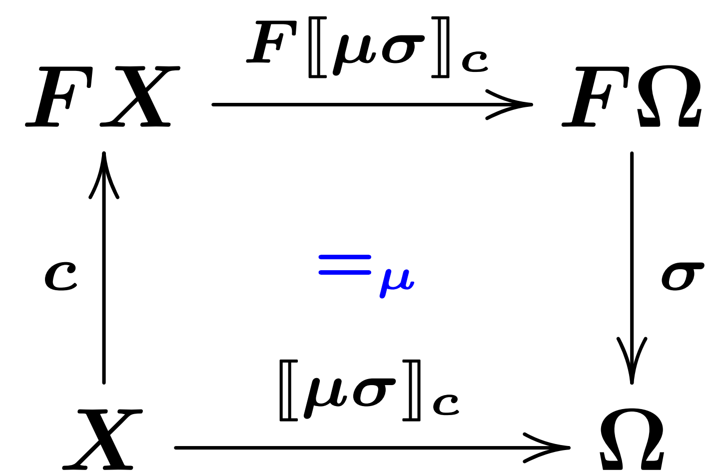
Coalgebra-Algebra Homomorphism

Def:

A ***coalgebra-algebra homomorphism*** from $c : X \to FX$ to $\sigma : F\Omega \to \Omega$ is a function $f : X \to \Omega$ s.t. $\sigma \circ Ff \circ c = f$



- Especially, the least **coalgebra-algebra homomorphism** $[\![\mu\sigma]\!]_c : X \to \Omega$ captures **reachability** of various systems



- Example:
  For nondeterministic systems, there exists $\sigma : F\{0,1\} \to \{0,1\}$ s.t.
  $[\![\mu\sigma]\!]_c(x) = 1 \iff$ an accepting state is reachable from x

Modality as an algebra
FΩ→Ω
(pred. lifting via Yoneda)

Coalgebra-Algebra Homomorphism is Fixed Point

- We define $\Phi_{c,\sigma} : \Omega^X \to \Omega^X$ by

$$\Phi_{c,\sigma} : \; X \xrightarrow{f} \Omega \;\; \longmapsto \;\; \begin{array}{ccc} FX & \xrightarrow{Ff} & F\Omega \\ {\scriptstyle c}\uparrow & & \downarrow{\scriptstyle \sigma} \\ X & & \Omega \end{array}$$

Prop:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & F\Omega \\ {\scriptstyle c}\uparrow & = & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{f} & \Omega \end{array} \;\; \Leftrightarrow \;\; f \text{ is a fixed point of } \Phi_{c,\sigma}$$

- It is known that reachability of various systems is characterized as the least **fixed point**

⇒ reachability as the
least **coalgebra-algebra homomorphism**

Categorical Ranking Function

Def:

A *ranking domain* wrt. $\boldsymbol{\sigma : F\Omega \to \Omega}$ is a triple

$$(\,\boldsymbol{r : FR \to R},\ \boldsymbol{q : R \to \Omega},\ \sqsubseteq_{\boldsymbol{R}}\,) \quad \text{s.t.}$$

1. $\boldsymbol{R}$ is a complete lattice and $\boldsymbol{\Phi_{c,r}}$ is monotone
2. $\boldsymbol{q}$ is monotone, $\bot$-preserving and continuous
3. $\boldsymbol{q \circ r \sqsubseteq \sigma \circ Fq}$ 4. $\boldsymbol{r}$ is corecursive

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An arrow $\boldsymbol{b : X \to R}$ is a *ranking arrow* wrt. $(\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})$ if:

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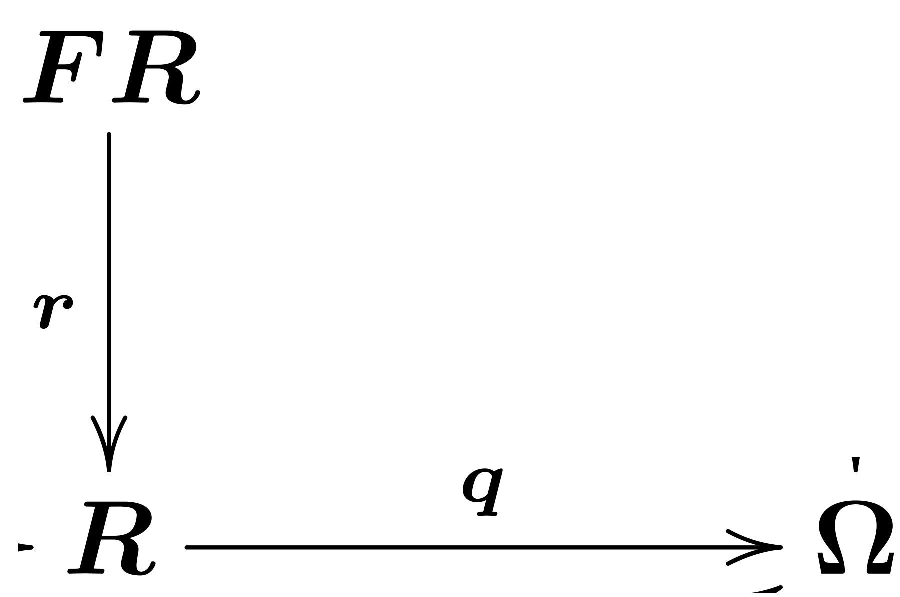

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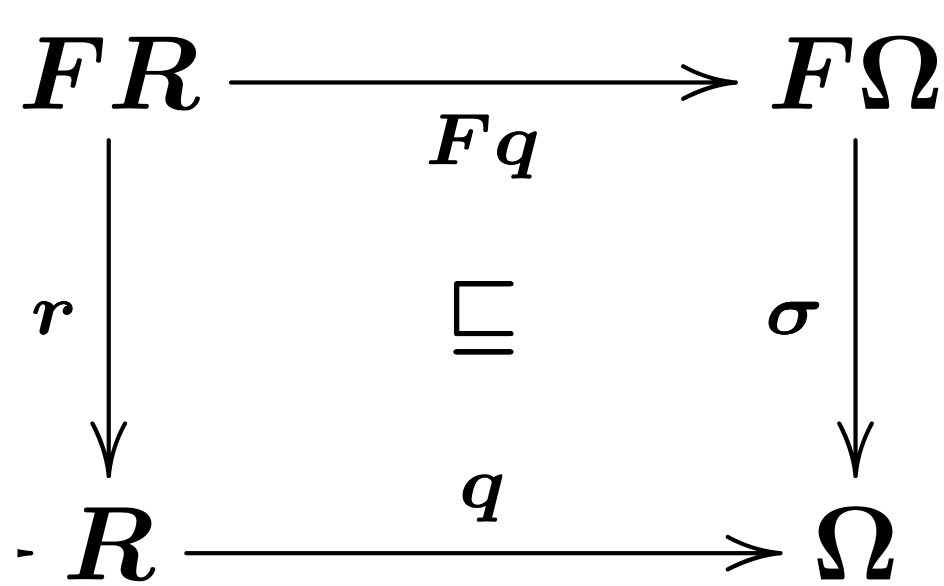

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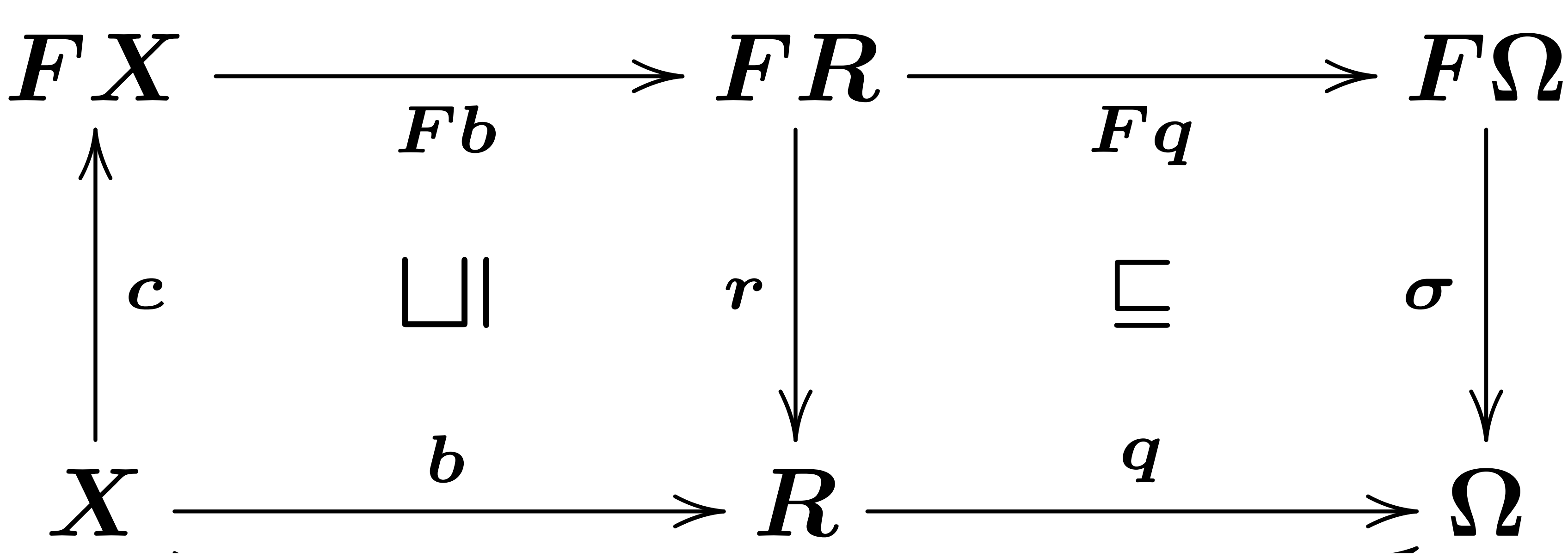

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“Well-fddness-aware refinement of truth values and modality”

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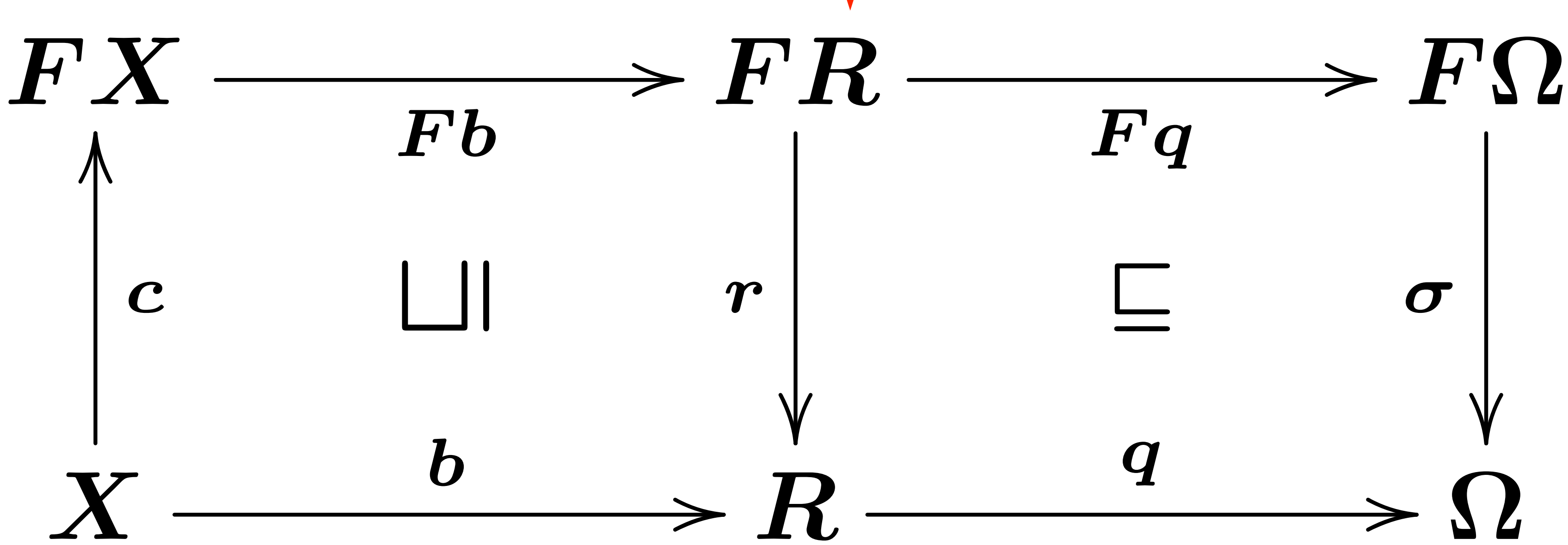

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ranking function as a "local" construct

"Well-fddness-aware refinement of truth values and modality"

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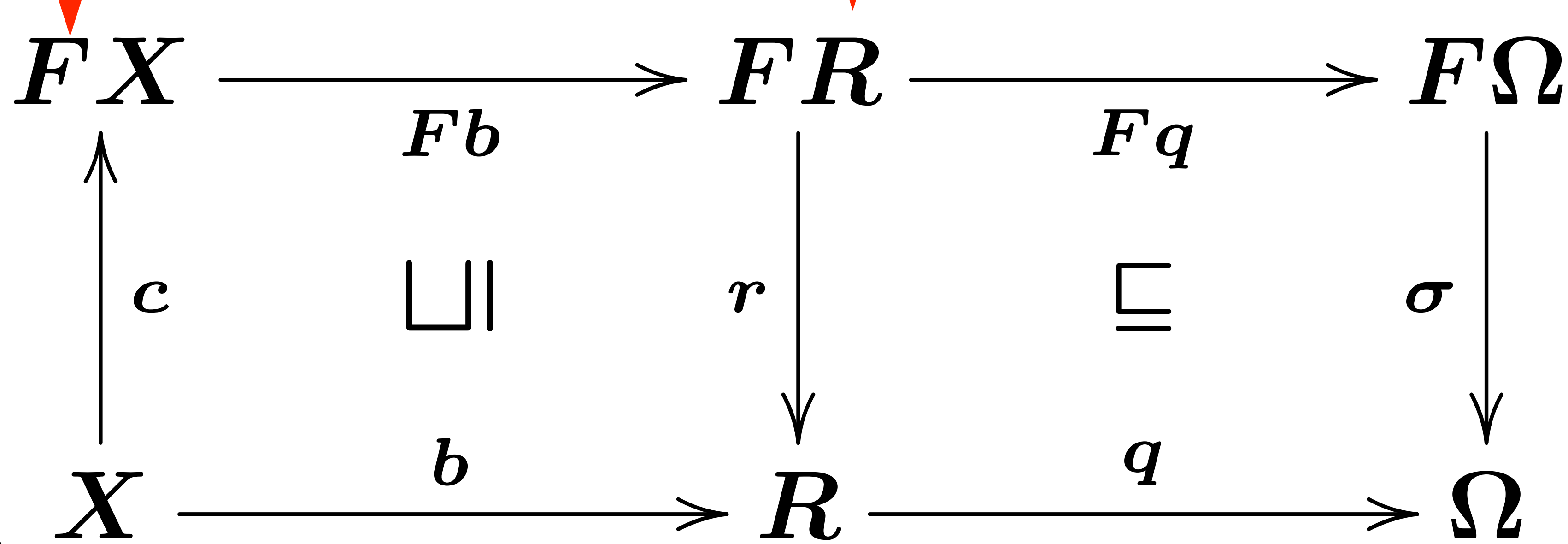

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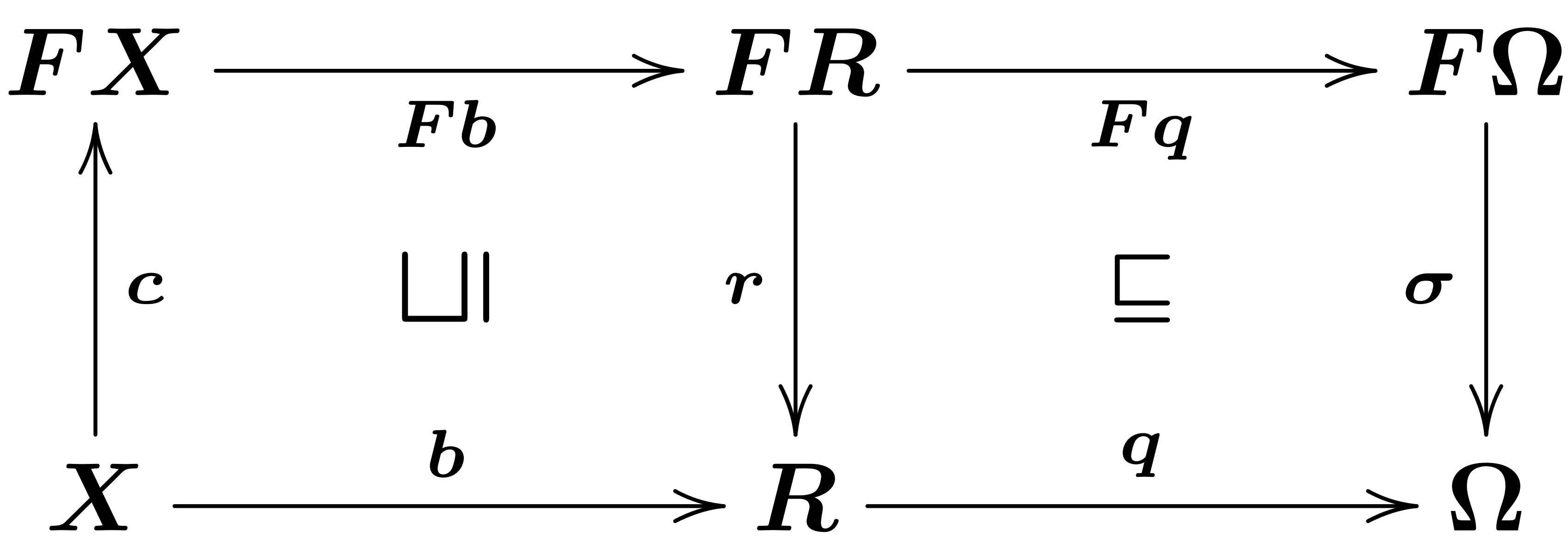

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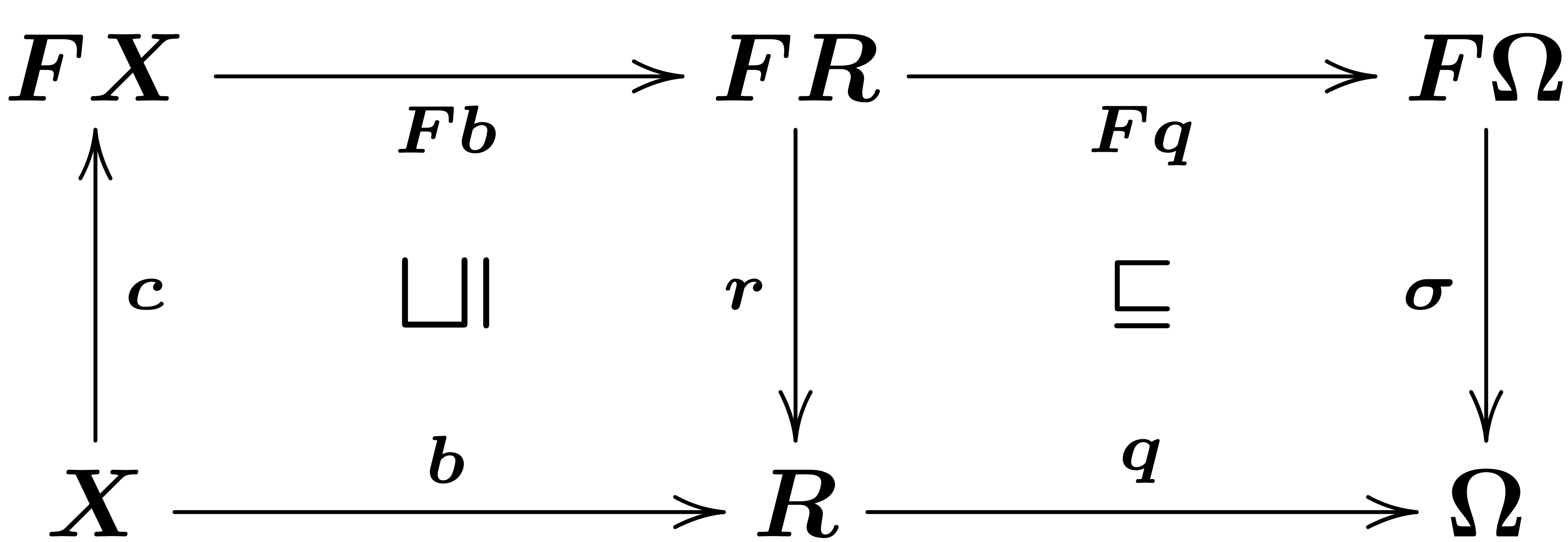


Corecursive Algebra

Def:

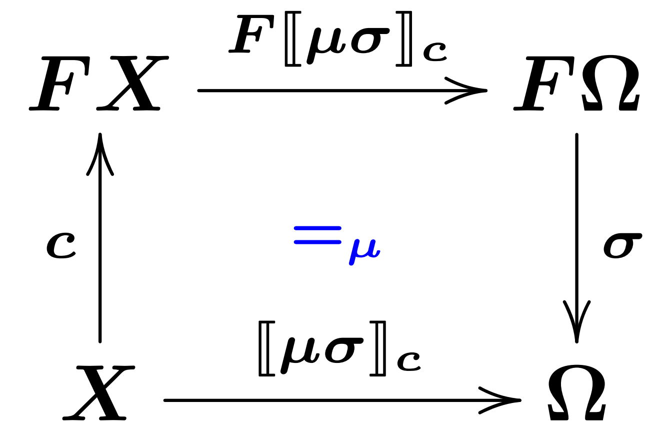
An algebra $r : FR \to R$ is **corecursive** if for all coalgebra $c : X \to FX$, a coalgebra-algebra homomorphism from c to r uniquely exists.

$$\begin{array}{ccc} FX & \overset{F(\!|r|\!)_c}{\dashrightarrow} & FR \\ c \uparrow & = & \downarrow r \\ X & \overset{(\!|r|\!)_c}{\dashrightarrow} & R \end{array}$$

- It has been used to ensure **productivity** of general structured corecursion [Capretta et al., SBMF '09]
- We use it to ensure **termination**

Intuition behind Corecursiveness

- We want to
  under-approximate the **least** **fixed point**



- The definition of ranking domain
  is "pre-fixed point like"

$$\min_{x \to x'} b(x') + 1 \le b(x)$$

⇒ It { over-approximates the least fixed point; or
under-approximates the greatest fixed point

⬇

we collapse the least and the greatest fixed points
into one point
(i.e. unique coalgebra-algebra homomorphism)

Categorical Soundness Theorem

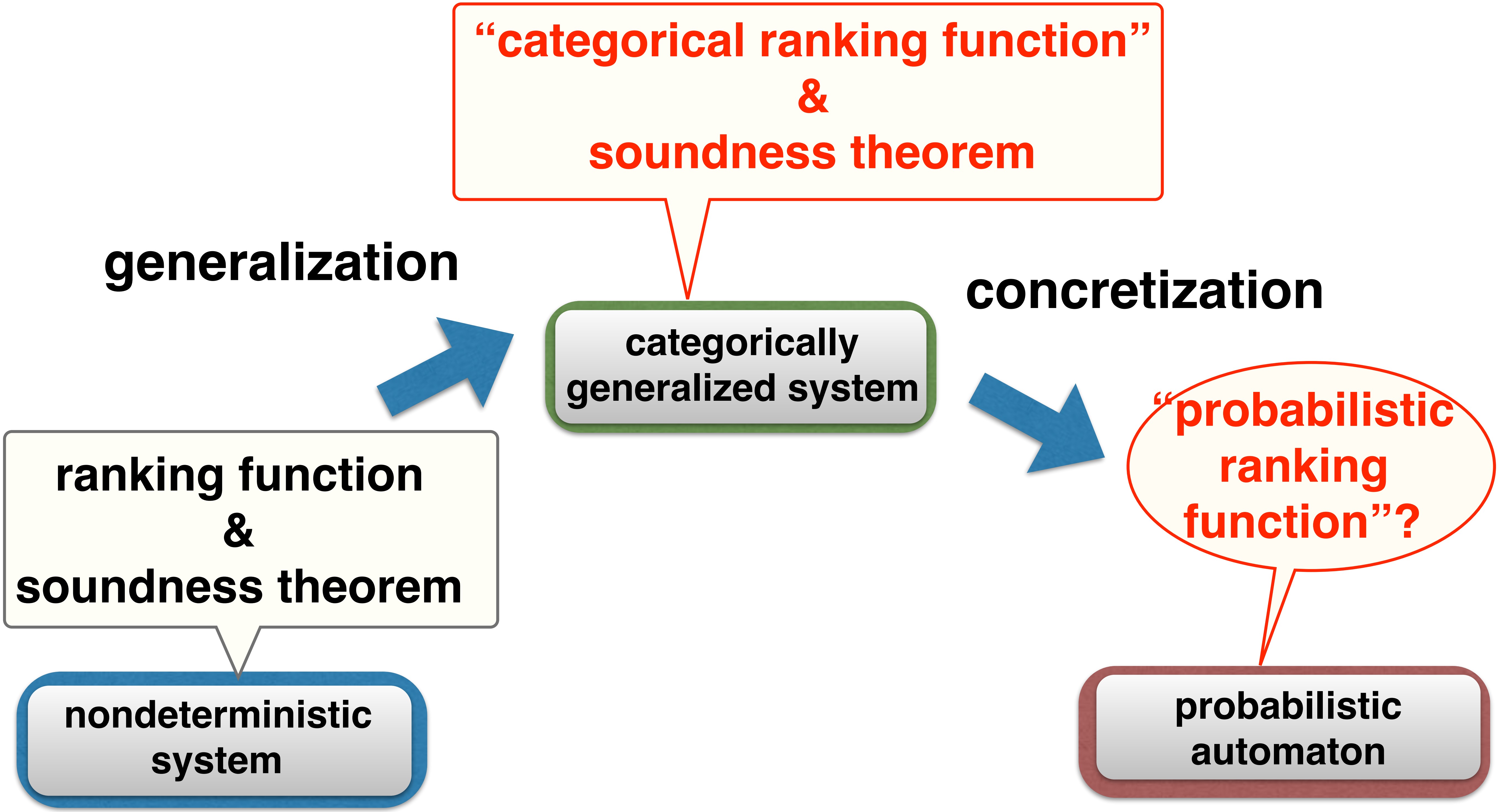
Thm: (see e.g. [Floyd, PSAM '67])

$\boldsymbol{b}$ is a ranking function $\Longrightarrow$ $\{\boldsymbol{x} \mid \boldsymbol{b}(\boldsymbol{x}) < \infty\} \subseteq \left\{ \boldsymbol{x} \;\middle|\; \begin{array}{r} \text{accepting states} \\ \text{reachable} \end{array} \right\}$

Thm (soundness):

$\boldsymbol{b}$ is a ranking arrow wrt. $(\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})$ $\Longrightarrow$ $\boldsymbol{q} \circ \boldsymbol{b} \sqsubseteq [\![\boldsymbol{\mu\sigma}]\!]_{\boldsymbol{c}}$

Concretization



$$c : X \to \mathcal{P}X \times \{0,1\}$$

$$c : X \to \mathcal{D}X \times \{0,1\}$$

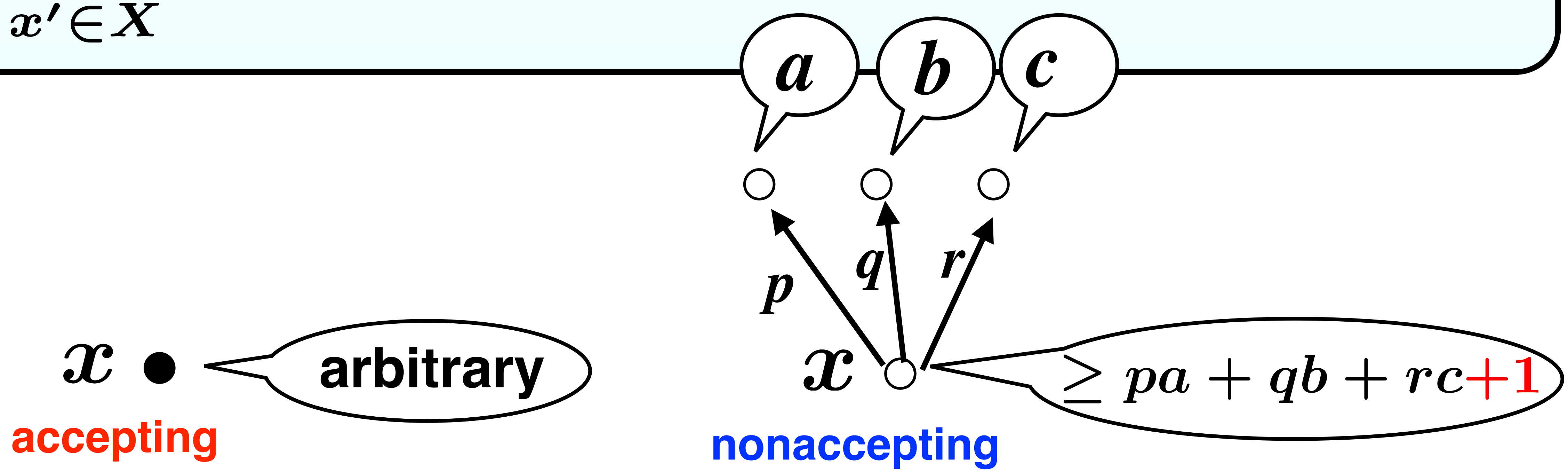
Ranking Supermartingale [Chakarov et al., '13]

- A method for checking almost-sure reachability on probabilistic systems

Def:

A function $b : X \to [0, \infty]$ is a **ranking supermartingale** if:

$$\sum_{x' \in X} \mathrm{Prob}(x \to x') \cdot b(x') + 1 \quad \leq \quad b(x)$$



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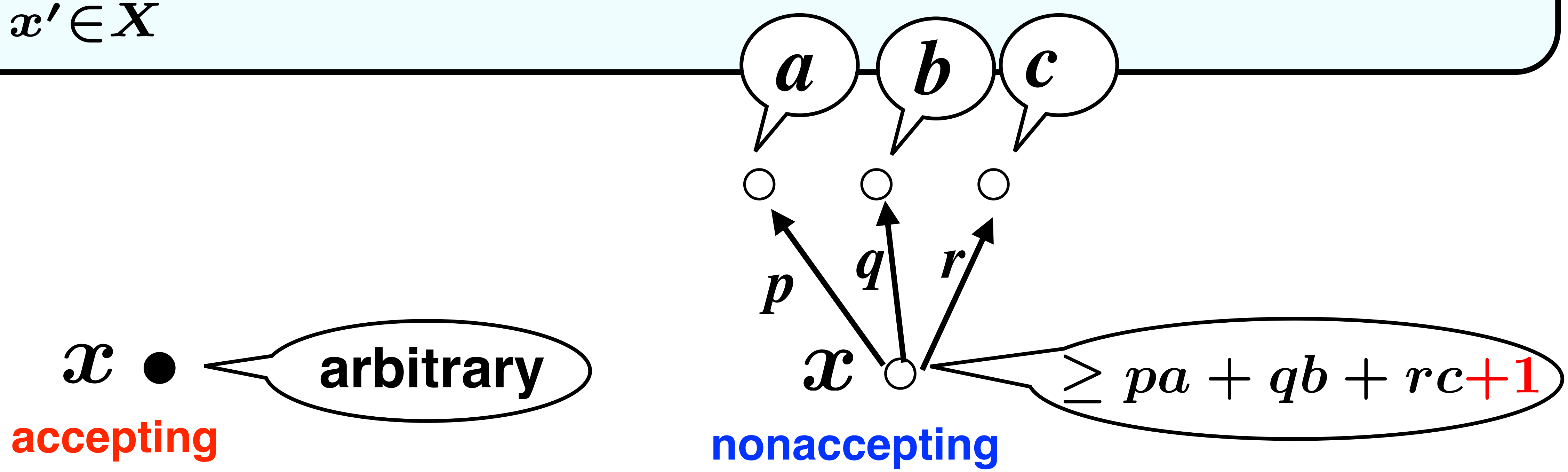


Thm:

b is a ranking supermartingale and $b(x) < \infty$ $\Longrightarrow \Pr\left(\begin{array}{r}\text{an accepting state}\\ \text{is reached}\end{array}\right) = 1$

Problem and Next Step

- We couldn't find a ranking domain $(\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})$ s.t.

$$\boldsymbol{b} \text{ is a ranking supermartingale} \iff \begin{array}{l}\boldsymbol{b} \text{ is a ranking arrow} \\ \text{wrt. } (\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})\end{array}$$

➡ We decided to **give up** describing ranking supermartingales

- Instead, we found **two ranking domains** for probabilistic systems

➡ They induces **new** definitions of ranking function for probabilistic systems (to the best of our knowledge)

Scaled Noncounting Ranking Supermartingale

Def:

For $\gamma \in (0,1)$, a function $b : X \to [0,1]$ is a γ **-scaled noncounting ranking supermartingale** if:

$$\gamma \cdot \sum_{x' \in X} \Pr(x \to x') \cdot b(x') \;\geq\; b(x)$$

By soundness of (categorical) ranking arrows,

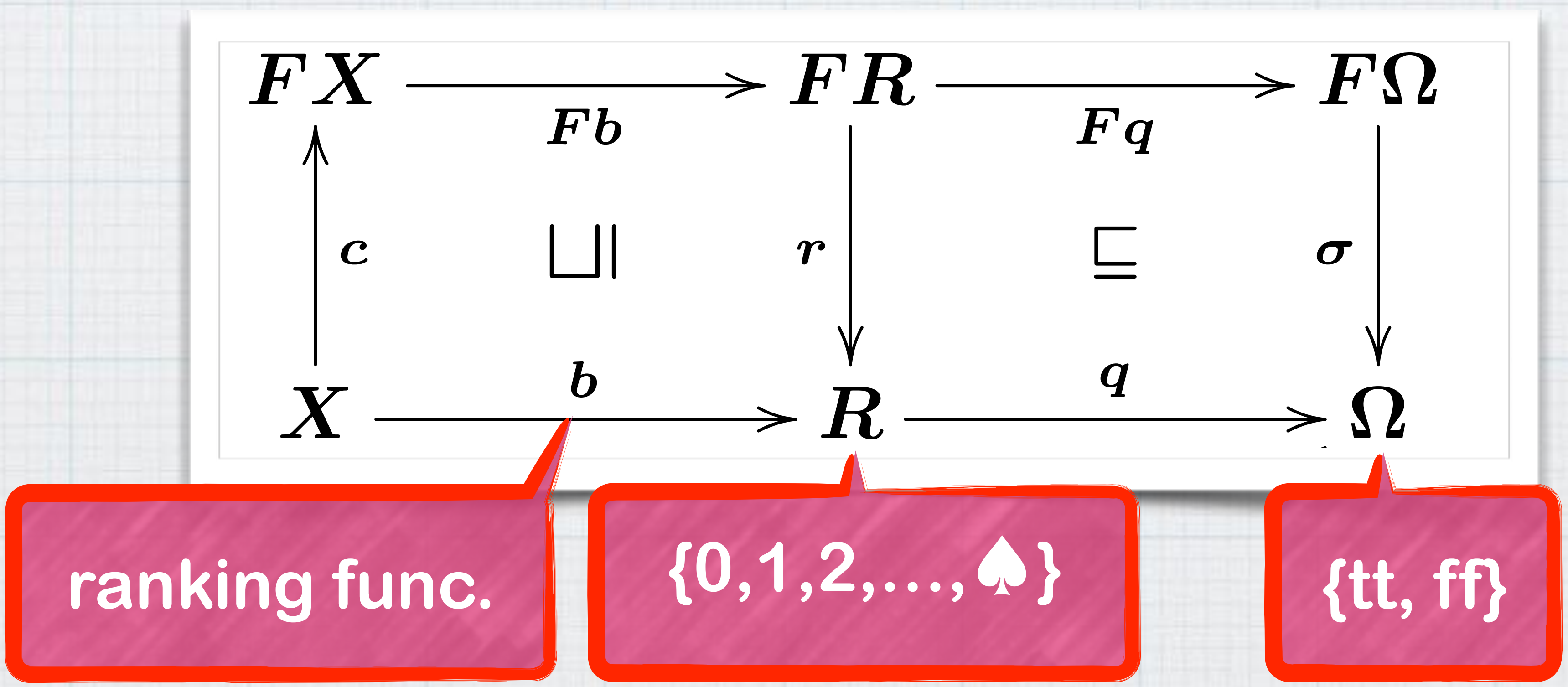
Thm:

$$b(x) \;\leq\; \Pr\begin{pmatrix}\text{an accepting state} \\ \text{is reached from } x\end{pmatrix}$$

⇒ **Quantitative reasoning**

Summary: Categorical Ranking Functions by Corecursive Algebras [LICS'17]

* Ranking function
  = (invariant-like) inductive constraint
    + well-foundedness
  = (co)algebraic simulation
    + corecursive algebra
      (that refines truth values and modality)



* New proof method for probabilistic liveness

Summary

* Significance of fixed-points other than greatest ("grand challenges")

* (Hierarchical) equational systems as syntax

* Lattice-theoretic foundation:
  Knaster-Tarski and Cousot-Cousot

* Buechi & parity in the Kleisli approach:
  departure from finality

* Ranking function =
  simulation with a corecursive algebra as its domain

~~We're Hiring!~~

Call for Collaboration

- ERATO Metamathematics for Systems Design Project
- 5.5 yrs (-2022.3),
  10-15 postdocs & senior researchers
- Formal methods for cyber-physical systems
  - … via logical & categorical metatheories
  - with serious applications w/ manufacturers (automotive) & autonomous driving project (autonomoose in Waterloo, CAN)

Formal Methods for Software

Specification

* **machine-manipulatable** specifications
* consistency check

Verification

* **proof** of correctness
* potential automation

Synthesis

* **goal-directed** (automatic) search

Formal Methods for Software

Cyber-Physical Systems

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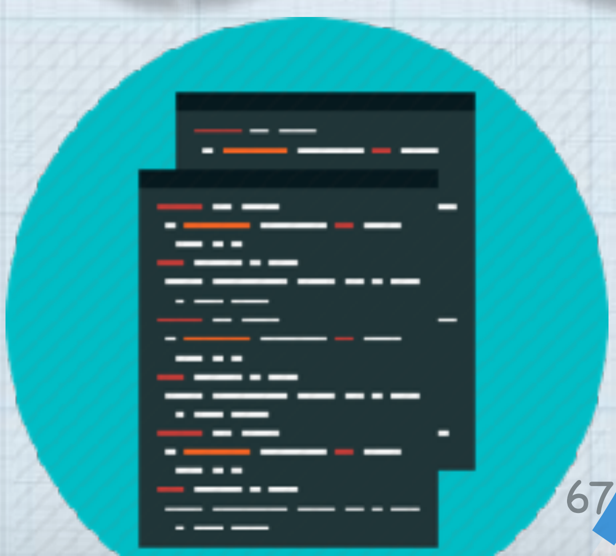
Formal Methods for Software



Cyber-Physical Systems

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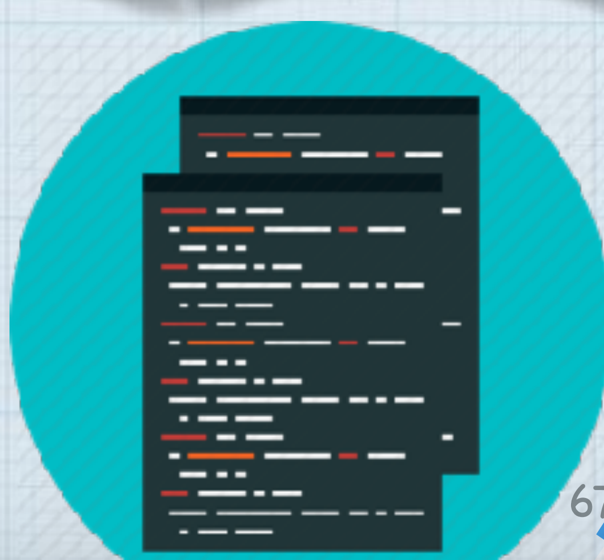


Cyber-Physical Systems

- Continuous physical dynamics
- Quantitative concerns (time, energy, ...)

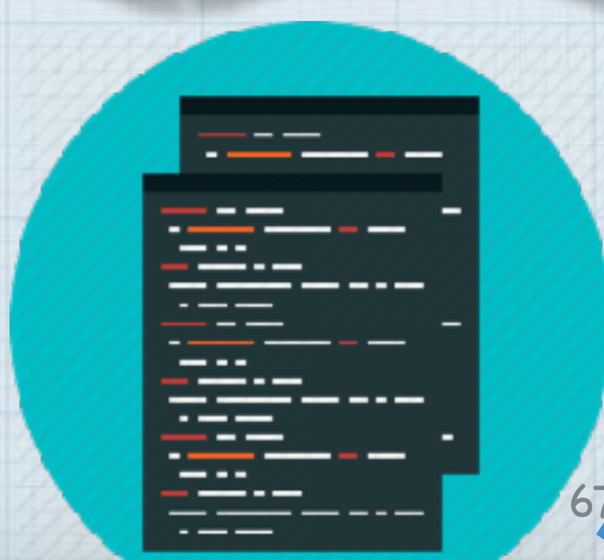
- Qualitative (yes/no)
- Discrete dynamics

Specification
- machine-manipulatable specifications
- consistency check

Verification
- proof of correctness
- potential automation

Synthesis
- goal-directed (automatic) search

Formal Methods for Software



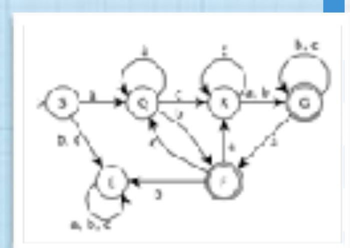
Hasuo

Heterogenizing SS, So Far

SS techniques

$$\frac{\{A \wedge b\}\ P_1\ \{A\}}{\{A\}\ \text{while } b\ P_1\ \{A \wedge \neg b\}}\ (\text{While})$$

* Verification by program logic
* Automata-theoretic synthesis



* Specification by temporal logics

$\mathsf{G}(P \supset \mathsf{F}Q)$

* …

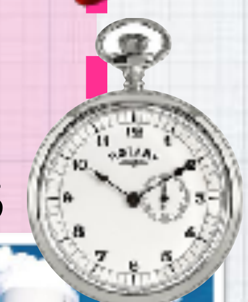
T

+

new concerns

* Continuous dynamics

* Probability



* Realtime constraints

* Energy
* …

e

Heterogenizing SS, So Far

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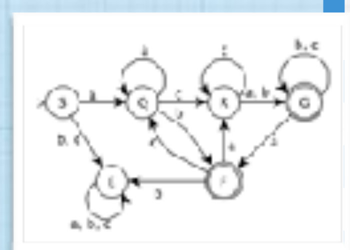

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$\mathsf{G}(P \supset \mathsf{F}Q)$

* ...

T

+

new concerns

* Continuous dynamics

* Probability

* Realtime constraints

* Energy

* ...

e

→

heterogenized techniques

* Probabilistic automata [Baier, Katoen, Hermanns, ...]
* Hybrid automata [Alur, Henzinger, ...]
* Differential dynamic logic [Platzer]

$T(e)$

* $T + e \to T(e)$, in a one-by-one manner
* Substantial theoretical efforts for each T, e

Metamathematical Transfer

$$T_1 + e_1 \rightarrow T_1(e_1)$$

$$T_2 + e_2 \rightarrow T_2(e_2)$$

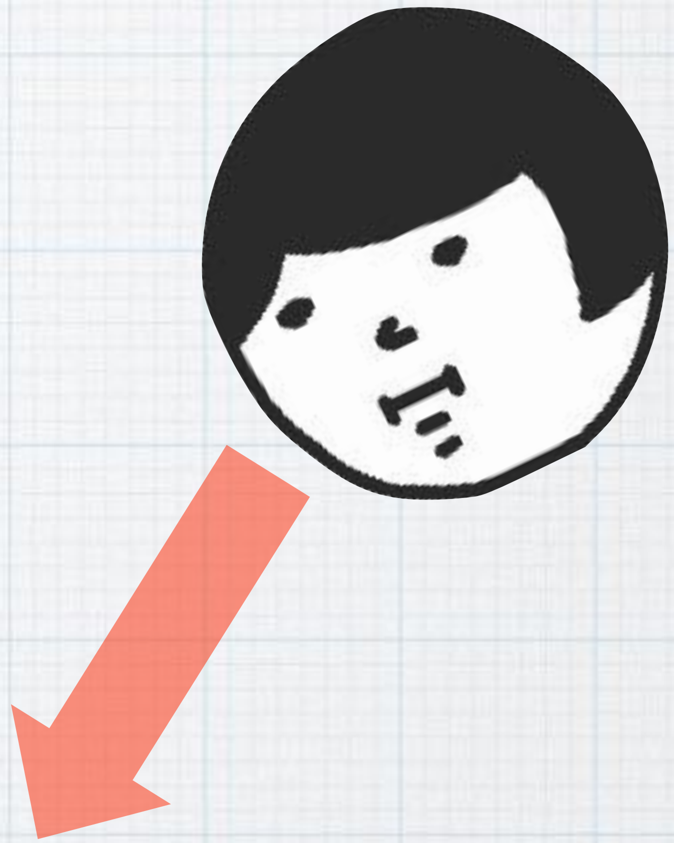
$$T_3 + e_3 \rightarrow T_3(e_3)$$

$$\vdots \qquad \vdots \qquad \vdots$$

SS techniques | new concerns | heterogenized techniques

Metamathematical Transfer

Meta-theoretician



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SS techniques | new concerns | heterogenized techniques

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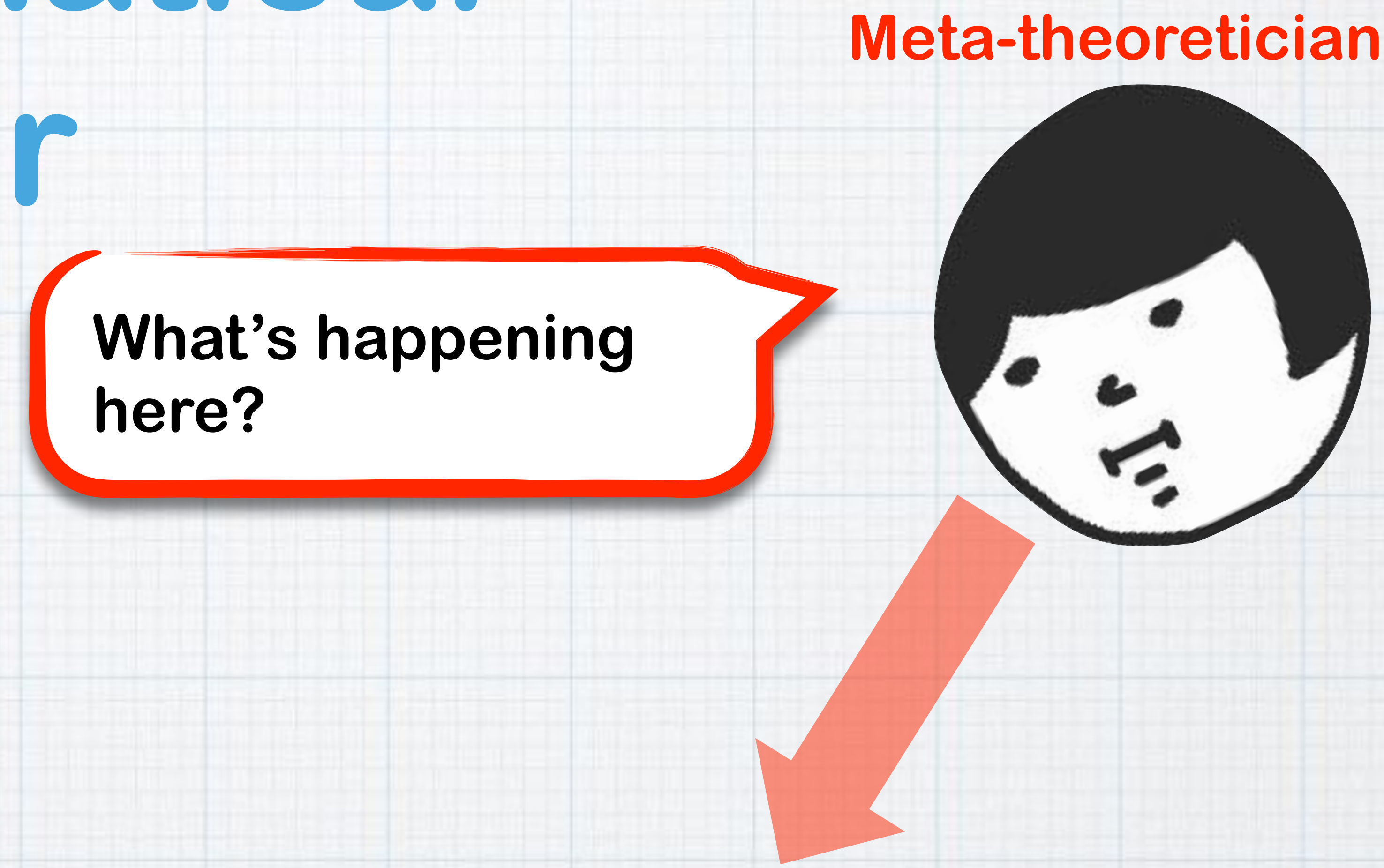
Metamathematical Transfer



$T_1 + e_1 \rightarrow T_1(e_1)$

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SS techniques | new concerns | heterogenized techniques

Hasuo

Metamathematical Transfer

Meta-theoretician

What's happening here?

... uniform & comprehensive construction

$T + e \to T(e)$

$T_1 + e_1 \to T_1(e_1)$

$T_2 + e_2 \to T_2(e_2)$

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SS techniques | new concerns | heterogenized techniques

Hasuo

Metamathematical Transfer

Meta-theoretician

What's happening here?

... **uniform** & **comprehensive** construction

$T + e \rightarrow T(e)$

$T_1 + e_1 \rightarrow T_1(e_1)$

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$T_3 + e_3 \rightarrow T_3(e_3)$

$\vdots \quad \vdots \quad \vdots$

SS techniques — new concerns — heterogenized techniques

Exploiting the languages of **modern abstract math.**, esp. **category theory** & **logic**

Our prev. results via

- **nonstandard transfer**
- **coalgebraic unfolding**

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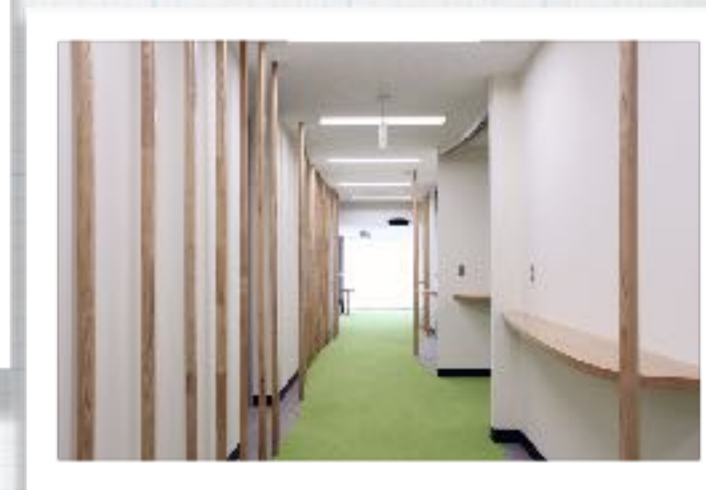
Call for Collaboration



- 5.5 yrs (-2022.3),
  10-15 postdocs & senior researchers
- Interdisciplinary
  - Control theory, software engineering, optimization, machine learning, user interface, …
  - Many new techniques, and many common techniques
- I. Hasuo (Director),
  S. Katsumata, K. Czarnecki, F. Ishikawa (Group Leaders),
  M. Hasegawa, T. Ushio (Site Leaders),
  D. Sprunger, J. Dubut (Postdocs), …
- Search "ERATO MMSD"