One Eilenberg Theorem to Rule Them All

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Abstract Eilenberg-type correspondences, relating varieties of languages (e.g. of finite words, infinite words, or trees) to pseudovarieties of finite algebras, form the backbone of algebraic language theory. Numerous such correspondences are known in the literature. We demonstrate that they all arise from the same recipe: one models languages and the algebras recognizing them by monads on an algebraic category, and applies a Stone-type duality. Our main contribution is a variety theorem that covers e.g. Wilke's and Pin's work on ∞-languages, the variety theorem for cost functions of Daviaud, Kuperberg, and Pin, and unifies the two previous categorical approaches of Bojańczyk and of Adámek et al. In addition it gives new results, such as an extension of the local variety theorem of Gehrke, Grigorieff, and Pin from finite to infinite words.

1 Introduction

Algebraic language theory investigates the behaviors of finite machines by relating them to finite algebraic structures. This has proved very fruitful. For example, regular languages are precisely the languages recognized by finite monoids, and the decidability of star-freeness rests on Schützenberger's theorem [33]: a regular language is star-free iff it is recognized by a finite aperiodic monoid. At the heart of algebraic language theory are results establishing generic correspondences of this kind. The prototype is Eilenberg's celebrated variety theorem [15]: it states that *varieties of languages* (classes of regular languages closed under boolean operations, derivatives, and homomorphic preimages) and *pseudovarieties of monoids* (classes of finite monoids closed under quotients, submonoids, and finite products) are in bijective correspondence. This together with Reiterman's theorem [28], stating that pseudovarieties of monoids can be specified by *profinite equations*, establishes a firm connection between automata, languages, and algebras.

In the past decades numerous further Eilenberg-type theorems were discovered for regular languages [17, 23, 27, 34], treating varieties with weaker closure properties, but also for machine behaviors beyond finite words, including weighted

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languages over a field [29], infinite words [24,35], words on linear orderings [8,9], ranked trees [5], binary trees [32], and cost functions [14]. This plethora of similar results has raised interest in category-theoretic approaches to algebraic language theory which allow to derive all the above results as special instances of only *one* general variety theorem (that therefore would *rule them all*). An important first step was achieved by Bojańczyk [11]. He extends the classical notion of language recognition by monoids to algebras for a *monad* on sorted sets, and proves a generic Eilenberg theorem. Our previous work in [1–3,13] takes an orthogonal approach: one keeps monoids but considers them in categories $\mathscr{D}$ of (ordered) algebras such as posets, semilattices, and vector spaces. In this way we uniformly covered five Eilenberg theorems for languages of finite words [15,23,27,29,34].

To obtain the *one* Eilenberg theorem, a unification of the two approaches is required. On the one hand, one needs to take the step from sets to more general categories $\mathscr{D}$ to capture the proper notion of language recognition; e.g. for the treatment of weighted languages [29] one needs to work over the category of vector spaces. On the other hand, to deal with machine behaviors beyond finite words, one has to replace monoids by other algebraic structures. The main contribution of this paper is a variety theorem that achieves the desired unification, and in addition encompasses many Eilenberg-type correspondences captured by neither of the previous generic results, including the work [8,9,14,24,32,35]. Thus, we hope to convince the reader that our results make a substantial step towards the desired one Eilenberg theorem. Our approach starts with the observation that all Eilenberg theorems in the literature emerge essentially from the same four steps:

1. Identify an algebraic theory such that the languages in mind are the ones recognized by finite algebras. For example, for regular languages take monoids.
2. Find a presentation of the finite algebras in terms of *unary* operations; e.g., monoids can be presented by left and right multiplication with fixed elements.
3. Infer the form of the syntactic algebras, i.e. the minimal recognizers of languages, and the derivatives under which varieties of languages are closed.
4. Establish a bijective correspondence between varieties of languages and pseudovarieties of algebras by relating languages to their syntactic algebras.

It turns out that all these steps can be facilitated or even completely automatized.

For Step 1, putting a common roof over Bojańczyk's and our own previous work, we consider a variety $\mathscr{D}$ and algebras for a *monad* $\mathbf{T}$ on $\mathscr{D}^S$, the category of S-sorted $\mathscr{D}$-algebras for some finite set S of sorts. For example, to capture regular languages one takes the monad $\mathbf{T}\Sigma = \Sigma^*$ on $\mathbf{Set}$ representing monoids. For regular ∞-languages one takes the monad $\mathbf{T}(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$ on $\mathbf{Set}^2$ representing ω-semigroups.

For Step 2, Bojańczyk gave a generic unary presentation for any monad on sorted sets. However, this presentation is often too unwieldy. For example, in the case of monoids it contains all unary operations associated to words with one variable, but one wants to restrict to words where the variable appears only *once*. Thus we make our setting parametric in a *choice* of a unary presentation of $\mathbf{T}$.

We emphasize that non-trivial work still lies in proving that the languages of interest are precisely those recognized by finite $\mathbf{T}$-algebras, and in finding a good unary presentation of $\mathbf{T}$. However, our work here shows that then Steps 3 and 4 are completely generic: after choosing a unary presentation, the syntactic algebras (Theorem 3.10) and the variety theorem (Theorem 5.7) come "for free". In fact, Theorem 3.10 even shows that a unary presentation is necessary and sufficient for constructing syntactic algebras. Our main result is the following

Variety Theorem. *Varieties of languages recognizable by finite* $\mathbf{T}$*-algebras are in bijective correspondence with pseudovarieties of* $\mathbf{T}$*-algebras.*

The proof relies on two main ingredients. The first one is *duality*: besides $\mathscr{D}$ we also consider a variety $\mathscr{C}$ that is dual to $\mathscr{D}$ on the level of *finite* algebras. Varieties of languages live in $\mathscr{C}$, while over $\mathscr{D}^S$ we form pseudovarieties of $\mathbf{T}$-algebras. This duality-based approach is inspired by the work of Gehrke, Grigorieff, and Pin [17] who interpret the original Eilenberg theorem [15] in terms of Stone duality ($\mathscr{C}$ = boolean algebras, $\mathscr{D}$ = sets). Our second ingredient is the *profinite monad* of $\mathbf{T}$, introduced in [12]. It generalizes the classical construction of the free profinite monoid, and allows for the introduction of topological methods to our setting. For example, Pippenger's result [26] that the boolean algebra of regular languages dualizes to the free profinite monoid holds at the level of monads (Theorem 3.3).

Together with our generalization of Reiterman's theorem in [12], showing that pseudovarieties of $\mathbf{T}$-algebras are presentable by *profinite equations*, the variety theorem establishes a conceptual and highly parametric framework for algebraic language theory. To illustrate its strength, we demonstrate in Section 6 that it instantiates to roughly a dozen Eilenberg correspondences known in the literature. In addition, it yields new results, e.g. an extension of the local variety theorem of [17] from finite to infinite words.

2 The Profinite Monad

We start by introducing our categorical framework for algebraic language theory. Readers should be familiar with basic concepts from category theory such as monads, limits, and duality [21]. The appendix contains a brief categorical toolkit.

Assumptions 2.1. Throughout this paper fix a variety $\mathscr{C}$ of algebras and a variety $\mathscr{D}$ of algebras or ordered algebras. Thus, $\mathscr{D}$ is presented by equations or inequations. We assume that (i) $\mathscr{C}$ and $\mathscr{D}$ are locally finite, i.e. all finitely generated algebras are finite; (ii) the full subcategories $\mathscr{C}_f$ and $\mathscr{D}_f$ on finite algebras are dually equivalent; (iii) the signature of $\mathscr{C}$ contains a constant; (iv) epimorphisms in $\mathscr{D}$ are surjective. Finally, fix a *finite* set S of sorts and a monad $\mathbf{T} = (T, \eta, \mu)$ on the product category $\mathscr{D}^S$ with T preserving epimorphisms.

Notation 2.2. Recall that an object of $\mathscr{D}^S$ is a family $D = (D_s)_{s\in S}$ of objects in $\mathscr{D}$, and a morphism $f\colon D \to D'$ in $\mathscr{D}^S$ is a family $(f_s\colon D_s \to D'_s)_{s\in S}$ of morphisms in $\mathscr{D}$. We usually write f for f_s if the sort s is clear from the context.

Example 2.3. The following categories $\mathscr{C}$ and $\mathscr{D}$ satisfy our assumptions:

1. $\mathscr{C} = \mathbf{BA}$ (boolean algebras) and $\mathscr{D} = \mathbf{Set}$: Stone duality [18] yields a dual equivalence $\mathbf{BA}_f^{op} \simeq \mathbf{Set}_f$, mapping a finite boolean algebra to its atoms.

2. $\mathscr{C} = \mathbf{DL}_{01}$ (distributive lattices with 0 and 1) and $\mathscr{D} = \mathbf{Pos}$ (posets): Birkhoff duality [10] gives a dual equivalence $(\mathbf{DL}_{01})_f^{op} \simeq \mathbf{Pos}_f$, mapping a finite distributive lattice to the poset of its join-irreducible elements.
3. $\mathscr{C} = \mathscr{D} = \mathbf{JSL}_0$ (join-semilattices with 0): there is a self-duality of $(\mathbf{JSL}_0)_f$ mapping a finite semilattice $(X, \vee)$ to its opposite semilattice $(X, \wedge)$.
4. $\mathscr{C} = \mathscr{D} = \mathbf{Vec}_K$ (vector spaces over a finite field K): the familiar self-duality maps a finite(-dimensional) space X to its dual space $X^* = \mathbf{Vec}_K(X, K)$.

Example 2.4. Our monads $\mathbf{T}$ of interest represent structures in language theory.

1. Let $\mathbf{T}_*$ be the free-monoid monad on $\mathbf{Set}$. Languages of finite words correspond to subsets of $T_*\Sigma = \Sigma^*$. The category of $\mathbf{T}_*$-algebras is isomorphic to the category of monoids.
2. Languages of finite and infinite words (i.e. ∞-*languages*) are represented by the monad $\mathbf{T}_\infty$ on $\mathbf{Set}^2$ associated to the algebraic theory of ω-semigroups. Recall that an ω-*semigroup* is a two-sorted set $A = (A_+, A_\omega)$ equipped with a binary product $A_+ \times A_+ \xrightarrow{\cdot} A_+$, a mixed binary product $A_+ \times A_\omega \xrightarrow{\cdot} A_\omega$ and an ω-ary product $A_+^\omega \xrightarrow{\pi} A_\omega$ satisfying all (mixed) associative laws [22]. The free ω-semigroup on (Σ, Γ) is $(\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$ with products given by concatenation. Thus $T_\infty(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$, and an ∞-language over the alphabet Σ corresponds to a two-sorted subset of $T_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$.
3. Weighted languages $L\colon \Sigma^* \to K$ over a finite field K are represented by the monad $\mathbf{T}_K$ on $\mathbf{Vec}_K$ constructing free K-algebras. Thus for the vector space K^Σ with finite basis Σ we have $T_K(K^\Sigma) = K[\Sigma]$, the space of polynomials $\sum_{i<n} k_i w_i$ with $k_i \in K$ and $w_i \in \Sigma^*$. Since $K[\Sigma]$ has the basis Σ^*, weighted languages correspond to linear maps from $T_K(K^\Sigma)$ to K.

Remark 2.5. The variety $\mathscr{D}$ has the factorization system of surjective morphisms and injective (resp. order-reflecting) morphisms, extending sortwise to $\mathscr{D}^S$. We denote by $\mathbf{Alg}_f\, \mathbf{T}$ and $\mathbf{Alg}\, \mathbf{T}$ the categories of (finite) $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms. Since T preserves epimorphisms, the factorization system of $\mathscr{D}^S$ lifts to $\mathbf{Alg}\, \mathbf{T}$: every $\mathbf{T}$-homomorphism factorizes into a sortwise surjective homomorphism followed by a sortwise injective (resp. order-reflecting) one. *Quotients* and *subalgebras* in $\mathbf{Alg}\, \mathbf{T}$ are taken in this factorization system.

Recall that the Stone space $\widehat{\Sigma^*}$ of *profinite words* over an alphabet Σ is formed as the inverse (a.k.a. cofiltered) limit of all finite quotient monoids of Σ^*. In [12] we generalized this construction from the free-monoid monad $\mathbf{T}_*$ on $\mathbf{Set}$ to arbitrary monads $\mathbf{T}$ as follows.

Notation 2.6. For a variety $\mathscr{D}$ of algebras, let $\mathbf{Stone}(\mathscr{D})$ denote the category of topological $\mathscr{D}$-algebras carrying a Stone topology, and continuous $\mathscr{D}$-morphisms. Similarly, if $\mathscr{D}$ is a variety of ordered algebras, let $\mathbf{Priest}(\mathscr{D})$ denote the category of ordered topological $\mathscr{D}$-algebras carrying a Priestley topology, and monotone continuous $\mathscr{D}$-morphisms. Denote by $\widehat{\mathscr{D}}$ the full subcategory of $\mathbf{Stone}(\mathscr{D})$ (resp. $\mathbf{Priest}(\mathscr{D})$) on *profinite* $\mathscr{D}$*-algebras*, i.e. inverse limits of algebras in $\mathscr{D}_f$. We view $\mathscr{D}_f$ as a full subcategory of $\widehat{\mathscr{D}}$, by identifying objects of $\mathscr{D}_f$ with profinite $\mathscr{D}$-algebras carrying the discrete topology.

Example 2.7. We have $\widehat{\mathbf{Set}} = \mathbf{Stone}$, $\widehat{\mathbf{Pos}} = \mathbf{Priest}$, $\widehat{\mathbf{JSL}_0} = \mathbf{Stone}(\mathbf{JSL}_0)$ and $\widehat{\mathbf{Vec}_K} = \mathbf{Stone}(\mathbf{Vec}_K)$, see Johnstone [18]. Thus, in all these examples every algebra in $\mathbf{Stone}(\mathscr{D})$ (resp. $\mathbf{Priest}(\mathscr{D})$ is profinite.

Construction 2.8 (see [12]). For any object $D \in \mathscr{D}_f^S$ form the poset $\mathsf{Quo}_f(\mathbf{T}D)$ of all finite quotient algebras $e\colon \mathbf{T}D \twoheadrightarrow (A,\alpha)$ of the free $\mathbf{T}$-algebra $\mathbf{T}D = (TD,\mu_D)$, ordered by $e \le e'$ iff e factors through e'. Define $\hat{T}D$ in $\widehat{\mathscr{D}}^S$ to be the inverse limit of the diagram $\mathsf{Quo}_f(\mathbf{T}D) \to \widehat{\mathscr{D}}^S$ mapping $(e\colon \mathbf{T}D \twoheadrightarrow (A,\alpha))$ to A. We denote the limit projection associated to e by $e^+\colon \hat{T}D \twoheadrightarrow A$. In particular, for any finite $\mathbf{T}$-algebra (A,α) we have the limit projection $\alpha^+ : \hat{T}A \to A$ because $\alpha : \mathbf{T}A \twoheadrightarrow (A,\alpha)$ is a surjective $\mathbf{T}$-homomorphism.

Theorem 2.9 (see [12]). *The object map $D \mapsto \hat{T}D$ from $\mathscr{D}_f^S$ to $\widehat{\mathscr{D}}^S$ extends (via inverse limits) to a functor $\hat{T}\colon \widehat{\mathscr{D}}^S \to \widehat{\mathscr{D}}^S$. Further, $\hat{T}$ can be equipped with the structure of a monad $\widehat{\mathbf{T}} = (\hat{T},\hat{\eta},\hat{\mu})$ called the* profinite monad *of* $\mathbf{T}$. *Its unit $\hat{\eta}_D$ and multiplication $\hat{\mu}_D$ for $D \in \mathscr{D}_f^S$ are determined by the commutative diagram* (2.1) *for all $e\colon \mathbf{T}D \twoheadrightarrow (A,\alpha)$ in $\mathsf{Quo}_f(\mathbf{T}D)$.*

$$\begin{tikzcd}
D \arrow[r, "\hat{\eta}_D"] \arrow[dr, "e\eta_D"'] & \hat{T}D \arrow[d, two heads, "e^+"] & \hat{T}\hat{T}D \arrow[l, "\hat{\mu}_D"'] \arrow[d, two heads, "\hat{T}e^+"] \\
& A & \hat{T}A \arrow[l, two heads, "\alpha^+"]
\end{tikzcd} \tag{2.1}$$

Example 2.10. The monad $\widehat{\mathbf{T}_*}$ on **Stone** assigns to each finite set (i.e. each finite discrete space) Σ the space $\widehat{\Sigma^*}$ of profinite words. Similarly, the monad $\widehat{\mathbf{T}_K}$ on $\mathbf{Stone}(\mathbf{Vec}_K)$ assigns to each finite vector space K^Σ the Stone-topological vector space obtained as the limit of all finite quotient spaces of $K[\Sigma]$.

Remark 2.11. 1. If (A,α) is a finite $\mathbf{T}$-algebra, (A,α^+) is a $\widehat{\mathbf{T}}$-algebra: putting $e=\alpha$ in (2.1) gives the unit and associative law. By [12, Prop. 3.10] this yields an isomorphism $\mathbf{Alg}_f\,\mathbf{T} \cong \mathbf{Alg}_f\,\widehat{\mathbf{T}}$ given by $(A,\alpha) \mapsto (A,\alpha^+)$ and $h \mapsto h$.

2. Let $V\colon \widehat{\mathscr{D}}^S \to \mathscr{D}^S$ denote the forgetful functor. If $D \in \widehat{\mathscr{D}}_f^S$ we often write D for VD. By [12, Rem. B.6] there is a natural transformation $\iota\colon TV \to V\hat{T}$ whose component $\iota_D\colon TVD \to V\hat{T}D$ for $D \in \mathscr{D}_f^S$ is determined by $Ve^+ \cdot \iota_D = e$ for all finite quotient algebras $e\colon \mathbf{T}D \twoheadrightarrow A$ of $\mathbf{T}D$ in $\mathbf{Alg}\,\mathbf{T}$. More generally, we call a finite quotient $e\colon TD \twoheadrightarrow A$ in $\mathscr{D}^S$ *extensible* if $V\hat{e}\cdot\iota_D = e$ for some $\hat{e}\colon \hat{T}D \twoheadrightarrow A$ in $\widehat{\mathscr{D}}^S$. Thus every finite quotient of $\mathbf{T}D$ in $\mathbf{Alg}\,\mathbf{T}$ is extensible.

Remark 2.12. 1. $\widehat{\mathscr{D}}$ is the *pro-completion* (the free completion under inverse limits) of $\mathscr{D}_f$, see [18, Remark VI.2.4]. Moreover, since $\mathscr{C}$ is locally finite, $\mathscr{C}$ is the *ind-completion* (the free completion under filtered colimits) of $\mathscr{C}_f$. Thus the dual equivalence between $\mathscr{C}_f$ and $\mathscr{D}_f$ extends to a dual equivalence between $\mathscr{C}$ and $\widehat{\mathscr{D}}$. We denote the equivalence functors by $P\colon \widehat{\mathscr{D}} \xrightarrow{\simeq} \mathscr{C}^{op}$ and $P^{-1}\colon \mathscr{C}^{op} \xrightarrow{\simeq} \widehat{\mathscr{D}}$. For $\mathscr{C} = \mathbf{BA}$ and $\mathscr{D} = \mathbf{Set}$ (with $\widehat{\mathscr{D}} = \mathbf{Stone}$), this is the classical Stone duality [18]: P maps a Stone space to the boolean algebra of clopens, and P^{-1} maps a boolean algebra to the Stone space of all ultrafilters.

2. We write $|{-}|$ for the forgetful functors of $\mathscr{C}$ and $\widehat{\mathscr{D}}$ into **Set**, and $\mathbb{1}$ for the free objects on one generator both in $\mathscr{C}$ and $\widehat{\mathscr{D}}$. The two finite objects $O_{\mathscr{C}} := P\mathbb{1}$ and $O_{\mathscr{D}} := P^{-1}\mathbb{1}$ play the role of a *dualizing object* (also called a *schizophrenic object* in [18]) of $\mathscr{C}$ and $\widehat{\mathscr{D}}$. This means that there is a natural isomorphism between $|{-}| \cdot P$ and $\widehat{\mathscr{D}}(-, O_{\mathscr{D}})$ given for all $D \in \widehat{\mathscr{D}}$ by
$$|PD| \cong \mathscr{C}(\mathbb{1}, PD) \cong \widehat{\mathscr{D}}(P^{-1}PD, O_{\mathscr{D}}) \cong \widehat{\mathscr{D}}(D, O_{\mathscr{D}}).$$
Analogously $|P^{-1}| \cong \mathscr{C}(-, O_{\mathscr{C}})$. In particular, the objects $O_{\mathscr{C}}$ and $O_{\mathscr{D}}$ have the same underlying set up to isomorphism, since $|O_{\mathscr{D}}| \cong \widehat{\mathscr{D}}(\mathbb{1}, O_{\mathscr{D}}) \cong |P\mathbb{1}| = |O_{\mathscr{C}}|$
3. *Subobjects* in the variety $\mathscr{C}$ are represented by monomorphisms (= injective morphisms). Dually, *quotients* in $\widehat{\mathscr{D}}$ are represented by epimorphisms, which can be shown to be precisely the surjective morphisms. *Quotients* of $\widehat{\mathbf{T}}$-algebras are thus represented by sortwise surjective $\widehat{\mathbf{T}}$-homomorphisms.

3 Recognizable Languages and Syntactic T-Algebras

A language $L \subseteq \Sigma^*$ may be identified with its characteristic function $L\colon \Sigma^* \to \{0,1\}$. To get a notion of *language* in our categorical setting, we replace the one-sorted alphabet Σ by an S-sorted alphabet Σ in $\mathbf{Set}_f^S$, and represent it in $\mathscr{D}^S$ via the free object $\mathbb{\Sigma} \in \mathscr{D}_f^S$ generated by Σ (w.r.t. the forgetful functor $|{-}|\colon \mathscr{D}^S \to \mathbf{Set}^S$). The set $\{0,1\}$ is replaced by a finite "object of outputs" in $\mathscr{D}_f^S$, viz. the object with $O_{\mathscr{D}} \in \mathscr{D}_f$ in each sort. We denote this object of $\mathscr{D}_f^S$ also by $O_{\mathscr{D}}$. This leads to the following definition, unifying concepts in [11] and [2].

Definition 3.1. A *language* over the alphabet $\Sigma \in \mathbf{Set}_f^S$ is a morphism $L\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ in $\mathscr{D}^S$. It is *recognized* by a $\mathbf{T}$-homomorphism $h\colon \mathbf{T}\mathbb{\Sigma} \to (A, \alpha)$ if there is a morphism $p\colon A \to O_{\mathscr{D}}$ in $\mathscr{D}^S$ with $L = p \cdot h$. A language is $\mathbf{T}$*-recognizable* if it is recognized by some $\mathbf{T}$-homomorphism with finite codomain. We denote the set of all $\mathbf{T}$-recognizable languages over Σ by $\mathsf{Rec}(\Sigma)$.

Example 3.2. 1. $\mathbf{T} = \mathbf{T}_*$ on **Set** with $O_{\mathbf{Set}} = \{0,1\}$: a language $L\colon T_*\Sigma \to \{0,1\}$ corresponds to a classical language $L \subseteq \Sigma^*$ of finite words. It is recognized by a monoid morphism $h\colon \Sigma^* \to A$ iff $L = h^{-1}[Y]$ for some subset $Y \subseteq A$. Recognizable languages coincide with regular languages, i.e. languages accepted by finite automata; see e.g. [25].
2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$ with $O_{\mathbf{Set}} = \{0,1\}$: since $T_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$, a language $L\colon T_\infty(\Sigma, \emptyset) \to \{0,1\}$ corresponds to an ∞-language $L \subseteq \Sigma^+ \cup \Sigma^\omega$. It is recognized by an ω-semigroup morphism $h\colon (\Sigma^+, \Sigma^\omega) \to A$ iff $L = h^{-1}[Y]$ for some two-sorted subset $Y \subseteq A$. Recognizable ∞-languages coincide with regular ∞-languages, i.e. languages accepted by finite Büchi automata [22].

A key observation for the topological approach to automata theory is that regular languages over Σ correspond to clopen subsets of the Stone space $\widehat{\Sigma^*}$ of profinite words, i.e. to continuous maps from $\widehat{\Sigma^*}$ into the discrete space $\{0,1\}$; see [25, Prop. VI.3.12]. This generalizes from the monad $\mathbf{T}_*$ on **Set** to arbitrary monads $\mathbf{T}$:

Theorem 3.3. *Recognizable languages over Σ correspond bijectively to morphisms from $\hat{T}\Sigma$ to $O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$.*

Proof (Sketch). For any recognizable language $L\colon T\Sigma \to O_{\mathscr{D}}$, choose a finite quotient algebra $e\colon \mathbf{T}\Sigma \twoheadrightarrow (A, \alpha)$ and a morphism $p\colon A \to O_{\mathscr{D}}$ with $L = e \cdot p$. This yields the morphism $\hat{L} := p \cdot e^+ \colon \hat{T}\Sigma \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$, where e^+ is the limit projection of Construction 2.8. Conversely, every morphism $\hat{L}\colon \hat{T}\Sigma \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$ restricts to the recognizable language $L := V\hat{L} \cdot \iota_{\Sigma} \colon T\Sigma \to O_{\mathscr{D}}$, cf. Remark 2.11.2. The maps $L \mapsto \hat{L}$ and $\hat{L} \mapsto L$ can be shown to be mutually inverse. ☐

Remark 3.4. From the above theorem and Remark 2.12.2 we get

$$\mathsf{Rec}(\Sigma) \cong \widehat{\mathscr{D}}^S(\hat{T}\Sigma, O_{\mathscr{D}}) \cong \prod_s \widehat{\mathscr{D}}((\hat{T}\Sigma)_s, O_{\mathscr{D}}) \cong \prod_s |P(\hat{T}\Sigma)_s| \tag{3.1}$$

Thus we can consider $\mathsf{Rec}(\Sigma)$ as an object of $\mathscr{C}$ isomorphic to $\prod_s P(\hat{T}\Sigma)_s$. One can show that $\mathsf{Rec}(\Sigma)$ forms a subobject of $\prod_s O_{\mathscr{C}}^{|T\Sigma|_s}$: the embedding $\mathsf{Rec}(\Sigma) \rightarrowtail \prod_s O_{\mathscr{C}}^{|T\Sigma|_s}$ maps a language $L\colon T\Sigma \to O_{\mathscr{D}}$ to the tuple $(|T\Sigma|_s \xrightarrow{|L|} |O_{\mathscr{D}}| \xrightarrow{\cong} |O_{\mathscr{C}}|)_{s\in S}$, using the bijection $|O_{\mathscr{D}}| \cong |O_{\mathscr{C}}|$ of Remark 2.12.2. Consequently the $\mathscr{C}$-algebraic structure of $\mathsf{Rec}(\Sigma)$ is determined by $O_{\mathscr{C}}$. For example, for $\mathscr{C} = \mathbf{BA}$ with $O_{\mathbf{BA}} = \{0, 1\}$, the boolean structure of $\mathsf{Rec}(\Sigma)$ is given by union, intersection and complement. For $\mathbf{T} = \mathbf{T}_*$ on $\mathbf{Set}$, we thus recover a result of Pippenger [26]: the boolean algebra of regular languages over Σ is dual to the Stone space $\widehat{\Sigma^*}$ of profinite words; in fact, in this one sorted case (3.1) states $\mathsf{Rec}(\Sigma) \cong P(\widehat{\Sigma^*})$.

An important tool for the algebraic approach to regular languages is the syntactic monoid of a language, viz. the smallest monoid recognizing it. We now introduce syntactic algebras for $\mathbf{T}$-recognizable languages, unifying the two corresponding concepts introduced in [11] and [2].

Definition 3.5. Let $L\colon T\Sigma \to O_{\mathscr{D}}$ be a recognizable language. A *syntactic* $\mathbf{T}$-*algebra of* L is a finite $\mathbf{T}$-algebra A_L together with a surjective $\mathbf{T}$-homomorphism $e_L\colon \mathbf{T}\Sigma \twoheadrightarrow A_L$ (called a *syntactic morphism* of L) such that (i) e_L recognizes L, and (ii) e_L factors through any surjective $\mathbf{T}$-homomorphism $e\colon \mathbf{T}\Sigma \twoheadrightarrow A$ recognizing L, i.e. $e_L = h \cdot e$ for some $h\colon A \twoheadrightarrow A_L$ in $\mathbf{Alg}\,\mathbf{T}$.

Example 3.6. 1. $\mathbf{T} = \mathbf{T}_*$ on $\mathbf{Set}$: the *syntactic monoid* [25] of a recognizable language $L\colon \Sigma^* \to \{0, 1\}$ is the quotient monoid $e_L\colon \Sigma^* \twoheadrightarrow \Sigma^*/{\equiv_L}$, where $\equiv_L$ is the monoid congruence on Σ^* defined by $v \equiv_L w$ iff $L(xvy) = L(xwy)$ for all $x, y \in \Sigma^*$.

2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$: the *syntactic ω-semigroup* [22] of a recognizable language $L\colon (\Sigma^+, \Sigma^\omega) \to \{0, 1\}$ is the quotient ω-semigroup $e_L\colon (\Sigma^+, \Sigma^\omega) \twoheadrightarrow (\Sigma^+, \Sigma^\omega)/{\equiv_L}$, where $\equiv_L$ is the following ω-semigroup congruence on $(\Sigma^+, \Sigma^\omega)$: for $v, w \in \Sigma^+$ put $v \equiv_L w$ iff $L(xvy) = L(xwy)$, $L(xvz) = L(xwz)$ and $L(x(vy)^\omega) = L(x(wy)^\omega)$ for all $x, y \in \Sigma^*$ and $z \in \Sigma^\omega$. And for $v, w \in \Sigma^\omega$ put $v \equiv_L w$ iff $L(xv) = L(xw)$ for all $x \in \Sigma^*$.

3. Let $\mathbf{T}$ be any monad on $\mathbf{Set}^S$. Generalizing work of Almeida [5] on algebras for a finitary signature, Bojańczyk [11] showed that every $\mathbf{T}$-recognizable language $L\colon T\Sigma \to \{0,1\}$ has a syntactic $\mathbf{T}$-algebra, constructed as follows. Denote by $1_s \in \mathbf{Set}^S$ the S-sorted set with one element in sort s and otherwise empty; thus a morphism $1_s \to A$ in $\mathbf{Set}^S$ chooses an element of A_s. A *polynomial* over Σ is a morphism $p\colon 1_{s'} \to T(\Sigma + 1_s)$ with $s, s' \in S$, i.e. a "term" of output sort s' in a variable of sort s. Every polynomial induces an evaluation map $(T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'}$ that inserts elements of $(T\Sigma)_s$ for the variable. The syntactic $\mathbf{T}$-algebra of L is given by $e_L\colon \mathbf{T}\Sigma \twoheadrightarrow \mathbf{T}\Sigma/{\equiv_L}$, where $\equiv_L$ is the equivalence relation defined on sort s by $x \equiv_L y$ iff $L \cdot [p]\,(x) = L \cdot [p]\,(y)$ for all polynomials $p\colon 1_{s'} \to T(\Sigma + 1_s)$ with $s' \in S$.

In each of the above examples, $\equiv_L$ is based on unary operations. For monoids one uses the operations $v \mapsto xvy$ on Σ^*. They determine the syntactic morphism because the monoid structure on any quotient of Σ^* can be recovered from them. For ω-semigroups, $\equiv_L$ uses the operations $v \mapsto xvy$ on Σ^+, $v \mapsto xvz$ from Σ^+ to Σ^ω, $v \mapsto x(vy)^\omega$ from Σ^+ to Σ^ω, and $v \mapsto xv$ on Σ^ω. They determine any finite ω-semigroup, see [22,35]. In the last example, the operations are $(T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'}$, and again this works as any finite quotient of $\mathbf{T}\Sigma$ is determined by the polynomials. Here is a categorical formulation of this phenomenon:

Definition 3.7. Let $\Sigma \in \mathbf{Set}_f^S$. By a *unary operation on* $\mathbf{T}\mathbb{\Sigma}$ is meant a morphism $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathscr{D}$, where s and s' are arbitrary sorts. A set $\mathbb{U}_\Sigma$ of unary operations on $\mathbf{T}\mathbb{\Sigma}$ is a *unary presentation of* $\mathbf{T}$ *over* Σ if for any extensible finite quotient $e\colon T\mathbb{\Sigma} \twoheadrightarrow A$ in $\mathscr{D}^S$ (see Rem. 2.11.2) the following are equivalent:

(i) e is a $\mathbf{T}$-*algebra congruence* on $\mathbf{T}\mathbb{\Sigma}$, i.e. there exists a $\mathbf{T}$-algebra structure (A, α) on A for which $e\colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow (A, \alpha)$ is a $\mathbf{T}$-homomorphism.

(ii) Each operation $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ has a lifting along e, i.e. a morphism $u_A\colon A_s \to A_{s'}$ in $\mathscr{D}$ with $e \cdot u = u_A \cdot e$.

Notation 3.8. For any set $\mathbb{U}_{\mathbb{\Sigma}}$ of unary operations on $\mathbf{T}\mathbb{\Sigma}$, we denote by $\mathbb{U}_\Sigma(s)$ the subset of all members with domain $(T\mathbb{\Sigma})_s$, and by $\overline{\mathbb{U}}_\Sigma$ the closure of $\mathbb{U}_\Sigma$ under composition and identity morphisms.

Definition 3.9. Let $\mathbb{U}_\Sigma$ be a set of unary operations on $\mathbf{T}\mathbb{\Sigma}$ and $L\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ be a language. If $\mathscr{D}$ is a variety of algebras, the *syntactic equivalence* of L (w.r.t. $\mathbb{U}_\Sigma$) is the S-sorted equivalence relation $\equiv_L$ on $|T\mathbb{\Sigma}|$ defined as follows: for elements $x, y \in |T\mathbb{\Sigma}|_s$ put

$$x \equiv_L y \quad \text{iff} \quad L \cdot u(x) = L \cdot u(y) \text{ for all } u \in \overline{\mathbb{U}}_\Sigma(s).$$

If $\mathscr{D}$ is a variety of ordered algebras, the *syntactic preorder* of L (w.r.t. $\mathbb{U}_\Sigma$) is the S-sorted preorder $\leq_L$ on $|T\mathbb{\Sigma}|$ defined on sort s by

$$x \leq_L y \quad \text{iff} \quad L \cdot u(x) \leq L \cdot u(y) \text{ for all } u \in \overline{\mathbb{U}}_\Sigma(s).$$

Let $\mathscr{D}$ be a variety of ordered algebras. Recall that a *congruence* on $D \in \mathscr{D}^S$ is an S-sorted preorder $\leq \,=\, (\leq_s)_{s\in S}$ on $|D|$ such that the preorder $\leq_s$ on $|D|_s$ extends the order of D_s and respects all operations of D_s. The ordered quotient algebra $\pi\colon D \twoheadrightarrow D/\!\leq$ induced by $\leq$ is carried by the equivalence classes of the S-sorted equivalence relation $\equiv \,=\, (\leq \cap \geq)$, with induced algebraic structure and order. Clearly $\leq_L$ is a congruence on $T\mathbf{\Sigma}$. Likewise, in the unordered case, $\equiv_L$ is a congruence on $T\mathbf{\Sigma}$. Thus one can form the quotient $e_L : T\mathbf{\Sigma} \twoheadrightarrow T\mathbf{\Sigma}/\!\leq_L$ (resp. $e_L : T\mathbf{\Sigma} \twoheadrightarrow T\mathbf{\Sigma}/\!\equiv_L$) in $\mathscr{D}^S$ and it is natural to ask when it forms a syntactic morphism for L. This turns out to hold whenever $\mathbb{U}_\Sigma$ is a unary presentation:

Theorem 3.10. *Let $\mathbb{U}_\Sigma$ be a set of unary operations on $\mathbf{T}\mathbf{\Sigma}$. Then the following statements are equivalent:*

- *(i) $\mathbb{U}_\Sigma$ is a unary presentation of $\mathbf{T}$ over Σ.*
- *(ii) For each recognizable language $L\colon T\mathbf{\Sigma} \to O_{\mathscr{D}}$, the morphism $e_L\colon T\mathbf{\Sigma} \twoheadrightarrow T\mathbf{\Sigma}/\!\equiv_L$ (resp. $e_L\colon T\mathbf{\Sigma} \twoheadrightarrow T\mathbf{\Sigma}/\!\leq_L$) is a $\mathbf{T}$-algebra congruence on $\mathbf{T}\mathbf{\Sigma}$ and forms a syntactic morphism of L.*

Example 3.11. 1. $\mathbf{T} = \mathbf{T}_*$ on $\mathbf{Set}$: by Example 3.6.1 and Theorem 3.10, we have for all $\Sigma \in \mathbf{Set}_f$ the unary presentation $\mathbb{U}_\Sigma = \{\, \Sigma^* \xrightarrow{x\cdot - \cdot y} \Sigma^* : x, y \in \Sigma^* \,\}$.

2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$: by Example 3.6.2 and Theorem 3.10, we have for all $\overline{\Sigma} = (\Sigma, \emptyset) \in \mathbf{Set}_f^2$ the unary presentation $\mathbb{U}_{\overline{\Sigma}}$ consisting of the maps $\Sigma^+ \xrightarrow{x\cdot - \cdot y} \Sigma^+$, $\Sigma^+ \xrightarrow{x\cdot - \cdot z} \Sigma^\omega$, $\Sigma^+ \xrightarrow{x\cdot(-\cdot y)^\omega} \Sigma^\omega$ and $\Sigma^\omega \xrightarrow{x\cdot -} \Sigma^\omega$ with $x, y \in \Sigma^*$ and $z \in \Sigma^\omega$.
3. Let $\mathbf{T}$ be any monad on $\mathbf{Set}^S$. By Example 3.6.3 and Theorem 3.10 we have for all $\Sigma \in \mathbf{Set}_f^S$ the unary presentation $\mathbb{U}_\Sigma = \{\, (T\Sigma)_s \xrightarrow{[p]} (T\Sigma)_{s'} : p$ is a polynomial over $\Sigma \,\}$.

4 Pseudovarieties of T-algebras

In this section we investigate pseudovarieties of $\mathbf{T}$-algebras, the "algebraic half" of any Eilenberg-type correspondence, and their connection to profinite $\widehat{\mathbf{T}}$-algebras.

Definition 4.1. A *Σ-generated $\mathbf{T}$-algebra* is a quotient $e\colon \mathbf{T}\mathbf{\Sigma} \twoheadrightarrow A$ of $\mathbf{T}\mathbf{\Sigma}$ in $\mathbf{Alg}\,\mathbf{T}$. The *subdirect product* of $e_i\colon \mathbf{T}\mathbf{\Sigma} \twoheadrightarrow A_i$ $(i = 0, 1)$ is the image $e\colon \mathbf{T}\mathbf{\Sigma} \twoheadrightarrow A$ of the $\mathbf{T}$-homomorphism $\langle e_0, e_1\rangle\colon \mathbf{T}\mathbf{\Sigma} \to A_0 \times A_1$. We call e_1 a *quotient* of e_0 if e_1 factors through e_0. A *local pseudovariety of Σ-generated $\mathbf{T}$-algebras* is a class of Σ-generated *finite* $\mathbf{T}$-algebras closed under subdirect products and quotients.

The class of local pseudovarieties over Σ is a complete lattice w.r.t. intersection.

Definition 4.2. A $\widehat{\mathbf{T}}$-algebra is *profinite* if it is an inverse limit of finite $\widehat{\mathbf{T}}$-algebras (cf. Remark 2.11.1). By a *Σ-generated profinite $\widehat{\mathbf{T}}$-algebra* is meant a quotient $e\colon \widehat{\mathbf{T}}\mathbf{\Sigma} \twoheadrightarrow A$ of $\widehat{\mathbf{T}}\mathbf{\Sigma}$ in $\mathbf{Alg}\,\widehat{\mathbf{T}}$ with A profinite. Σ-generated profinite $\widehat{\mathbf{T}}$-algebras are ordered by $e \leq e'$ iff e factors through e'.

Proposition 4.3. *For each $\Sigma \in \mathbf{Set}_f^S$, the lattices of local pseudovarieties of Σ-generated $\mathbf{T}$-algebras and Σ-generated profinite $\widehat{\mathbf{T}}$-algebras are isomorphic.*

Proof (Sketch). For any Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $e\colon \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$, form the local pseudovariety $\mathscr{P}^e$ consisting of all Σ-generated finite $\mathbf{T}$-algebras arising as quotients of A (cf. Remark 2.11.1). Then $e \mapsto \mathscr{P}^e$ gives the isomorphism. □

Remark 4.4. If $\mathscr{D}$ is a variety of ordered algebras, Proposition 4.3 can be interpreted in terms of profinite inequations. By a *profinite inequation over* Σ is meant a pair of elements $u, v \in |\hat{T}\mathbb{\Sigma}|_s$ in some sort s. A Σ-generated finite $\mathbf{T}$-algebra $e\colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ *satisfies the equation* $u \le v$ if $e^+(u) \le e^+(v)$. From 4.3 it easily follows that local pseudovarieties are precisely the classes of Σ-generated finite $\mathbf{T}$-algebras presentable by profinite inequations over Σ. Likewise, if $\mathscr{D}$ is a variety of algebras, local pseudovarieties are presentable by *profinite equations.*

Eilenberg's variety theorem deals, in lieu of languages over a fixed alphabet, with all alphabets at once. We will do the same in all our one-sorted applications. However, Example 2.4.2 shows that in a many-sorted setting one often needs to make a suitable *choice* of alphabets in $\mathbf{Set}_f^S$.

Notation 4.5. For the rest of this paper, we fix a class $\mathbb{A} \subseteq \mathbf{Set}_f^S$ of alphabets.

Definition 4.6. A $\mathbf{T}$-algebra A is $\mathbb{A}$*-generated* if there exists a surjective $\mathbf{T}$-homomorphism $e\colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ for some $\Sigma \in \mathbb{A}$. By a *pseudovariety of* $\mathbf{T}$*-algebras* is meant a class of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras closed under quotients and $\mathbb{A}$-generated subalgebras of finite products.

Remark 4.7. In most applications all finite products of $\mathbb{A}$-generated $\mathbf{T}$-algebras are $\mathbb{A}$-generated. In this case the definition of a pseudovariety simplifies: it is a class of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras closed under quotients, $\mathbb{A}$-generated subalgebras, and finite products.

Example 4.8. 1. Every finite $\mathbf{T}$-algebra (A, α) is $\mathbf{Set}_f^S$-generated: since $\mathscr{D}$ is locally finite, there exists an epimorphism $e\colon \mathbb{\Sigma} \twoheadrightarrow A$ with $\Sigma \in \mathbf{Set}_f^S$, so (A, α) is a quotient of $\mathbf{T}\mathbb{\Sigma}$ via $(\mathbf{T}\mathbb{\Sigma} \xrightarrow{Te} \mathbf{T}A \xrightarrow{\alpha} (A, \alpha))$. Thus, for $\mathbb{A} = \mathbf{Set}_f^S$, a pseudovariety of $\mathbf{T}$-algebras is a class of finite $\mathbf{T}$-algebras closed under quotients, subalgebras, and finite products. This concept was studied in [12]. For the monad $\mathbf{T}_*$ on $\mathbf{Set}$ we get the original concept of Eilenberg: a class of finite monoids closed under quotients, submonoids, and finite products.

2. Let $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$. As suggested by Example 2.4.2, we choose $\mathbb{A} = \{\, (\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f \,\}$. A finite $\mathbf{T}_\infty$-algebra (= finite ω-semigroup) A is $\mathbb{A}$-generated iff it is *complete*, i.e. every element $a \in A_\omega$ can be expressed as an infinite product $a = \pi(a_0, a_1, \ldots)$ for some $a_i \in A_+$. Clearly complete ω-semigroups are closed under finite products. Thus a pseudovariety of $\mathbf{T}_\infty$-algebras is a class of finite complete ω-semigroups closed under quotients, complete ω-subsemigroups, and finite products. This concept is due to Wilke [35]; see also [22].

The following definition generalizes a notion introduced for monoids in [13].

Remark 4.9. Every $\mathbf{T}$-homomorphism $g\colon \mathbf{T}D' \to \mathbf{T}D$ with $D, D' \in \mathscr{D}_f^S$ extends to a $\widehat{\mathbf{T}}$-homomorphism $\hat{g}\colon \widehat{\mathbf{T}}D' \to \widehat{\mathbf{T}}D$ with $\iota_D \cdot g = V\hat{g} \cdot \iota_{D'}$, cf. Rem. 2.11.2.

Definition 4.10. A *profinite theory* is a family $\varphi = (\varphi_\Sigma \colon \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$ of Σ-generated profinite $\widehat{\mathbf{T}}$-algebras such that for every $\mathbf{T}$-homomorphism $g \colon \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$ with $\Sigma, \Delta \in \mathbb{A}$ there exists a $\widehat{\mathbf{T}}$-homomorphism $g_P \colon P_\Delta \to P_\Sigma$ with $\varphi_\Sigma \cdot \hat{g} = g_P \cdot \varphi_\Delta$. We put $\varphi \leq \varphi'$ iff φ_Σ factors through φ'_Σ for each $\Sigma \in \mathbb{A}$.

Proposition 4.11. *The lattice of pseudovarieties of* $\mathbf{T}$*-algebras (ordered by inclusion) is isomorphic to the lattice of profinite theories.*

Proof (Sketch). For any profinite theory $\varphi = (\varphi_\Sigma \colon \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$, form the pseudovariety $\mathscr{V}_\varphi$ of all finite $\mathbf{T}$-algebras (A, α) for which (A, α^+) (cf. Rem. 2.11.1) is a quotient of some P_Σ. The map $\varphi \mapsto \mathscr{V}_\varphi$ gives the isomorphism. ☐

Remark 4.12. If $\mathscr{D}$ is a variety of ordered algebras, this result can again be interpreted via inequations. A finite $\mathbf{T}$-algebra A *satisfies* a profinite inequation $u \leq v$ over $\Sigma \in \mathbb{A}$ if $e^+(u) \leq e^+(v)$ for all surjective $\mathbf{T}$-homomorphisms $e \colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$. Pseudovarieties are the classes of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras presentable by profinite inequations over $\mathbb{A}$. In the unordered case, take equations $u = v$ in lieu of inequations. For $\mathbb{A} = \mathbf{Set}_f^S$, this was proved in [12, Thm. 4.12 and Rem. 5.7].

5 The Variety Theorem

In this section we present our variety theorem. We assume throughout that, for each $\Sigma \in \mathbf{Set}_f^S$, a unary presentation $\mathbb{U}_\Sigma$ of $\mathbf{T}$ over Σ is given.

Remark 5.1. Recall that the variety $\mathscr{C}$ is assumed to have a constant in the signature. Choosing a constant gives a natural transformation from $C_{\mathbb{1}} \colon \mathscr{C} \to \mathscr{C}$, the constant functor on $\mathbb{1} \in \mathscr{C}$, to the identity functor $\mathsf{Id}_{\mathscr{C}}$. It dualizes to a natural transformation $\bot \colon \mathsf{Id}_{\widehat{\mathscr{D}}} \to C_{O_{\mathscr{D}}}$. The idea is that $\bot$ models the empty set. For the categories $\mathscr{D}$ of Example 2.3 we have $O_{\mathbf{Set}} = \{0, 1\}$, $O_{\mathbf{Pos}} = O_{\mathbf{JSL_0}} = \{0 < 1\}$ (the two-chain) and $O_{\mathbf{Vec}_K} = K$, and in each case we choose $\bot \colon D \to O_{\mathscr{D}}$ for $D \in \widehat{\mathscr{D}}$ to be the constant morphism with value 0.

Definition 5.2. Let $L \colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ be a language over $\Sigma \in \mathbf{Set}_f^S$.

1. The *derivative* $u^{-1}L$ of L w.r.t. an operation $u \colon (T\Sigma)_s \to (T\Sigma)_{s'}$ in $\mathbb{U}_\Sigma$ is the language over Σ given on sort s by $(T\mathbb{\Sigma})_s \xrightarrow{u} (T\mathbb{\Sigma})_{s'} \xrightarrow{L_{s'}} O_{\mathscr{D}}$ and on sorts $t \neq s$ by $(T\mathbb{\Sigma})_t \xrightarrow{\iota_{\mathbb{\Sigma}}} (V\hat{T}\mathbb{\Sigma})_t \xrightarrow{V\bot} O_{\mathscr{D}}$.
2. The *preimage* $g^{-1}L$ of L under a $\mathbf{T}$-homomorphism $g \colon \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$ is the language over Δ defined by $T\mathbb{\Delta} \xrightarrow{g} T\mathbb{\Sigma} \xrightarrow{L} O_{\mathscr{D}}$.

Example 5.3. 1. $\mathbf{T} = \mathbf{T}_*$ on $\mathbf{Set}$: let $\mathbb{U}_\Sigma$ as in Example 3.11.1. The derivatives of $L \subseteq \Sigma^*$ w.r.t. $\mathbb{U}_\Sigma$ are the languages $x^{-1}Ly^{-1} = \{\, v \in \Sigma^* : xvy \in L \,\}$ for $x, y \in \Sigma^*$. These are the classical derivatives for languages of finite words.

2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$: let $\mathbb{U}_{\overline{\Sigma}}$ (with $\overline{\Sigma} = (\Sigma, \emptyset)$) as in Example 3.11.2. The derivatives of $L \subseteq \Sigma^+ \cup \Sigma^\omega$ w.r.t. the operations in $\mathbb{U}_{\overline{\Sigma}}$ are the languages $\{\, v \in \Sigma^+ : xvy \in L \,\}$, $\{\, v \in \Sigma^+ : xvz \in L \,\}$, $\{\, v \in \Sigma^+ : x(vy)^\omega \in L \,\}$, and $\{\, v \in \Sigma^\omega : xv \in L \,\}$, where $x, y \in \Sigma^*$ and $z \in \Sigma^\omega$. These are the derivatives for ∞-languages studied by Wilke [35].

3. Let $\mathbf{T}$ be a monad on $\mathbf{Set}^S$, and take the polynomial presentation $\mathbb{U}_\Sigma$ of Example 3.11.3. The derivatives of a language $L \subseteq T\Sigma$ w.r.t. $\mathbb{U}_\Sigma$ are the languages $p^{-1}L \subseteq T\Sigma$ with $(p^{-1}L)_s = \{\, v \in (T\Sigma)_s : [p](v) \in L_{s'} \,\}$ and $(p^{-1}L)_t = \emptyset$ for $t \neq s$, where $p\colon 1_{s'} \to T(\Sigma + 1_s)$ is a polynomial over Σ. These are the *polynomial derivatives* studied by Bojańczyk [11].

Proposition 5.4. *Derivatives and preimages of recognizable languages are recognizable.*

Remark 5.5. Recall the isomorphism $\mathsf{Rec}(\Sigma) \cong \prod_s P(\hat{T}\mathbb{\Sigma})_s$ of Remark 3.4. In the following we study subobjects $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ in $\mathscr{C}$. However, for technical reasons we restrict ourselves to subobjects of the form $\prod_s m_s\colon \prod_s (W'_\Sigma)_s \rightarrowtail \prod_s P(\hat{T}\mathbb{\Sigma})_s$, where $m_s\colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\mathbb{\Sigma})_s$ is a monomorphism in $\mathscr{C}$:

$$\begin{tikzcd}
W_\Sigma \arrow[r, tail, "\subseteq"] \arrow[d, "\cong"'] & \mathsf{Rec}(\Sigma) \arrow[d, "\cong"] \\
\prod_s (W'_\Sigma)_s \arrow[r, tail, "\prod_s m_s"'] & \prod_s P(\hat{T}\mathbb{\Sigma})_s
\end{tikzcd}$$

Such subobjects are called *admissible.* Clearly, for $S = 1$, any subobject of $\mathsf{Rec}(\Sigma)$ is admissible. More importantly, if $\mathscr{C}$ is one of the categories of Example 2.3 and $\mathbb{U}_\Sigma$ contains all identity morphisms, one can show that any subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ closed under derivatives (i.e. $L \in W_\Sigma$ implies $u^{-1}L \in W_\Sigma$ for all $u \in \mathbb{U}_\Sigma$) is admissible. Thus, in these cases the admissibility condition in Definition 5.6.1 below can be dropped. For Definition 5.6.2, recall from the previous section that we work with a fixed class $\mathbb{A} \subseteq \mathbf{Set}_f^S$ of alphabets.

Definition 5.6. 1. A *local variety of languages* over an alphabet Σ is an admissible subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ closed under derivatives.

2. A *variety of languages* is a family of local varieties $(W_\Sigma \subseteq \mathsf{Rec}(\Sigma))_{\Sigma \in \mathbb{A}}$ closed under preimages, i.e. $L \in W_\Sigma$ implies $g^{-1}L \in W_\Delta$ for all $\Sigma, \Delta \in \mathbb{A}$ and all $\mathbf{T}$-homomorphisms $g\colon \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$.

We are ready to state our main result, which holds under the Assumptions 2.1.

Theorem 5.7 (Variety Theorem).

1. *The lattice of local varieties of languages over $\Sigma \in \mathbf{Set}_f^S$ (ordered by inclusion) is isomorphic to the lattice of local pseudovarieties of Σ-generated $\mathbf{T}$-algebras.*
2. *The lattice of varieties of languages (ordered by inclusion) is isomorphic to the lattice of pseudovarieties of $\mathbf{T}$-algebras.*

Proof (Sketch). Duality! For the first isomorphism one shows that an admissible subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$, represented by a morphism $(\ m_s\colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\mathbb{\Sigma})_s\ )_{s\in S}$ in $\mathscr{C}^S$, is closed under derivatives iff its dual $(\ P^{-1}m_s\colon (\hat{T}\mathbb{\Sigma})_s \twoheadrightarrow P^{-1}(W'_\Sigma)_s\ )_{s\in S}$ in $\widehat{\mathscr{D}}^S$ carries a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra. Then Proposition 4.3 gives the isomorphism. For the second isomorphism, one shows that a family $(W_\Sigma)_{\Sigma\in\mathbb{A}}$ of local varieties is closed under preimages iff its dual family forms a profinite theory. Then Proposition 4.11 gives the isomorphism. $\square$

Remark 5.8. Straubing [34] studied C-*varieties of regular languages* which are defined as Eilenberg's varieties of regular languages, except that closure under preimages is required only w.r.t. a given class C of monoid morphisms. By making a class C of $\mathbf{T}$-homomorphisms an additional parameter of our framework, Theorem 5.7 (and its duality-based proof) easily generalize to a monad version of Straubing's variety theorem for C-varieties.

6 Applications

(a) Languages of finite words. Let $\mathscr{D}$ be a *commutative* variety of algebras or ordered algebras, i.e. for any two objects $A, B \in \mathscr{D}$ the hom-set $\mathscr{D}(A, B)$ carries a subobject of $B^{|A|}$ in $\mathscr{D}$. All varieties $\mathscr{D}$ of Example 2.3 are commutative. A $\mathscr{D}$-*monoid* is an object $D \in \mathscr{D}$ with a monoid structure $(|D|, \bullet, 1)$ on the underlying set such that the multiplication is a *bimorphism*; that is, for every $x \in |D|$ the maps $x \bullet - \colon |D| \to |D|$ and $- \bullet x \colon |D| \to |D|$ carry endomorphisms on D. Let $\mathbf{T}_M$ be the monad on $\mathscr{D}$ constructing free $\mathscr{D}$-monoids. In [1] we showed that the free $\mathscr{D}$-monoid on $\mathbb{\Sigma} \in \mathscr{D}$ is $(\mathbb{\Sigma}^*, \bullet, \varepsilon)$, where $\mathbb{\Sigma}^*$ is the free $\mathscr{D}$-object on the set Σ^*, the multiplication $\bullet$ extends the concatenation of words, and the unit ε is the empty word. Thus $T_M \mathbb{\Sigma} = \mathbb{\Sigma}^*$. A language $L \colon T_M \mathbb{\Sigma} \to O_{\mathscr{D}}$ is $\mathbf{T}_M$-recognizable iff its adjoint transpose $L' \colon \Sigma^* \to |O_{\mathscr{D}}|$ (via the right adjoint $|{-}| \colon \mathscr{D} \to \mathbf{Set}$) is regular, i.e. computed by some finite Moore automaton with output set $|O_{\mathscr{D}}|$. Generalizing Example 3.6.1, we showed in [2] that each recognizable language $L \colon \mathbb{\Sigma}^* \to O_{\mathscr{D}}$ has a syntactic $\mathscr{D}$-monoid $e_L \colon \mathbb{\Sigma}^* \twoheadrightarrow \mathbb{\Sigma}^*/{\equiv_L}$, where $v \equiv_L w$ iff $L(x \bullet v \bullet y) = L(x \bullet w \bullet y)$ for all $x, y \in \Sigma^*$. For ordered varieties $\mathscr{D}$, e.g. $\mathbf{Pos}$, one uses in lieu of $\equiv_L$ the preorder $\leq_L$ on $\mathbb{\Sigma}^*$ defined by $v \leq_L w$ iff $L(x \bullet v \bullet y) \leq L(x \bullet w \bullet y)$ for all $x, y \in \Sigma^*$, and forms the induced poset $\mathbb{\Sigma}^*/{\leq_L}$. Theorem 3.10 gives the unary presentation $\mathbb{U}_\Sigma = \{\, \mathbb{\Sigma}^* \xrightarrow{x \bullet - \bullet y} \mathbb{\Sigma}^* : x, y \in \Sigma^* \,\}$ for all $\Sigma \in \mathbf{Set}_f$. Instantiating Definition 5.6 to $\mathbf{T} = \mathbf{T}_M$, a *variety of regular languages in* $\mathscr{C}$ associates to each Σ a set of regular languages over Σ closed under $\mathscr{C}$-algebraic operations (see Remark 3.4), derivatives (see Example 5.3.1) and preimages of $\mathscr{D}$-monoid morphisms. Theorem 5.7 then specializes to the main results of our papers [1, 3, 13]:

Theorem 6.1. *The lattice of (local) varieties of regular languages in $\mathscr{C}$ is isomorphic to the lattice of (local) pseudovarieties of $\mathscr{D}$-monoids.*

For the categories of Example 2.3 we recover the Eilenberg theorems listed in the table below. The third column describes the $\mathscr{C}$-algebraic operations under which (local) varieties of languages are closed, and the fourth column states what $\mathscr{D}$-monoids instantiate to. All these correspondences are known in the literature, and are uniformly covered by Theorem 6.1.

$\mathscr{C}$	$\mathscr{D}$	(local) var. of lang. closed under $\cong$	(local) pseudovarieties of	proved in
$\mathbf{BA}$	$\mathbf{Set}$	boolean operations	monoids	[15, 17]
$\mathbf{DL}_{01}$	$\mathbf{Pos}$	union and intersection	ordered monoids	[17, 23]
$\mathbf{JSL}_0$	$\mathbf{JSL}_0$	union	idempotent semirings	[27]
$\mathbf{Vec}_K$	$\mathbf{Vec}_K$	addition of weighted languages	K-algebras	[29]

(b) Polynomial varieties. Let $\mathbf{T}$ be any monad on $\mathbf{Set}^S$. Choose $\mathbb{A} = \mathbf{Set}_f^S$ and $\mathbb{U}_\Sigma$ as in Example 3.11.3. A *polynomial variety of* $\mathbf{T}$*-recognizable languages* associates to each $\Sigma \in \mathbf{Set}_f^S$ a set of $\mathbf{T}$-recognizable languages over Σ closed under boolean operations, polynomial derivatives (see Example 5.3.3), and preimages of $\mathbf{T}$-homomorphisms. Theorem 5.7 yields the following Eilenberg correspondence. Its non-local part is due to Bojańczyk [11].

Theorem 6.2. *The lattice of (local) polynomial varieties of* $\mathbf{T}$*-recognizable languages is isomorphic to the lattice of (local) pseudovarieties of* $\mathbf{T}$*-algebras.*

Next, we consider correspondences that are *not* covered by Theorem 6.1 and 6.2, but are either instances of our Theorem 5.7, or emerge by introducing new parameters to our setting.

(c) Languages of ∞-words. Let $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$ with $\mathbb{A} = \{\, (\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f \,\}$, and consider the unary presentation of Example 3.11.2. A *variety of* ∞*-languages* associates to each $\Sigma \in \mathbf{Set}_f$ a set of regular ∞-languages over Σ closed under boolean operations, derivatives (see Example 5.3.2) and preimages of ω-semigroup morphisms. Theorem 5.7 gives

Theorem 6.3. *The lattice of (local) varieties of* ∞*-languages is isomorphic to the lattice of (local) pseudovarieties of* ω*-semigroups.*

The non-local part is Wilke's theorem for ∞-languages [35] (in the formulation of [22]), while the local part is a new result, extending the corresponding result of Gehrke, Grigorieff, and Pin [17] for finite words. Similarly, one can take the monad $\mathbf{T}_{\infty,\leq}$ on $\mathscr{D} = \mathbf{Pos}$ representing *ordered* ω*-semigroups.* Since $\mathscr{C} = \mathbf{DL}_{01}$, we obtain *positive varieties of* ∞*-languages*, emerging from Wilke's concept by dropping closure under complement. Then Theorem 5.7 gives the result below. Its non-local part is due to Pin [24], and the local part is again a new result.

Theorem 6.4. *The lattice of (local) positive varieties of* ∞*-languages is isomorphic to the lattice of (local) pseudovarieties of ordered* ω*-semigroups.*

Let us outline three further examples that could be treated with the same techniques as above; we postpone the details to a journal version of this paper.

(d) Ordered words. A natural generalization of ∞-words are words on linear orderings, for which Bedon et al. [8, 9] establish two variety theorems. Both are instances of Theorem 5.7.

(e) Tree languages. Languages of binary trees are represented by the monad $\mathbf{T}$ on $\mathbf{Set}^3$ associated to Wilke's *tree algebras* [36]. The free tree algebra on $(\Sigma, \emptyset, \emptyset)$ is $T(\Sigma, \emptyset, \emptyset) = (\Sigma, T_\Sigma, C_\Sigma)$ where T_Σ is the set of Σ-labeled finite binary trees (labeled at every node) and C_Σ is the set of *contexts*, i.e. $(\Sigma + \{*\})$-labeled binary trees where $*$ appears only at a single leaf. We take $\mathbb{A} = \{\, (\Sigma, \emptyset, \emptyset) : \Sigma \in \mathbf{Set}_f \,\}$. Tree languages are subsets of T_Σ, or equivalently, subsets of $T(\Sigma, \emptyset, \emptyset)$ that are empty in the first and third sort. On the algebraic side, one needs to restrict to *reduced* tree algebras. These are $\mathbb{A}$-generated $\mathbf{T}$-algebras A determined by the second sort, in the sense that a quotient $e \colon A \twoheadrightarrow B$ is an isomorphism whenever it is bijective in the second sort. The variety theorem of Salehi and Steinby [32] establishes a bijective correspondence between varieties of tree languages and pseudovarieties of reduced tree algebras. This is not a direct instance of Theorem 5.7, as languages are restricted to a subset of the sorts. However, by making our setting parametric in a subset $S_0 \subseteq S$, we can cover this result with our methods.

(f) Cost functions. Daviaud, Kuperberg, and Pin [14] study varieties of *regular cost functions*, a quantitative version of regular languages. The corresponding algebras are called *stabilization algebras*. These are ordered algebras whose axioms involve inequations but also an implication. Consequently stabilization algebras do not form a variety of ordered algebras and are not represented by a monad on **Pos**. However, one can take the monad $\mathbf{T}_S$ on **Pos** associated to the theory of stabilization algebras *minus* the implication. Then, as shown in [14], regular cost functions correspond to languages $L\colon T_S\Sigma \to \{0 < 1\}$ recognized by finite stabilization algebras (rather than *arbitrary* finite $\mathbf{T}_S$-algebras).

To cover stabilization algebras in our categorical setting, we need an additional parameter: a *quasivariety* $\mathscr{Q} \subseteq \mathbf{Alg}_f\,\mathbf{T}$ of finite $\mathbf{T}$-algebras, i.e. a subclass closed under subalgebras and finite products. (In the above example, $\mathscr{Q}$ is taken to be the quasivariety of all finite stabilization algebras, that is, finite $\mathbf{T}_S$-algebras satisfying the implication.) In lieu of the profinite monad $\widehat{\mathbf{T}}$ we form the *pro-$\mathscr{Q}$ monad* $\widehat{\mathbf{T}}_{\mathscr{Q}}$ on $\widehat{\mathscr{D}}^S$, where $\hat{T}_{\mathscr{Q}}D$, for $D \in \mathscr{D}_f^S$, is the inverse limit of all quotients of $\mathbf{T}D$ in $\mathscr{Q}$. Profinite $\widehat{\mathbf{T}}$-algebras are replaced by *pro-$\mathscr{Q}$ algebras* for $\widehat{\mathbf{T}}_{\mathscr{Q}}$, i.e. quotient algebras of $\widehat{\mathbf{T}}_{\mathscr{Q}}$ arising as inverse limits of algebras in $\mathscr{Q}$. A *pseudovariety of* $\mathbf{T}$*-algebras relative to* $\mathscr{Q}$ is a subclass of $\mathscr{Q}$ closed under quotients (in $\mathscr{Q}$) and $\mathbb{A}$-generated subalgebras of finite products. Theorem 5.7 and its proof then easily generalize to a correspondence between varieties of $\mathscr{Q}$-recognizable languages and pseudovarieties of $\mathbf{T}$-algebras relative to $\mathscr{Q}$. For the above monad $\mathbf{T}_S$ on **Pos** and $\mathscr{Q}$ = finite stabilization algebras, we recover the variety theorem of [14]: varieties of cost functions correspond to pseudovarieties of stabilization algebras.

7 Conclusions and Future Work

We presented a duality-based framework for algebraic language theory that captures the bulk of Eilenberg theorems in the literature. Besides working out the details of (d)–(f) above, there are several interesting directions for future work. We aim to investigate additional parameters, e.g. use an abstract factorization system in $\mathscr{D}$ and $\widehat{\mathscr{D}}$, and use in lieu of free objects $\mathbb{\Sigma}$ arbitrary (finite) objects as "alphabets". This would put even more examples under the roof of our theory, e.g. infinitary Eilenberg-type correspondences as in [6, 31] that relate varieties of (not necessarily finite) algebras to varieties of (not necessarily recognizable) languages. By studying the free-category monad on the category of graphs, we expect a variety theorem for languages of finite paths vs. pseudovarieties of categories, a counterpart to the Reiterman theorem for finite categories of Jones [19]. It should also be interesting to investigate whether by using ideas from our framework it is possible to obtain a variety theory for data languages based on *nominal* Stone duality [16]. Putting all this under one roof might then truly yield the *One Eilenberg Theorem to Rule Them All.*

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This appendix contains all proofs and additional details we omitted due to space restrictions.

A Categorical toolkit

We review some concepts from category theory we will use throughout this paper. For details we refer to standard textbooks such as [21], and also to [4] for an introduction to locally presentable categories.

A.1 Monads. A *monad* on a category $\mathscr{A}$ is a triple $\mathbf{T} = (T, \eta, \mu)$ consisting of an endofunctor $T\colon \mathscr{A} \to \mathscr{A}$ and two natural transformations $\eta\colon \mathsf{Id} \to T$ and $\mu\colon TT \to T$ (called the *unit* and *multiplication* of $\mathbf{T}$) such that the following diagrams commute:

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "id"'] & TT \arrow[d, "\mu"] \\
& T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "id"'] & TT \arrow[d, "\mu"] \\
& T
\end{tikzcd}
\qquad
\begin{tikzcd}
TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"'] & T
\end{tikzcd}$$

Given two monads $\mathbf{S} = (S, \eta^{\mathbf{S}}, \eta^{\mathbf{S}})$ and $\mathbf{T} = (T, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$ on $\mathscr{A}$, a *monad morphism* $\varphi\colon \mathbf{S} \to \mathbf{T}$ is a natural transformation $\varphi\colon S \to T$ making the following diagrams commute:

$$\begin{tikzcd}
\mathsf{Id} \arrow[r, "\eta^{\mathbf{S}}"] \arrow[dr, "\eta^{\mathbf{T}}"'] & S \arrow[d, "\varphi"] \\
& T
\end{tikzcd}
\qquad
\begin{tikzcd}
SS \arrow[r, "S\varphi"] \arrow[d, "\mu^{\mathbf{S}}"'] & ST \arrow[r, "\varphi T"] & TT \arrow[d, "\mu^{\mathbf{T}}"] \\
S \arrow[rr, "\varphi"'] & & T
\end{tikzcd}$$

A.2 Algebras for a monad. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathscr{A}$. By a $\mathbf{T}$-*algebra* is meant a pair (A, α) of an object $A \in \mathscr{A}$ and a morphism $\alpha\colon TA \to A$ satisfying the *unit* and *associative laws*:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "id"'] & TA \arrow[d, "\alpha"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
TTA \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\
TA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

Given two $\mathbf{T}$-algebras (A, α) and (B, β), a $\mathbf{T}$-*homomorphism* $h\colon (A, \alpha) \to (B, \beta)$ is a morphism $h\colon A \to B$ in $\mathscr{A}$ such that $h \cdot \alpha = \beta \cdot Th$. Denote by $\mathbf{Alg}\,\mathbf{T}$ the category of $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms. There is a forgetful functor $U\colon \mathbf{Alg}\,\mathbf{T} \to \mathscr{A}$ given by $(A, \alpha) \mapsto A$ on objects and $h \mapsto h$ on morphisms. It has a left adjoint assigning to each object A of $\mathscr{A}$ the $\mathbf{T}$-algebra $\mathbf{T}A = (TA, \mu_A)$, called the *free* $\mathbf{T}$-*algebra on* A, and to each morphism $h\colon A \to B$ the $\mathbf{T}$-homomorphism $Th\colon \mathbf{T}A \to \mathbf{T}B$. Note that for any $\mathbf{T}$-algebra (A, α) the associative law states precisely that $\alpha\colon \mathbf{T}A \to (A,\alpha)$ is a $\mathbf{T}$-homomorphism. Moreover, the unit law implies that α is a (split) epimorphism in $\mathscr{A}$.

A.3 Limits of T-algebras. The forgetful functor $U\colon \mathbf{Alg}\,\mathbf{T} \to \mathscr{A}$ preserves limits, being a right adjoint (see A.2). More importantly, it also *creates limits*. That is, given a diagram $D\colon \mathscr{S} \to \mathbf{Alg}\,\mathbf{T}$ and a limit cone $(p_s\colon A \to UD_s)_{s\in\mathscr{S}}$ over UD in $\mathscr{A}$, there exists a unique $\mathbf{T}$-algebra structure (A,α) on A such that all p_s are $\mathbf{T}$-homomorphisms, and moreover $(p_s\colon (A,\alpha) \to D_s)_{s\in\mathscr{S}}$ forms a limit cone over D in $\mathbf{Alg}\,\mathbf{T}$. In case $\mathscr{A}$ is complete, it follows that $\mathbf{Alg}\,\mathbf{T}$ is complete and that U *reflects limits*. That is, a cone $(p_s : (A,\alpha) \to D_s)_{s\in S}$ over D is a limit cone whenever $(p_s : A \to UD_s)_{s\in S}$ is a limit cone over UD.

A.4 Comma categories. Let $F\colon \mathscr{B} \to \mathscr{A}$ be a functor and A an object in $\mathscr{A}$. The *comma category* $(A \downarrow F)$ has as objects all morphisms $(A \xrightarrow{f} FB)$ in $\mathscr{A}$ with $B \in \mathscr{B}$, and its morphisms from $(A \xrightarrow{f_1} FB_1)$ to $(A \xrightarrow{f_2} FB_2)$ are morphisms $h\colon B_1 \to B_2$ in $\mathscr{B}$ with $f_2 = Fh \cdot f_1$. If $F\colon \mathscr{B} \hookrightarrow \mathscr{A}$ is the inclusion of a subcategory $\mathscr{B}$, we write $(A \downarrow \mathscr{B})$ for $(A \downarrow F)$.

A.5 Kan extensions. The *right Kan extension* of a functor $F\colon \mathscr{A} \to \mathscr{C}$ along $K\colon \mathscr{A} \to \mathscr{B}$ is a functor $R\colon \mathscr{B} \to \mathscr{C}$ together with a universal natural transformation $\varepsilon\colon RK \to F$, i.e. for every functor $G\colon \mathscr{B} \to \mathscr{C}$ and every natural transformation $\gamma\colon GK \to F$ there exists a unique natural transformation $\gamma^\dagger : G \to R$ with $\gamma = \varepsilon \cdot \gamma^\dagger K$. If $\mathscr{A}$ is small and $\mathscr{C}$ is complete, this extension exists, and the object RB for $B \in \mathscr{B}$ is the limit of the diagram

$$(B \downarrow K) \xrightarrow{Q^B} \mathscr{A} \xrightarrow{F} \mathscr{C}$$

that maps $(B \xrightarrow{f} KA)$ to FA and $h\colon (B \xrightarrow{f_1} KA_1) \to (B \xrightarrow{f_2} KA_2)$ to Fh.

A.6 Codensity monads. Let $\varepsilon\colon RK \to K$ be the right Kan extension of a functor $K\colon \mathscr{A} \to \mathscr{B}$ along itself. Then R can be equipped with a monad structure $\mathbf{R} = (R, \eta^{\mathbf{R}}, \mu^{\mathbf{R}})$ where the unit $\eta^{\mathbf{R}}$ is $(id_K)^\dagger\colon \mathsf{Id} \to R$ and the multiplication $\mu^{\mathbf{R}}$ is $(\varepsilon \cdot R\varepsilon)^\dagger\colon RR \to R$. The monad $\mathbf{R}$ is called the *codensity monad* of K, see e.g. [20].

A.7 Final functors. Let $\mathscr{K}$ be a cofiltered category (see A.8). A functor $F\colon \mathscr{K} \to \mathscr{B}$ is called *final* if

(i) for any object B of $\mathscr{B}$, there exists a morphism $f\colon FK \to B$ for some $K \in \mathscr{K}$, and
(ii) given two parallel morphisms $f, g\colon FK \to B$ with $K \in \mathscr{K}$, there exists a morphism $k\colon K' \to K$ in $\mathscr{K}$ with $f \cdot Fk = g \cdot Fk$.

The importance of final functors is that they facilitate the construction of limits. If $F : \mathscr{K} \to \mathscr{B}$ is final, then a diagram $D\colon \mathscr{B} \to \mathscr{A}$ has a limit iff the diagram $DF : \mathscr{K} \to \mathscr{A}$ has a limit, and in this case the two limit objects agree.

Specifically, any limit cone $(p_B\colon A \to D_B)_{B\in\mathscr{B}}$ over D restricts to a limit cone $(p_{FK}\colon A \to D_{FK})_{K\in\mathscr{K}}$ over DF.

A.8 Cofiltered limits and inverse limits. A category $\mathscr{K}$ is *cofiltered* if for every finite subcategory $D\colon \mathscr{K}' \rightarrowtail \mathscr{K}$ there exists a cone over D. This is equivalent to the following three conditions:

(i) $\mathscr{K}$ is nonempty.
(ii) For any two objects Y and Z of $\mathscr{K}$, there exist two morphisms $f\colon X \to Y$ and $g\colon X \to Z$ with a common domain X.
(iii) For any two parallel morphisms $f, g\colon Y \to Z$ in $\mathscr{K}$, there exists a morphism $e\colon X \to Y$ with $f \cdot e = g \cdot e$.

A *cofiltered limit* in a category $\mathscr{A}$ is a limit of a diagram $\mathscr{K} \to \mathscr{A}$ with cofiltered scheme $\mathscr{K}$. It is also called an *inverse limit* if $\mathscr{K}$ is a (co-directed) poset. For any small cofiltered category $\mathscr{K}$, there exists a final functor $F : \mathscr{K}_0 \to \mathscr{K}$ where $\mathscr{K}_0$ is a small co-directed poset. Consequently, a category has cofiltered limits iff it has inverse limits, and a functor preserves cofiltered limits iff it preserves inverse limits.

The dual concept of a cofiltered limit is a *filtered colimit*.

A.9 Finitely copresentable objects. An object A of a category $\mathscr{A}$ is called *finitely copresentable* if the hom-functor $\mathscr{A}(-, A)\colon \mathscr{A} \to \mathbf{Set}^{op}$ preserves cofiltered limits. Equivalently, for any cofiltered limit cone $(p_i\colon B \to B_i)_{i\in I}$ in $\mathscr{A}$ the following two statements hold:

(i) Every morphism $f\colon B \to A$ factors through some p_i.
(ii) For any $i \in I$ and any two morphisms $s, s'\colon B_i \to A$ with $s \cdot p_i = s' \cdot p_i$, there exists a connecting morphism $b_{ji}\colon B_j \to B_i$ in the given diagram with $s \cdot b_{ji} = s' \cdot b_{ji}$.

A.10 Locally finitely copresentable categories. A category $\mathscr{A}$ is called *locally finitely copresentable* if it satisfies the following three properties:

(i) $\mathscr{A}$ is complete;
(ii) the full subcategory $\mathscr{A}_f$ of finitely copresentable objects is essentially small, i.e. the objects of $\mathscr{A}_f$ (taken up to isomorphism) form a set;
(iii) any object A of $\mathscr{A}$ is a cofiltered limit of finitely copresentable objects; that is, there exists a cofiltered limit cone $(A \to A_i)_{i\in I}$ in $\mathscr{A}$ with $A_i \in \mathscr{A}_f$ for all $i \in I$.

If $\mathscr{A}$ is locally finitely copresentable, so is any functor category $\mathscr{A}^{\mathscr{S}}$, where $\mathscr{S}$ is an arbitrary small category. In particular, this holds for any product category $\mathscr{A}^S$ (where S is set) and for the arrow category $\mathscr{A}^{\to}$. The latter has as objects all morphisms of $\mathscr{A}$, and as morphisms from $(A \xrightarrow{f} B)$ to $(C \xrightarrow{g} D)$ all pairs of morphisms $(a : A \to C, b : B \to D)$ in $\mathscr{A}$ with $b \cdot f = g \cdot a$. The finitely copresentable objects of $\mathscr{A}^{\to}$ are precisely the arrows with finitely copresentable domain and codomain.

A.11 Cofiltered limits in locally finitely copresentable categories. Let $\mathscr{A}$ be a locally finitely copresentable category. A cone $(p_i\colon B \to B_i)_{i\in I}$ over a cofiltered diagram in $\mathscr{A}$ is a limit cone iff

(i) every morphism $f\colon B \to A$ with $A \in \mathscr{A}_f$ factors through some p_i, and
(ii) this factorization is essentially unique: given $i \in I$ and $s, s'\colon B_i \to A$ with $s \cdot p_i = s' \cdot p_i$, there exists a morphism $b_{ji}\colon B_j \to B_i$ in the diagram with $s \cdot b_{ji} = s' \cdot b_{ji}$.

Note that if all p_i's are epimorphisms, condition (ii) is trivial.

A.12 Canonical diagrams. Let $\mathscr{A}$ be a locally finitely copresentable category. Then for each object $A \in \mathscr{A}$ the comma category $(A \downarrow \mathscr{A}_f)$ is essentially small and cofiltered. The *canonical diagram of* A is the cofiltered diagram $K_A\colon (A \downarrow \mathscr{A}_f) \to \mathscr{A}$ that maps an object $(A \xrightarrow{f} A_1)$ to A_1 and a morphism $h\colon (A \xrightarrow{f_1} A_1) \to (A \xrightarrow{f_2} A_2)$ to $h\colon A_1 \to A_2$. Every object A of $\mathscr{A}$ is the cofiltered limit of its canonical diagram, that is, K_A has the limit cone

$$(f\colon A \to K_A f)_{f\in(A\downarrow\mathscr{A}_f)}.$$

A.13 Pro-completions. Let $\mathscr{B}$ be a small category. By a *pro-completion* (or a *free completion under cofiltered limits*) of $\mathscr{B}$ is meant a category $\mathsf{Pro}\,\mathscr{B}$ together with a full embedding $I\colon \mathscr{B} \rightarrowtail \mathsf{Pro}\,\mathscr{B}$ such that

(i) $\mathsf{Pro}\,\mathscr{B}$ has cofiltered limits.
(ii) For any functor $F\colon \mathscr{B} \to \mathscr{C}$ into a category $\mathscr{C}$ with cofiltered limits, there exists a functor $\overline{F}\colon \mathsf{Pro}\,\mathscr{B} \to \mathscr{C}$, unique up to natural isomorphism, such that $\overline{F}$ preserves cofiltered limits and $\overline{F} \cdot I$ is naturally isomorphic to F.

The universal property determines $\mathsf{Pro}\,\mathscr{B}$ uniquely up to equivalence of categories. If the category $\mathscr{B}$ has finite limits, then $\mathsf{Pro}\,\mathscr{B}$ is locally finitely copresentable, and its finitely copresentable objects are up to isomorphism the objects IB $(B \in \mathscr{B})$. Conversely, every locally finitely copresentable category $\mathscr{A}$ arises in this way: we have $\mathscr{A} = \mathsf{Pro}\,\mathscr{A}_f$.

The dual concept of a pro-completion is an *ind-completion*, i.e. the free completion under filtered colimits.

A.14 Factorization systems. A *factorization system* in a category $\mathscr{A}$ is a pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms of $\mathscr{A}$ with the following properties:

(i) Both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms.
(ii) Every morphism f of $\mathscr{A}$ has a factorization $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

(iii) The *diagonal fill-in* property holds: given a commutative square as shown below with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism d making both triangles commute.

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\
C \arrow[r, tail, "m"'] & D
\end{tikzcd}$$

We will use three standard facts about factorization systems:

(a) Suppose that $\mathcal{M}$ is a class of monomorphisms. If $(p_i\colon A \to A_i)_{i\in I}$ is a limit cone in $\mathscr{A}$, then the factorization $p_i = (A \xrightarrow{e_i} A_i' \xrightarrow{m_i} A_i)$ with $e_i \in \mathcal{E}$ and $m_i \in \mathcal{M}$ yields another limit cone $(e_i\colon A \twoheadrightarrow A_i')_{i\in I}$ over the same scheme.

(b) Suppose that $\mathcal{E}$ is a class of epimorphisms. If $\mathbf{T}$ is a monad on $\mathscr{A}$ that preserves $\mathcal{E}$, i.e. $e \in \mathcal{E}$ implies $Te \in \mathcal{E}$, then $\mathbf{Alg}\,\mathbf{T}$ has the factorization system of $\mathcal{E}$-carried and $\mathcal{M}$-carried $\mathbf{T}$-homomorphisms.

(c) Every locally finitely copresentable category $\mathscr{A}$ has the (epi, strong mono) factorization system. Its arrow category $\mathscr{A}^{\to}$, see A.10, has the factorization system of componentwise epimorphic and strongly monomorphic morphisms.

B Topological toolkit

The following lemmas give important properties of cofiltered limits in the category of compact Hausdorff spaces and continuous maps. The proofs of the first three lemmas can be found in Chapter 1 of [30].

Lemma B.1. *Let $\tau\colon D_1 \to D_2$ be a natural transformation between cofiltered diagrams (over the same scheme) in the category of compact Hausdorff spaces. If each $\tau_i\colon D_1 i \twoheadrightarrow D_2 i$ is surjective, so is the mediating map $\operatorname{Lim}\tau\colon \operatorname{Lim} D_1 \to \operatorname{Lim} D_2$. In particular, if $(\tau_i\colon X \twoheadrightarrow Di)$ is a cone of surjections over a cofiltered diagram D, then the mediating map $X \to \operatorname{Lim} D$ is surjective.*

Lemma B.2. *Let D be a cofiltered diagram in the category of compact Hausdorff spaces. If all connecting maps $D(i \xrightarrow{f} j)$ are surjective, so is each limit projection $\varrho_i\colon \operatorname{Lim} D \to Di$.*

Lemma B.3. *Let D be a cofiltered diagram of non-empty spaces in the category of compact Hausdorff spaces. Then $\operatorname{Lim} D$ is non-empty.*

Lemma B.4. *Let $(p_i : X \to X_i)_{i\in I}$ be a cofiltered limit in the category of compact Hausdorff spaces, where all X_i's are finite (and thus discrete). For each $i \in I$, there is some $j \in I$ and a connecting map $g_{ji} : X_j \to X_i$ with $p_i[X] = g_{ji}[X_j]$.*

Proof. By A.8 we may assume that I is a codirected poset.

(i) For each $x \in X_i \setminus p_i[X]$, there exists some $j \leq i$ such that $x \notin g_{ji}[X_j]$. To see this, suppose the contrary. Then, for each $j \leq i$, the set $X'_j := \{\, y \in X_j : g_{ji}(y) = x \,\}$ is non-empty. Moreover, for $k \leq j \leq i$, the connecting map $g_{kj} : X_k \to X_j$ restricts to X'_k and X'_j. Thus $(X'_j)_{j \leq i}$ forms a subdiagram of $(X_j)_{j \leq i}$, and by Lemma B.3 its limit $(p'_j : X' \to X'_j)_{j \leq i}$ is non-empty. Consider the unique map $g : X' \to X$ with $p_j \cdot g = s_j \cdot p'_j$ for all $j \leq i$, where $s_j : X'_j \rightarrowtail X_j$ is the inclusion. Then, choosing any $x \in X'$, we have $x = p_i(g(x'))$, contradicting the assumption that $x \notin p_i[X]$.

(ii) Since $X_i \setminus p_i[X]$ is a finite, by (i) and codirectedness of I there is some $j \leq i$ such that $g_{ji}[X_j] \subseteq p_i[X]$. Moreover, we have $g_{ji}[X_j] \supseteq p_i[X]$ because (p_i) is a cone. This proves the claim.

□

C Details for Section 2

Proposition C.1. *$\widehat{\mathscr{D}}$ is the pro-completion of $\mathscr{D}_f$.*

A proof is sketched in [18, Remark VI.2.4] for the unordered case, and the argument given there works analogously for the ordered case. For convenience, we present a complete proof for the latter.

Proof. Let $\mathscr{D}$ be a variety of ordered algebras. Clearly $\widehat{\mathscr{D}}$ is complete (with limits formed on the level of **Set**) and, by definition, every object of $\widehat{\mathscr{D}}$ is a cofiltered limit of objects in $\mathscr{D}_f$. Thus, by A.13, it only remains to show that every object $D \in \mathscr{D}_f$ is finitely copresentable in $\widehat{\mathscr{D}}$: given a cofiltered limit cone $(p_i : X \to X_i)_{i \in I}$ in $\widehat{\mathscr{D}}$ and a morphism $f : X \to D$, we need to show that f factors through the cone essentially uniquely. The uniqueness is clear since, forgetting the $\mathscr{D}$-algebraic structure, D is finitely copresentable in **Priest**. Thus we only need to show the existence of a factorization.

(1) Suppose first that all X_i's are finite. Since (p_i) is a cofiltered limit cone in **Priest** and D is a finite poset with discrete topology (and thus finitely copresentable in **Priest**), there exists an $i \in I$ and a monotone map $f' : X_i \to D$ with $f' \cdot p_i = f$. Choose $j \in I$ and a connecting morphism $g : X_j \to X_i$ with $g[X_j] = p_i[X]$, see Lemma B.4. We claim that the composite $h = f' \cdot g$ is a morphism of $\widehat{\mathscr{D}}$, i.e. preserves all $\mathscr{D}$-operations. Indeed, given an n-ary operation symbol σ in the signature of $\mathscr{D}$ and $x_1, \ldots, x_n \in X_j$, choose $x'_k \in X$

with $g(x_k) = p_i(x'_k)$ for $k = 1, \dots, n$. Then

$$\begin{aligned}
h(\sigma(x_1, \dots, x_n)) &= f'(g(\sigma(x_1, \dots, x_n))) && (h = f'g)\\
&= f'(\sigma(g(x_1), \dots, g(x_n))) && (g \text{ morphism of } \mathscr{D})\\
&= f'(\sigma(p_i(x'_1), \dots, p_i(x'_n))) && (g(x'_k) = p_i(x'_k))\\
&= f'(p_i(\sigma(x'_1, \dots, x'_n))) && (p_i \text{ morphism of } \mathscr{D})\\
&= f(\sigma(x'_1, \dots, x'_n)) && (f'p_i = f)\\
&= \sigma(f(x'_1), \dots, f(x'_n)) && (f \text{ morphism of } \widehat{\mathscr{D}})\\
&= \sigma(f' \cdot p_i(x'_1), \dots, f' \cdot p_i(x'_n)) && (f = f'p_i)\\
&= \sigma(f' \cdot g(x_1), \dots, f' \cdot g(x_n)) && (p_i(x'_k) = g(x_k))\\
&= \sigma(h(x_1), \dots, h(x_n)) && (h = f'g)
\end{aligned}$$

Thus h lies in $\widehat{\mathscr{D}}$. Moreover, we have $f = f' \cdot p_i = f' \cdot g \cdot p_j = h \cdot p_j$, i.e. f factors through p_j.

(2) Now let the X_i's be arbitrary. We may assume that I is a codirected poset, see A.8. The connecting morphism for $i \leq j$ is denoted by $g_{ij} : X_i \to X_j$. Form the codirected poset

$$J = \{\, (i, e) : i \in I \text{ and } e : X_i \twoheadrightarrow A_e \text{ is a finite quotient of } X_i \text{ in } \widehat{\mathscr{D}} \,\}$$

ordered by

$$(i, e) \leq (j, q) \quad \text{iff} \quad i \leq j \text{ and } q \cdot g_{ij} = g \cdot e \text{ for some } g : A_e \to A_q.$$

Note that g is necessarily unique. It is easy to verify that the diagram $Q : J \to \widehat{\mathscr{D}}$ given by

$$(i, e) \mapsto A_e \quad \text{and} \quad ((i, e) \leq (j, q)) \mapsto g$$

has the limit cone

$$(e \cdot p_i : X \to A_e)_{(i,e) \in J}.$$

By (1), there exists an $(i, e) \in J$ and a morphism $f' : A_e \to D$ with $f' \cdot e \cdot p_i = f$. Thus f factors through p_i. □

C.1 Details for Remark 2.12.2

The forgetful functor $|-|\colon \widehat{\mathscr{D}} \to \mathbf{Set}$ is representable by $\mathbb{1}$, i.e. it is is naturally isomorphic to $\widehat{\mathscr{D}}(\mathbb{1}, -)$ via the isomorphisms

$$|D| \cong \mathbf{Set}(\{*\}, |D|) \cong \widehat{\mathscr{D}}(\mathbb{1}, D),$$

which are natural in $D \in \widehat{\mathscr{D}}$. Then, the natural isomorphism $|P| \cong \widehat{\mathscr{D}}(-, O_{\mathscr{D}})$ follows from the observation that the diagram below commutes for all $h\colon D' \to D$

in $\widehat{\mathscr{D}}$:

$$\begin{tikzcd}
\mathscr{C}(\mathbb{1}, PD) \arrow[r, "\mathscr{C}(\mathbb{1}, Ph)"] \arrow[d, "\cong"'] & \mathscr{C}(\mathbb{1}, PD') \arrow[d, "\cong"] \\
\widehat{\mathscr{D}}(D, O_{\mathscr{D}}) \arrow[r, "\widehat{\mathscr{D}}(h, O_{\mathscr{D}})"'] & \widehat{\mathscr{D}}(D', O_{\mathscr{D}})
\end{tikzcd}$$

and similarly for $|P^{-1}| \cong \mathscr{C}(-, O_{\mathscr{C}})$. □

C.2 Details for Remark 2.12.3

Notation C.2. Let V denote the two forgetful functors $V : \widehat{\mathscr{D}} \to \mathscr{D}$ and $V : \widehat{\mathscr{D}}^S \to \mathscr{D}^S$.

Remark C.3. 1. By Proposition C.1 and A.13, the category $\widehat{\mathscr{D}}$ is locally finitely copresentable, and its finitely copresentable objects are the objects of $\mathscr{D}_f$. Since the set S of sorts is assumed to be finite, this implies that the product category $\mathscr{D}^S$ is also locally finitely copresentable and its finitely copresentable objects are precisely the objects of $\mathscr{D}_f^S$. Hence, by A.13 again, the category $\widehat{\mathscr{D}}^S$ is the pro-completion of $\mathscr{D}_f^S$.

2. The forgetful functor $V : \widehat{\mathscr{D}} \to \mathscr{D}$ is a right adjoint and thus preserves limits, see [12, Proposition 2.8]. Since limits in $\widehat{\mathscr{D}}^S$ and $\mathscr{D}^S$ are computed sortwise, the same holds for the S-sorted forgetful functor $V : \widehat{\mathscr{D}}^S \to \mathscr{D}^S$.

Proposition C.4. *Epimorphisms in $\widehat{\mathscr{D}}$ are surjective.*

Proof. 1. First, observe that all monomorphisms in $\widehat{\mathscr{D}}$ are injective because the right adjoint $V : \widehat{\mathscr{D}} \to \mathscr{D}$, see Remark C.3.2, preserves monomorphisms.

2. We show that any epimorphism $e\colon A \twoheadrightarrow B$ in $\mathscr{D}_f$ is surjective. Since epimorphisms in $\mathscr{D}$ are assumed to be precisely the surjective morphisms, it suffices to show that e is an epimorphism in $\mathscr{D}$. Suppose that $f, g\colon B \to C$ are morphisms in $\mathscr{D}$ with $f \cdot e = g \cdot e$. Express C as a directed union $(c_i\colon C_i \rightarrowtail C)_{i \in I}$ of finite subobjects, using that $\mathscr{D}$ is a locally finite variety. Since B is finite and the union is directed, the morphisms f and g factor through some c_i, i.e. there exist morphisms f', g' with $f = c_i \cdot f'$ and $g = c_i \cdot g'$. Then

$$c_i \cdot f' \cdot e = f \cdot e = g \cdot e = c_i \cdot g' \cdot e,$$

and since c_i is monic, it follows that $f' \cdot e = g' \cdot e$. Since e is an epimorphism in $\mathscr{D}_f$ and C_i is finite, this implies $f' = g'$ and therefore $f = g$.

3. We prove that every epimorphism $e\colon A \twoheadrightarrow B$ in $\widehat{\mathscr{D}}$ with finite codomain is surjective. To see this, factorize e as $e = m \cdot q$ with q surjective and m injective (resp. order-reflecting). By part 1, the morphism m has finite domain, and moreover m is an epimorphism since e is. Thus, part 2 shows that m is surjective, which implies that e is surjective.

4. Now let $e : A \twoheadrightarrow B$ be an arbitary epimorphism in $\widehat{\mathscr{D}}$. By A.10, one can express e in the locally finitely copresentable category $\widehat{\mathscr{D}}^{\to}$ as a cofiltered limit $((a_i, b_i) : e \to f_i)_{i\in I}$ of morphisms $f_i : A_i \to B_i$ in $\mathscr{D}_f$. Take the (epi, strong mono) factorizations of a_i and b_i, see A.14(c). Diagonal fill-in gives a morphism e_i as in the diagram below:

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, two heads, "a_i'"'] \arrow[dd, bend right=60, "a_i"'] & B \arrow[d, two heads, "b_i'"] \arrow[dd, bend left=60, "b_i"] \\
A_i' \arrow[r, dashed, two heads, "e_i"] \arrow[d, tail, "a_i''"'] & B_i' \arrow[d, tail, "b_i''"] \\
A_i \arrow[r, "f_i"'] & B_i
\end{tikzcd}$$

The objects A_i' and B_i' are finite by part 1 of the proof. Moreover, since e and b_i' are epimorphic, so is e_i, and thus part 2 shows that e_i is surjective. Moreover, by part 3, also a_i' is surjective. Finally, observe that $((a_i', b_i') : e \to e_i)_{i\in I}$ is a cofiltered limit cone in $\widehat{\mathscr{D}}^{\to}$ by A.14(a),(c). Since limits in $\widehat{\mathscr{D}}^{\to}$ are computed componentwise, Lemma B.1 shows that e is surjective.

□

Remark C.5. From the fact that $\widehat{\mathscr{D}}$ is locally finitely copresentable and epimorphisms in $\widehat{\mathscr{D}}$ are precisely the surjective morphisms, it follows that the factorization system (epi, strong mono) of $\widehat{\mathscr{D}}$ coincides with (surjective, injective) if $\mathscr{D}$ is a variety of algebras, and with (surjective, order-reflecting) if $\mathscr{D}$ is a variety of ordered algebras. Thus, dually, the variety $\mathscr{C}$ has the factorization system (strong epi, mono) = (surjective, injective).

Remark C.6. We list some further properties of the profinite monad $\widehat{\mathbf{T}}$. See [12] for proofs.

1. For any $\mathscr{D} \in \mathscr{D}_f^S$, denote by $(\mathbf{T}D \downarrow \mathbf{Alg}_f\,\mathbf{T})$ the comma category of all $\mathbf{T}$-homomorphisms $h\colon \mathbf{T}D \to A$ with finite codomain, see A.4, and by $\mathsf{Quo}_f(\mathbf{T}D)$ its full subcategory on *surjective* homomorphims. The inclusion functor $\mathsf{Quo}_f(\mathbf{T}D) \hookrightarrow (\mathbf{T}D \downarrow \mathbf{Alg}_f\,\mathbf{T})$ is final, cf. A.7. Therefore $\hat{T}D$, see Construction 2.8, is also the cofiltered limit of the larger diagram

$$(\mathbf{T}D \downarrow \mathbf{Alg}_f\,\mathbf{T}) \to \widehat{\mathscr{D}}^S, \quad (\mathbf{T}D \xrightarrow{h} (A, \alpha)) \to A.$$

The limit projections are denoted by $h^+ \colon \hat{T}D \to A$. The following squares commute for all $\mathbf{T}$-homomorphisms $h\colon \mathbf{T}D \to (A, \alpha)$ with $(A, \alpha) \in \mathbf{Alg}_f\,\mathbf{T}$:

$$\begin{tikzcd}
D \arrow[r, "\hat{\eta}_D"] \arrow[dr, "h\eta_D"'] & \hat{T}D \arrow[d, "h^+"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{T}\hat{T}D \arrow[r, "\hat{\mu}_D"] \arrow[d, "\hat{T}h^+"'] & \hat{T}D \arrow[d, "h^+"] \\
\hat{T}A \arrow[r, "\alpha^+"'] & A
\end{tikzcd}
\tag{C.1}$$

2. The profinite monad $\widehat{\mathbf{T}}$ is the codensity monad (see A.6) of the forgetful functor
$$\mathbf{Alg}_f\,\mathbf{T} \to \mathscr{D}_f^S \xrightarrow{\cong} \widehat{\mathscr{D}}_f^S \hookrightarrow \widehat{\mathscr{D}}^S.$$
The limit formula for right Kan extensions (see A.5) yields the construction of $\hat{T}D$ and the commutative diagrams (C.1) in C.6.1.
3. Recall from Remark 2.11.1 the isomorphism $\mathbf{Alg}_f\,\mathbf{T} \xrightarrow{\cong} \mathbf{Alg}_f\,\widehat{\mathbf{T}}$. Its inverse is given by $(B,\beta) \mapsto (B, V\beta \cdot \iota_B)$ and $h \mapsto h$. In the following we will often tacitly identity finite $\mathbf{T}$-algebras with their corresponding finite $\widehat{\mathbf{T}}$-algebras.
4. Every finite $\widehat{\mathbf{T}}$-algebra is finitely copresentable in $\mathbf{Alg}\,\widehat{\mathbf{T}}$, see A.9.
5. Recall from Remark 2.11.2 the natural transformation $\iota\colon TV \to V\hat{T}$. Every $\widehat{\mathbf{T}}$-homomorphism $h\colon \widehat{\mathbf{T}}D \to (A,\alpha^+)$ with $(A,\alpha) \in \mathbf{Alg}_f\,\mathbf{T}$ and $D \in \mathscr{D}_f^S$ restricts to a $\mathbf{T}$-homomorphism $Vh \cdot \iota_D\colon \mathbf{T}D \to (A,\alpha)$.
6. For any $D \in \mathscr{D}_f^S$ the morphism $\iota_D\colon TVD \to V\hat{T}D$ is *dense*, i.e. for each sort s the image of
$$\iota_D\colon (TVD)_s \to (V\hat{T}D)_s = V(\hat{T}D)_s$$
is a dense subset of the profinite $\mathscr{D}$-algebra $(\hat{T}D)_s \in \widehat{\mathscr{D}}$. This implies that for any surjective morphism $e\colon \hat{T}D \twoheadrightarrow A$ in $\widehat{\mathscr{D}}^S$ with $A \in \widehat{\mathscr{D}}_f^S$, the restricted map $Ve \cdot \iota_D : TD \twoheadrightarrow A$ is also surjective, as this map is dense and A is discrete. *We will use this property frequently.*
7. The functor $\hat{T}$ preserves epimorphisms (= sortwise surjective morphisms) of $\widehat{\mathscr{D}}^S$. Thus the factorization system of $\mathscr{D}^S$ lifts to $\mathbf{Alg}\,\widehat{\mathbf{T}}$: every $\widehat{\mathbf{T}}$-homomorphism factorizes into a sortwise surjective homomorphism followed by a sortwise injective one.

Lemma C.7. *Every $\mathbf{T}$-homomorphism $g\colon \mathbf{T}D' \to \mathbf{T}D$ with $D, D' \in \mathscr{D}_f^S$ extends uniquely to a $\widehat{\mathbf{T}}$-homomorphism $\hat{g}\colon \widehat{\mathbf{T}}D' \to \widehat{\mathbf{T}}D$ such that the following diagrams commute for all $\mathbf{T}$-homomorphisms $h\colon \mathbf{T}D \to A$ with $A \in \mathbf{Alg}_f\,\mathbf{T}$:*

$$\begin{tikzcd}
TD' \arrow[r, "g"] \arrow[d, "\iota_{D'}"'] & TD \arrow[d, "\iota_D"] \\
V\hat{T}D' \arrow[r, "V\hat{g}"'] & V\hat{T}D
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{T}D' \arrow[r, "\hat{g}"] \arrow[dr, "(hg)^+"'] & \hat{T}D \arrow[d, "h^+"] \\
& A
\end{tikzcd}
\tag{C.2}$$

Proof. The morphisms $(hg)^+$ form a compatible family over the diagram defining $\hat{T}D$, i.e. for all $\mathbf{T}$-homomorphisms $k\colon A \to A'$ in $\mathbf{Alg}_f\,\mathbf{T}$ we have $(khg)^+ = k \cdot (hg)^+$. Indeed, this holds when precomposed with the dense map $\iota_{D'}$:
$$V(khg)^+ \cdot \iota_{D'} = khg = k \cdot V(hg)^+ \cdot \iota_{D'}.$$
Thus there exists a unique $\hat{g} : \hat{T}D' \to \hat{T}D$ with $(hg)^+ = h^+ \cdot \hat{g}$ for all h, i.e. the right-hand diagram of (C.2) commutes. This also implies that the left-hand diagram commutes. Indeed, it commutes when postcomposed with every

morphism Vh^+:

$$\begin{tikzcd}
TD' \arrow[r, "g"] \arrow[d, "\iota_{D'}"'] \arrow[rrdd, bend right=60, "hg"'] & TD \arrow[d, "\iota_D"'] \arrow[rdd, "h"] & \\
V\hat{T}D' \arrow[r, "V\hat{g}"] \arrow[rrd, "V(hg)^+"'] & V\hat{T}D \arrow[rd, "Vh^+"] & \\
 & & A
\end{tikzcd}$$

By Remark C.3.2, V preserves limits, thus the Vh^+ form a jointly monomorphic family and so we are done. ☐

D Details for Section 3

D.1 Proof of Theorem 3.3

We first show that the language $L := V\hat{L} \cdot \iota_{\Sigma} \colon T\Sigma \to O_{\mathscr{D}}$ is recognizable for any morphism $\hat{L} \colon \hat{T}\Sigma \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$. Since $O_{\mathscr{D}}$ is finitely copresentable in $\widehat{\mathscr{D}}^S$, see Remark C.3, the morphism $\hat{L}$ factors through the cofiltered limit cone defining $\hat{T}\Sigma$, i.e. there exists a $\mathbf{T}$-homomorphism $h \colon \mathbf{T}\Sigma \to A$ with $A \in \mathbf{Alg}_f\,\mathbf{T}$ and a morphism $p \colon A \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$ with $\hat{L} = p \cdot h^+$. It follows that L is recognized by h via p, see the diagram below:

$$\begin{tikzcd}
T\Sigma \arrow[r, "L"] \arrow[d, "\iota_{\Sigma}"'] \arrow[rd, "h"] & O_{\mathscr{D}} \\
V\hat{T}\Sigma \arrow[r, "Vh^+"'] & A \arrow[u, "p=Vp"']
\end{tikzcd} \tag{D.1}$$

Conversely, let $L \colon T\Sigma \to O_{\mathscr{D}}$ be any recognizable language. Choose a $\mathbf{T}$-homomorphism $h \colon \mathbf{T}\Sigma \to A$ with A finite and a morphism $p \colon A \to O_{\mathscr{D}}$ with $L = p \cdot h$. This yields the following morphism in $\widehat{\mathscr{D}}^S$:

$$\hat{L} = (\hat{T}\Sigma \xrightarrow{h^+} A \xrightarrow{p} O_{\mathscr{D}}).$$

Since $L = V\hat{L} \cdot \iota_{\Sigma}$ and ι_{Σ} is dense by Remark C.6.6, the morphism $\hat{L}$ is independent of the choice of h and p. Clearly the maps $\hat{L} \mapsto L$ and $L \mapsto \hat{L}$ are mutually inverse, which proves the claim. ☐

D.2 Details for Remark 3.4

We verify that $\mathsf{Rec}(\Sigma)$, viewed as an object of $\mathscr{C}$ isomorphic to $\prod_s P(\hat{T}\Sigma)_s$, is a subobject of $\prod_s O_{\mathscr{C}}^{|T\Sigma|_s}$ via the map

$$(T\Sigma \xrightarrow{L} O_{\mathscr{D}}) \quad \mapsto \quad (|T\Sigma|_s \xrightarrow{|L|} |O_{\mathscr{D}}| \xrightarrow{\cong} |O_{\mathscr{C}}|)_{s \in S}$$

1. We first show that, for each sort s, the object $P(\hat{T}\mathbb{\Sigma})_s$ forms a subobject of $O_{\mathscr{C}}^{|T\mathbb{\Sigma}|_s}$. For each element $x\colon \mathbb{1} \to (T\mathbb{\Sigma})_s$ of $|T\mathbb{\Sigma}|_s$, the $\mathscr{D}$-morphism $(\mathbb{1} \xrightarrow{x} (T\mathbb{\Sigma})_s \xrightarrow{\iota_{\mathbb{\Sigma}}} V(\hat{T}\mathbb{\Sigma})_s)$ is continuous, because $\mathbb{1}$ is finite and thus discrete. That is, there exists a morphism $\hat{x}\colon \mathbb{1} \to (\hat{T}\mathbb{\Sigma})_s$ in $\widehat{\mathscr{D}}$ with $V\hat{x} = \iota_{\mathbb{\Sigma}} \cdot x$. Since the morphisms x are jointly surjective, and $\iota_{\mathbb{\Sigma}}$ is dense by Remark C.6.6, the family $(\hat{x})_{x\colon \mathbb{1}\to(T\mathbb{\Sigma})_s}$ forms a jointly epimorphic family in $\widehat{\mathscr{D}}$. Thus the dual family $(P\hat{x}\colon P(\hat{T}\mathbb{\Sigma})_s \to O_{\mathscr{C}})$ in $\mathscr{C}$ is jointly monomorphic, which implies that the induced morphism m_s into the product (making the triangle below commute for all x) is monomorphic.

$$\begin{tikzcd}
P(\hat{T}\mathbb{\Sigma})_s \arrow[r, tail, "m_s"] \arrow[dr, "P\hat{x}"'] & O_{\mathscr{C}}^{|T\mathbb{\Sigma}|_s} \arrow[d, "\pi_x"] \\
 & O_{\mathscr{C}}
\end{tikzcd}$$

2. It follows that $\mathsf{Rec}(\Sigma)$ is a subobject of $\prod_s O_{\mathscr{C}}^{|T\mathbb{\Sigma}|_s}$ via the embedding

$$\mathsf{Rec}(\Sigma) \cong \widehat{\mathscr{D}}^S(\hat{T}\mathbb{\Sigma}, O_{\mathscr{D}}) \cong \prod_s \widehat{\mathscr{D}}((\hat{T}\mathbb{\Sigma})_s, O_{\mathscr{D}}) \cong \prod_s |P(\hat{T}\mathbb{\Sigma})_s| \xrightarrow{\prod_s m_s} \prod_s O_{\mathscr{C}}^{|\hat{T}\mathbb{\Sigma}|_s}.$$

By applying the definitions of the above three bijections and of the morphisms m_s, one easily verifies that this embedding maps a recognizable language $L\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ to the element $(|T\mathbb{\Sigma}|_s \xrightarrow{|L|} |O_{\mathscr{D}}| \xrightarrow{\cong} |O_{\mathscr{C}}|)_{s\in S}$ of $\prod_s O_{\mathscr{C}}^{|T\mathbb{\Sigma}|_s}$, as claimed. □

D.3 Details for Example 3.6.3

Every polynomial $p : 1_{s'} \to T(\Sigma + 1_s)$ induces an evaluation map $[p] : (T\Sigma)_s \to (T\Sigma)_{s'}$ that sends an element $x\colon 1_s \to T\Sigma$ of $(T\Sigma)_s$ to the following element of $(T\Sigma)_{s'}$:

$$1_{s'} \xrightarrow{p} T(\Sigma{+}1_s) \xrightarrow{T(\Sigma+x)} T(\Sigma{+}T\Sigma) \xrightarrow{T(\eta+T\Sigma)} T(T\Sigma{+}T\Sigma) \xrightarrow{T[id,id]} TT\Sigma \xrightarrow{\mu_\Sigma} T\Sigma.$$

D.4 Proof of Theorem 3.10

Lemma D.1. *1. If $\mathscr{D}$ is a variety of algebras, then for any object $D \in \mathscr{D}_f^S$ and any two elements $x, y \in |D|_s$ with $s \in S$ we have*

$$x = y \quad \textit{iff} \quad \forall(D \xrightarrow{k} O_{\mathscr{D}}) : k(x) = k(y).$$

2. If $\mathscr{D}$ be a variety of ordered algebras, then for any object $D \in \mathscr{D}_f^S$ and any two elements $x, y \in |D|_s$ with $s \in S$ we have

$$x \leq y \quad \textit{iff} \quad \forall(D \xrightarrow{k} O_{\mathscr{D}}) : k(x) \leq k(y).$$

Proof. To prove (b), suppose first that $S = 1$. Given $D \in \mathscr{D}_f$, its dual object $PD \in \mathscr{C}_f$ is finite und thus finitely generated, so there exists a surjective morphism (i.e. a strong epimorphism) $\coprod_{i\in I} \mathbb{1} \twoheadrightarrow PD$ in $\mathscr{C}_f$, where $I = |PD|$ is finite. Its dual morphism $m : D \rightarrowtail \prod_{i\in I} O_{\mathscr{D}}$ in $\mathscr{D}_f$ is a strong monomorphism and thus order-reflecting. Then $x \not\leq y$ in $|D|$ implies $m(x) \not\leq m(y)$, and thus $\pi_i m(x) \not\leq \pi_i m(y)$ for some $i \in I$ since the product projections $\pi_i \colon \prod_i O_{\mathscr{D}} \to O_{\mathscr{D}}$ are jointly order-reflecting. This shows that the morphism $k := \pi_i \cdot m \colon D \to O_{\mathscr{D}}$ separates x and y, as desired.

Now let S be arbitrary and $x \not\leq y \in |D|_s$. By the above argument there exists a morphism $k_s \colon D_s \to O_{\mathscr{D}}$ in $\mathscr{D}$ with $k_s(x) \not\leq k_s(y)$. For any sort $t \neq s$, pick an arbitrary morphism $k_t \colon D_t \to O_{\mathscr{D}}$. Such a morphism exists because, by our Assumption 2.1 that the signature of $\mathscr{C}$ has a constant, we dually have a morphism $\mathbb{1} \to PD_t$ in $\mathscr{C}_f$. Thus $k \colon D \to O_{\mathscr{D}}$ is a morphism in $\mathscr{D}^S$ with $k(x) \not\leq k(y)$.

The proof of (a) is analogous, using equations in lieu of inequations. □

Proof (Theorem 3.10). We only treat the case where $\mathscr{D}$ is a variety of ordered algebras; for the unordered case, just replace inequations by equations throughout the proof. In our proof we will repeatedly use the *homomorphism theorem*: given $e \colon A \twoheadrightarrow B$ and $f \colon A \to C$ in $\mathscr{D}^S$ with e sortwise surjective, there exists a morphism $g \colon B \to C$ with $g \cdot e = f$ iff, for all sorts s and $a, a' \in |A|_s$, $e(a) \leq e(a')$ implies $f(a) \leq f(a')$. Put $A_L := T\mathbb{\Sigma}/\leq_L$.

(i)⇒(ii) Suppose that $\mathbb{U}_\Sigma$ is a unary presentation of $\mathbf{T}$ over Σ, and let $L \colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ be a recognizable language.

(a) We show that there exists a morphism $p_L : A_L \to O_{\mathscr{D}}$ in $\mathscr{D}^S$ with $L = p_L \cdot e_L$, using the homomorphism theorem. Let $x, y \in |T\mathbb{\Sigma}|_s$ with $e_L(x) \leq e_L(y)$, i.e. $x \leq_L y$. Since $\overline{\mathbb{U}}_\Sigma$ contains all identities, putting $u := id_{(T\mathbb{\Sigma})_s}$ in the definition of $\leq_L$ (see Definition 3.9) yields $L(x) \leq L(y)$. The homomorphism theorem gives the desired p_L.

(b) Since L is recognizable, there is a surjective $\mathbf{T}$-homomorphism $e : T\mathbb{\Sigma} \twoheadrightarrow A$ into a finite $\mathbf{T}$-algebra A and a morphism $p : A \to O_{\mathscr{D}}$ in $\mathscr{D}^S$ with $L = p \cdot e$. Furthermore, since $\mathbb{U}_\Sigma$ forms a unary presentation, we can choose for each $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ a lifting $u_A : A_s \to A_{s'}$ along e, that is, $e \cdot u = u_A \cdot e$. We claim that there exists a morphism $h : A \to A_L$ in $\mathscr{D}^S$ with $e_L = h \cdot e$. This follows from the homomorphism theorem: let $x, y \in |T\mathbb{\Sigma}|_s$ with $e(x) \leq e(y)$. Then, for all sorts s' and $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$,

$$L \cdot u(x) = p \cdot e \cdot u(x) = p \cdot u_A \cdot e(x) \leq p \cdot u_A \cdot e(y) = p \cdot e \cdot u(y) = L \cdot u(y).$$

Thus $x \leq_L y$, or equivalently $e_L(x) \leq e_L(y)$, and the homomorphism theorem gives the desired h.

(c) We show that (I) e_L is extensible, and (II) every morphism $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ has a lifting along e_L. This implies the claim: since $\mathbb{U}_\Sigma$ is a unary presentation, A_L then carries a $\mathbf{T}$-algebra structure making e_L a $\mathbf{T}$-homomorphism. And part (a) and (b) show that e_L recognizes L and has the universal property of a syntactic morphism.

For (I), the following commutative diagram shows that e_L is extensible:

$$\begin{tikzcd}
& T\Sigma \arrow[dl, "\iota_{\Sigma}"'] \arrow[d, two heads, "e"] \arrow[dr, two heads, "e_L"] & \\
V\hat{T}\Sigma \arrow[r, two heads, "Ve^{+}"'] & A \arrow[r, two heads, "h"'] & A_L
\end{tikzcd}$$

Indeed, the left-hand triangle commutes by Remark 2.11.2, and the right-hand one by (b). Thus e_L has the continuous extension $h \cdot e^{+}$.

For (II), by the homomorphism theorem we need to show that for all $x, y \in |T\Sigma|_s$ with $x \leq_L y$ we have $u(x) \leq_L u(y)$. Note that for all sorts s'' and all $u' : (T\Sigma)_{s'} \to (T\Sigma)_{s''}$ in $\mathbb{U}_{\Sigma}$ we have $u' \cdot u \in \overline{\mathbb{U}}_{\Sigma}$ because $\overline{\mathbb{U}}_{\Sigma}$ is closed under composition. Thus $x \leq_L y$ implies

$$L\cdot(u'\cdot u)(x) \leq L\cdot(u'\cdot u)(x) \quad \text{for all sorts } s'' \text{ and } u' : (T\Sigma)_{s'} \to (T\Sigma)_{s''} \text{ in } \mathbb{U}_{\Sigma},$$

which means precisely that $u(x) \leq_L u(y)$.

(ii)⇒(i) Suppose that, for any recognizable language L over Σ, the morphism e_L is a **T**-algebra congruence, and moreover $e_L : \mathbf{T}\Sigma \twoheadrightarrow A_L$ is a syntactic morphism of L. We verify that $\mathbb{U}_{\Sigma}$ is a unary presentation, i.e. the equivalence of (i) and (ii) in Definition 3.7 for any extensible finite quotient $e : T\Sigma \twoheadrightarrow A$ in $\mathscr{D}^S$.

3.7.(i)⇒3.7.(ii) Suppose that A carries a **T**-algebra structure making $e : \mathbf{T}\Sigma \twoheadrightarrow A$ a **T**-homomorphism. We need to show that every morphism $u : (T\Sigma)_s \to (T\Sigma)_{s'}$ in $\mathbb{U}_{\Sigma}$ has a lifting along e, i.e. there exists a morphism $u_A : A_s \to A_{s'}$ with $e\cdot u = u_A\cdot e$. This requires another use of the homomorphism theorem. For any morphism $k : A \to O_{\mathscr{D}}$ in $\mathscr{D}^S$ we have the recognizable language $L_k := k \cdot e : T\Sigma \to O_{\mathscr{D}}$, which by hypothesis has the syntactic morphism $e_{L_k} : \mathbf{T}\Sigma \twoheadrightarrow A_{L_k}$. Since e_{L_k} recognizes L_k, there exists a morphism $p_{L_k} : A_{L_k} \to O_{\mathscr{D}}$ with $L_k = p_{L_k} \cdot e_{L_k}$. Furthermore, the universal property of the syntactic morphism e_{L_k} gives a unique **T**-homomorphism $h_k : A \twoheadrightarrow A_{L_k}$ with $e_{L_k} = h_k \cdot e$. Then for all $x, y \in |T\Sigma|_s$ we have the following implications:

$$\begin{aligned}
e(x) \leq e(y) \;&\Rightarrow\; \forall (k : A \to O_{\mathscr{D}}) : e_{L_k}(x) \leq e_{L_k}(y) && (e_{L_k} = h_k \cdot e)\\
&\Leftrightarrow\; \forall k : x \leq_{L_k} y && (\text{def. } e_{L_k})\\
&\Rightarrow\; \forall k : L_k \cdot u(x) \leq L_k \cdot u(y) && (\text{def. } \leq_{L_k})\\
&\Leftrightarrow\; \forall k : k \cdot e \cdot u(x) \leq k \cdot e \cdot u(y) && (\text{def. } L_k)\\
&\Leftrightarrow\; e \cdot u(x) \leq e \cdot u(y) && (\text{Lemma D.1}).
\end{aligned}$$

Thus the homomorphism theorem gives the desired lifting u_A.

3.7.(ii)⇒3.7.(i) Let $e = V\hat{e} \cdot \iota_{\Sigma} : T\Sigma \twoheadrightarrow A$ be an extensible finite quotient in $\mathscr{D}^S$, and suppose that every $u : (T\Sigma)_s \to (T\Sigma)_{s'}$ in $\mathbb{U}_{\Sigma}$ has a lifting $u_A : A_s \to A_{s'}$ along e. We need to show that A carries a **T**-algebra structure making e a **T**-homomorphism. For each $k : A \to O_{\mathscr{D}}$ in $\mathscr{D}^S$ the language $L_k := k \cdot e : T\Sigma \to O_{\mathscr{D}}$ is recognizable by Theorem 3.3, since $L_k = V(k \cdot \hat{e}) \cdot \iota_{\Sigma}$. Thus by hypothesis we have the syntactic morphism $e_{L_k} : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A_{L_k}$. Since e_{L_k} recognizes L_k, there exists a morphism $p_{L_k} : A_{L_k} \to O_{\mathscr{D}}$ with $L_k = p_{L_k} \cdot e_{L_k}$.

We claim that e_{L_k} factors through e. To see this, we use the homomorphism theorem. Given $x, y \in |T\Sigma|_s$ with $e(x) \leq e(y)$, we have for all $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_{\Sigma}$:

$$
\begin{aligned}
L_k \cdot u(x) &= k \cdot e \cdot u(x) && \text{(def. } L_k\text{)}\\
&= k \cdot u_A \cdot e(x) && \text{(def. } u_A\text{)}\\
&\leq k \cdot u_A \cdot e(y) && (e(x) \leq e(y))\\
&= k \cdot e \cdot u(y) && \text{(def. } u_A\text{)}\\
&= L_k \cdot u(y) && \text{(def. } L_k\text{).}
\end{aligned}
$$

Thus $x \leq_{L_k} y$, or equivalently $e_{L_k}(x) \leq e_{L_k}(y)$. The homomorphism theorem yields a morphism h_k with $e_{L_k} = h_k \cdot e$.

We are ready to define the desired $\mathbf{T}$-algebra structure (A, α) on A for which e is a $\mathbf{T}$-homomorphism. Since T preserves epimorphisms, it suffices to find a morphism $\alpha : TA \to A$ in $\mathscr{D}^S$ making the following square commute:

$$\begin{tikzcd}
TT\mathbb{\Sigma} \arrow[r, "\mu_{\mathbb{\Sigma}}"] \arrow[d, two heads, "Te"'] & T\mathbb{\Sigma} \arrow[d, two heads, "e"] \\
TA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

The construction of α once again rests on the homomorphism theorem. The proof is illustrated by the diagram below, where α_{L_k} is the $\mathbf{T}$-algebra structure of A_{L_k}.

$$\begin{tikzcd}
TT\mathbb{\Sigma} \arrow[r, "\mu_{\mathbb{\Sigma}}"] \arrow[d, two heads, "Te"'] \arrow[dd, bend right=60, "Te_{L_k}"'] & T\mathbb{\Sigma} \arrow[d, two heads, "e"] \arrow[dr, "L_k"] & \\
TA \arrow[r, dashed, "\alpha"'] \arrow[d, two heads, "Th_k"'] & A \arrow[r, "k"] \arrow[d, two heads, "h_k"] & O_{\mathscr{D}} \\
TA_{L_k} \arrow[r, "\alpha_{L_k}"'] & A_{L_k} \arrow[ur, "p_{L_k}"'] &
\end{tikzcd}$$

For all $x, y \in |TT\mathbb{\Sigma}|_s$ with $Te(x) \leq Te(y)$ we have

$$
\begin{aligned}
k \cdot e \cdot \mu_{\mathbb{\Sigma}}(x) &= L_k \cdot \mu_{\mathbb{\Sigma}}(x) && \text{(def. } L_k\text{)}\\
&= p_{L_k} \cdot e_{L_k} \cdot \mu_{\mathbb{\Sigma}}(x) && \text{(def. } p_{L_k}\text{, } e_{L_k}\text{)}\\
&= p_{L_k} \cdot \alpha_{L_k} \cdot Te_{L_k}(x) && (e_{L_k} \text{ is } \mathbf{T}\text{-hom.})\\
&= p_{L_k} \cdot \alpha_{L_k} \cdot Th_k \cdot Te(x) && \text{(def. } h_k\text{)}\\
&\leq p_{L_k} \cdot \alpha_{L_k} \cdot Th_k \cdot Te(y) && (Te(x) \leq Te(y))\\
&= \cdots && \text{(compute backwards)}\\
&= k \cdot e \cdot \mu_{\mathbb{\Sigma}}(y).
\end{aligned}
$$

Since this holds for all $k : A \to O_{\mathscr{D}}$, Lemma D.1 implies that $e \cdot \mu_{\mathbb{\Sigma}}(x) \leq e \cdot \mu_{\mathbb{\Sigma}}(y)$. Thus the homomorphism theorem yields the desired $\mathbf{T}$-algebra structure α. $\square$

E Details for Section 4

E.1 Profinite $\widehat{\mathbf{T}}$-algebras

In this subsection we develop a few technical results on profinite $\widehat{\mathbf{T}}$-algebras that are subsequently used.

Remark E.1. Every free $\widehat{\mathbf{T}}$-algebra $\widehat{\mathbf{T}}D$ with $D \in \mathscr{D}_f^S$ is profinite. Indeed, since the forgetful functor from $\mathbf{Alg}\,\widehat{\mathbf{T}}$ to $\widehat{\mathscr{D}}^S$ reflects limits, see A.3, the right-hand square of (C.1) shows that the $\widehat{\mathbf{T}}$-algebra $\widehat{\mathbf{T}}D$ is the cofiltered limit of the diagram

$$(\mathbf{T}D \downarrow \mathbf{Alg}_f\, \mathbf{T}) \to \mathbf{Alg}\,\widehat{\mathbf{T}}, \quad (h : \mathbf{T}D \to (A, \alpha)) \mapsto (A, \alpha^+),$$

with limit projections h^+.

Lemma E.2. *A $\widehat{\mathbf{T}}$-algebra A is profinite iff A is the limit of the cofiltered diagram*

$$(A \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}}) \to \mathbf{Alg}\,\widehat{\mathbf{T}}, \quad (h : A \to A') \mapsto A',$$

with limit projections $h : A \to A'$.

Proof. The "if" direction is trivial. For the "only if" direction, suppose that A is profinite, i.e. there exists a cofiltered limit cone $(p_i : A \to A_i)$ in $\mathbf{Alg}\,\widehat{\mathbf{T}}$ with $A_i \in \mathbf{Alg}_f\, \widehat{\mathbf{T}}$. Since the forgetful functor $\hat{U} : \mathbf{Alg}\,\widehat{\mathbf{T}} \to \widehat{\mathscr{D}}^S$ reflects limits, it suffices to show that the cofiltered cone $(h : \hat{U}A \to \hat{U}A')$ in $\widehat{\mathscr{D}}^S$ is a limit cone. To this end we verify the criterion of A.11.

For (i), let $f : \hat{U}A \to B$ be a morphism in $\widehat{\mathscr{D}}^S$ with $B \in \widehat{\mathscr{D}}_f^S$. Since $\hat{U}$ preserves limits, we have the limit cone $(p_i : \hat{U}A \to \hat{U}A_i)$ in $\widehat{\mathscr{D}}^S$. Moreover, since B is finitely copresentable in $\widehat{\mathscr{D}}^S$, see Remark C.3, there exists an i and morphism $f' : \hat{U}A_i \to B$ with $f = f' \cdot p_i$. This proves that f factors through the cone $(h : \hat{U}A \to \hat{U}A')$ via $h = p_i$ and f', as desired.

For (ii), suppose that $h : A \to A'$ in $(A \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ and $f', f'' : \hat{U}A' \to B$ are given with $f' \cdot h = f'' \cdot h$. Since A' is finitely copresentable in $\mathbf{Alg}\,\widehat{\mathbf{T}}$ by Remark 2.11.4, there exists an i and a $\widehat{\mathbf{T}}$-homomorphism $h' : A_i \to A'$ with $h = h' \cdot p_i$. Then $(f' \cdot h') \cdot p_i = (f'' \cdot h') \cdot p_i$. Thus, by (ii) applied to the cofiltered limit cone $(p_i : \hat{U}A \to \hat{U}A_i)$, we have a connecting morphism $a_{ji} : A_j \to A_i$ in the diagram with $f' \cdot h' \cdot a_{ji} = f'' \cdot h' \cdot a_{ji}$. Thus $h' \cdot a_{ji} : p_j \to h$ is a morphism in $(A \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ that merges f' and f'', as desired.

$$\begin{tikzcd}
 & & \hat{U}A \arrow[dll, "p_j"'] \arrow[dl, "p_i"] \arrow[d, "h"] \arrow[dr, "f"] & \\
\hat{U}A_j \arrow[r, "a_{ji}"'] & \hat{U}A_i \arrow[r, "h'"'] & \hat{U}A' \arrow[r, shift left, "f'"] \arrow[r, shift right, "f''"'] & B
\end{tikzcd}$$

□

Notation E.3. Let $(A \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ be the full subcategory of $(A \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ on all surjective $\widehat{\mathbf{T}}$-homomorphisms $e : A \twoheadrightarrow A'$ with finite codomain.

Corollary E.4. *A $\widehat{\mathbf{T}}$-algebra A is profinite iff A is the limit of the cofiltered diagram*

$$(A \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}}) \to \mathbf{Alg}\,\widehat{\mathbf{T}}, \quad (e\colon A \twoheadrightarrow A') \mapsto A',$$

with limit projections $e\colon A \twoheadrightarrow A'$.

Proof. Using Remark C.6.7, one shows that $(A \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}})$ is a final cofiltered subcategory of $(A \downarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}})$. □

Lemma E.5. *Let $\mathbb{U}_\Sigma$ be a unary presentation of $\mathbf{T}$ over Σ, and let $e\colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ and $k\colon A \twoheadrightarrow B$ be surjective $\mathbf{T}$-homomorphisms with $A, B \in \mathbf{Alg}_f\,\mathbf{T}$. Then the following diagram commutes for all $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$, where u_A and u_B are the liftings of u along e and $k \cdot e$, respectively.*

$$\begin{tikzcd}
(T\mathbb{\Sigma})_s \arrow[r, two heads, "e"] \arrow[d, "u"'] & A_s \arrow[r, two heads, "k"] \arrow[d, "u_A"] & B_s \arrow[d, "u_B"] \\
(T\mathbb{\Sigma})_{s'} \arrow[r, two heads, "e"'] & A_{s'} \arrow[r, two heads, "k"'] & B_{s'}
\end{tikzcd}$$

Proof. Clear since e is an epimorphism. □

Lemma E.6. *Let $\mathbb{U}_\Sigma$ be a unary presentation of $\mathbf{T}$ over Σ. Then every $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ has a unique extension to a morphism $\hat{u}\colon (\hat{T}\mathbb{\Sigma})_s \to (\hat{T}\mathbb{\Sigma})_{s'}$ in $\widehat{\mathscr{D}}$ making the following square commute.*

$$\begin{tikzcd}
(T\mathbb{\Sigma})_s \arrow[r, "u"] \arrow[d, "\iota_{\mathbb{\Sigma}}"'] & (T\mathbb{\Sigma})_{s'} \arrow[d, "\iota_{\mathbb{\Sigma}}"] \\
V(\hat{T}\mathbb{\Sigma})_s \arrow[r, "V\hat{u}"'] & V(\hat{T}\mathbb{\Sigma})_{s'}
\end{tikzcd}$$

Proof. For each $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$, the morphisms $u_A \cdot e^+\colon (\hat{T}\mathbb{\Sigma})_s \to A_{s'}$ (where e ranges over surjective $\mathbf{T}$-homomorphisms $e\colon \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ with $A \in \mathbf{Alg}_f\,\mathbf{T}$ and u_A is the lifting of u along e) form a compatible family over the diagram defining $(\hat{T}\mathbb{\Sigma})_{s'}$ by Lemma E.5. Hence there exists a unique morphism $\hat{u}\colon (\hat{T}\mathbb{\Sigma})_s \to (\hat{T}\mathbb{\Sigma})_{s'}$ in $\widehat{\mathscr{D}}$ with $e^+ \cdot \hat{u} = u_A \cdot e^+$ for all e. Therefore in the diagram below the outside and all parts except, perhaps, for the upper square commute:

$$\begin{tikzcd}
(T\mathbb{\Sigma})_s \arrow[r, "u"] \arrow[d, "\iota_{\mathbb{\Sigma}}"'] \arrow[dd, two heads, bend right=60, "e"'] & (T\mathbb{\Sigma})_{s'} \arrow[d, "\iota_{\mathbb{\Sigma}}"] \arrow[dd, two heads, bend left=60, "e"] \\
V(\hat{T}\mathbb{\Sigma})_s \arrow[r, "V\hat{u}"] \arrow[d, two heads, "Ve^+"'] & V(\hat{T}\mathbb{\Sigma})_{s'} \arrow[d, two heads, "Ve^+"] \\
A_s \arrow[r, "u_A"'] & A_{s'}
\end{tikzcd}$$

It follows that the upper square commutes when postcomposed with the morphisms Ve^+. Since by Remark C.3 the functor V preserves limits (and thus the morphisms Ve^+ are jointly monomorphic), the upper square commutes. Moreover, $\hat{u}$ is unique with this property because $\iota_{\mathbb{\Sigma}}$ is dense (see Remark 2.11.2) and $A_{s'}$ is a Hausdorff space. ☐

Remark E.7. It follows that, for any extensible finite quotient $e = V\hat{e} \cdot \iota_{\mathbb{\Sigma}} : T\mathbb{\Sigma} \twoheadrightarrow A$, a morphism $u_A : A_s \to A_{s'}$ is a lifting of $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ along e iff it is a lifting of $\hat{u}$ along $\hat{e}$, i.e. the following square commutes:

$$\begin{tikzcd}
(\hat{T}\mathbb{\Sigma})_s \arrow[r, "\hat{u}"] \arrow[d, two heads, "\hat{e}"'] & (\hat{T}\mathbb{\Sigma})_{s'} \arrow[d, two heads, "\hat{e}"] \\
A_s \arrow[r, "u_A"'] & A_{s'}
\end{tikzcd}$$

The following lemma shows that the lifting property of a unary presentation extends from finite to profinite algebras:

Lemma E.8. *Let $\mathbb{U}_\Sigma$ be a unary presentation of $\mathbf{T}$ over Σ. Then for any surjective morphism $\hat{e} : \hat{T}\mathbb{\Sigma} \twoheadrightarrow A$ in $\widehat{\mathscr{D}}^S$ the following statements are equivalent:*

(i) There exists a $\widehat{\mathbf{T}}$-algebra structure on A making $\hat{e} : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$ a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra.

(ii) Each $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ has a lifting along $\hat{e}$, i.e. there exists a morphism $u_A : A_s \to A_{s'}$ in $\widehat{\mathscr{D}}$ for which the following square commutes:

$$\begin{tikzcd}
(\hat{T}\mathbb{\Sigma})_s \arrow[r, "\hat{u}"] \arrow[d, two heads, "\hat{e}"'] & (\hat{T}\mathbb{\Sigma})_{s'} \arrow[d, two heads, "\hat{e}"] \\
A_s \arrow[r, "u_A"'] & A_{s'}
\end{tikzcd} \tag{E.1}$$

Proof. (i)⇒(ii) Let $\hat{e} : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$ be a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra. For any finite quotient algebra $h : A \twoheadrightarrow A'$ in $\mathbf{Alg}_f\,\widehat{\mathbf{T}}$ we have the surjective $\mathbf{T}$-homomorphism $e := V(h \cdot \hat{e}) \cdot \iota_{\mathbb{\Sigma}} : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A'$, see Remark C.6.5. Since $\mathbb{U}_\Sigma$ is a unary presentation, each $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ has a lifting $u_{A'} : A'_s \to A'_{s'}$ along e.

Since A is profinite, A is the cofiltered limit of the diagram of all finite quotients $h : A \twoheadrightarrow A'$, see Lemma E.2. The morphisms $u_{A'} \cdot h : A_s \to A'_{s'}$ form a compatible family over this diagram by Lemma E.5. Therefore there exists a morphism $u_A : A_s \to A_{s'}$ in $\widehat{\mathscr{D}}$ with $h \cdot u_A = u_{A'} \cdot h$ for all h. It follows that the square (E.1) commutes, as it commutes by Remark E.7 when postcomposed with the limit projections h.

$$\begin{tikzcd}
(\hat{T}\mathbb{\Sigma})_s \arrow[r, "\hat{u}"] \arrow[d, two heads, "\hat{e}"'] & (\hat{T}\mathbb{\Sigma})_{s'} \arrow[d, two heads, "\hat{e}"] \\
A_s \arrow[r, dashed, "u_A"] \arrow[d, two heads, "h"'] & A_{s'} \arrow[d, two heads, "h"] \\
A'_s \arrow[r, "u_{A'}"'] & A'_{s'}
\end{tikzcd}$$

(ii)⇒(i) Let $\hat{e} : \hat{T}\mathbb{\Sigma} \twoheadrightarrow A$ be an epimorphism in $\widehat{\mathscr{D}}^S$ with the lifting property (E.1). We need to show that A has a $\widehat{\mathbf{T}}$-algebra structure such that A is profinite and $\hat{e}$ is a $\widehat{\mathbf{T}}$-homomorphism.

(a) We first prove an auxiliary result. Let $g : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow B$ be a surjective $\widehat{\mathbf{T}}$-homomorphism with $B \in \mathbf{Alg}_f\,\widehat{\mathbf{T}}$, and form the pushout $p = g' \cdot \hat{e} = e' \cdot g : \hat{T}\mathbb{\Sigma} \twoheadrightarrow P$ of $\hat{e}$ and g in $\widehat{\mathscr{D}}^S$. We claim that P carries a $\widehat{\mathbf{T}}$-algebra structure making p a $\widehat{\mathbf{T}}$-homomorphism. To see this consider the diagram below:

$$\begin{tikzcd}
 & & (\hat{T}\mathbb{\Sigma})_{s'} \arrow[dll, two heads, "\hat{e}"'] \arrow[drr, two heads, "g"] & & \\
 & & (\hat{T}\mathbb{\Sigma})_s \arrow[u, "\hat{u}"'] \arrow[dl, two heads, "\hat{e}"'] \arrow[dr, two heads, "g"] \arrow[dd, two heads, "p"] & & \\
A_{s'} \arrow[ddrr, two heads, "g'"'] & A_s \arrow[l, "u_A"'] \arrow[dr, two heads, "g'"'] & & B_s \arrow[r, "u_B"] \arrow[dl, two heads, "e'"] & B_{s'} \arrow[ddll, two heads, "e'"] \\
 & & P_s \arrow[d, dashed, "u_P"] & & \\
 & & P_{s'} & &
\end{tikzcd}$$

By hypothesis there exists for each $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ a morphism u_A making the upper left-hand square commute. Likewise, since $\mathbb{U}_\Sigma$ is a unary presentation, there exists by Remark E.7 a morphism u_B making the upper right-hand square commute. Then the morphisms $g' \cdot u_A$ and $e' \cdot u_B$ form a compatible family, so by the universal property of the pushout there exists a unique morphism $u_P : P_s \to P_{s'}$ in $\widehat{\mathscr{D}}$ making the two lower squares commute. Thus the whole diagram above commutes, which shows that $u_P \cdot p = p \cdot \hat{u}$ for all $u : (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$. Since $\mathbb{U}_\Sigma$ forms a unary presentation and P is finite, it follows from Remark E.7 that P carries a $\widehat{\mathbf{T}}$-algebra structure making p a $\widehat{\mathbf{T}}$-homomorphism, as desired.

(b) Let $\hat{U} : \mathbf{Alg}_f\,\widehat{\mathbf{T}} \to \widehat{\mathscr{D}}^S$ denote the forgetful functor, and let $\mathscr{S}$ be the full subcategory of $(A \downarrow \hat{U})$ on all surjective morphisms $h : A \twoheadrightarrow \hat{U}(A', \alpha')$ for which $h \cdot \hat{e} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow (A', \alpha')$ is a $\widehat{\mathbf{T}}$-homomorphism. Let us first verify that the category $\mathscr{S}$ is cofiltered by establishing the three conditions in A.8. First, $\mathscr{S}$ is nonempty because the image of the unique morphism $f : A \to 1$ into the terminal $\widehat{\mathbf{T}}$-algebra lies in $\mathscr{S}$. Second, any two $h_i : A \twoheadrightarrow \hat{U}(A'_i, \alpha'_i)$ $(i = 0, 1)$ in $\mathscr{S}$ have a common predecessor. To see this, form the product $\pi_i : A'_0 \times A'_1 \to A'_i$ in $\mathbf{Alg}\,\widehat{\mathbf{T}}$ and factorize the morphism $\langle h_0, h_1 \rangle : A \to A'_0 \times A'_1$ in $\widehat{\mathscr{D}}^S$ as $\langle h_0, h_1 \rangle = m \cdot h$ with h surjective and m injective.

$$\begin{tikzcd}
& \hat{T}\Sigma \arrow[d, two heads, "\hat{e}"] & \\
& A \arrow[d, two heads, "h"] \arrow[ddl, two heads, "h_0"'] \arrow[ddr, two heads, "h_1"] & \\
& A' \arrow[d, tail, "m"] & \\
A'_0 & A'_0 \times A'_1 \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & A'_1
\end{tikzcd}$$

Since the projections π_i are jointly monomorphic and $\pi_i \cdot m \cdot h \cdot \hat{e} = h_i \cdot \hat{e}$ is a $\widehat{\mathbf{T}}$-homomorphism, so is $m \cdot h \cdot \hat{e}$. Furthermore, since the factorization system of $\widehat{\mathscr{D}}^S$ lifts to $\mathbf{Alg}\,\widehat{\mathbf{T}}$, see Remark C.6.7, there exists a $\widehat{\mathbf{T}}$-algebra structure (A', α') on A' such that $h \cdot \hat{e}$ and m are $\widehat{\mathbf{T}}$-homomorphisms. Thus $h : A \twoheadrightarrow \hat{U}(A', \alpha')$ lies in $\mathscr{S}$ and is the desired predecessor of h_0 and h_1. The third condition in A.8 is trivially satisfied because $\mathscr{S}$ is a poset.
We claim that A is the cofiltered limit of the diagram

$$\mathscr{S} \xrightarrow{\pi} \widehat{\mathscr{D}}^S, \quad (h : A \twoheadrightarrow U(A', \alpha')) \to A' \tag{E.2}$$

with limit projections $h : A \twoheadrightarrow A'$. To this end we verify the criterion of A.11, i.e. we show that any morphism $f : A \to X$ with $X \in \mathscr{D}_f^S$ factors through some h. The proof is illustrated by the diagram below:

$$\begin{tikzcd}
\hat{T}\Sigma \arrow[rr, two heads, "g^+"] \arrow[dr, two heads, "p"] \arrow[dd, two heads, "\hat{e}"'] & & B \arrow[dl, two heads, "e'"'] \arrow[dd, "s"] \\
& P \arrow[dr, dashed, "s'"] & \\
A \arrow[ur, two heads, "h"] \arrow[rr, "f"'] & & X
\end{tikzcd}$$

The morphism $f \cdot \hat{e}$ factors through the cofiltered limit cone defining $\hat{T}\Sigma$, because X is finitely copresentable in $\widehat{\mathscr{D}}^S$ (see Remark C.3). That is, there exists a surjective $\mathbf{T}$-homomorphism $g : \mathbf{T}\Sigma \twoheadrightarrow B$ with $B \in \mathbf{Alg}_f\,\mathbf{T}$ and a morphism $s : B \to X$ in $\widehat{\mathscr{D}}^S$ with $s \cdot g^+ = f \cdot \hat{e}$. Form the pushout $p = h \cdot \hat{e} = e' \cdot g^+ : \hat{T}\Sigma \twoheadrightarrow P$ of $\hat{e}$ and g^+ in $\widehat{\mathscr{D}}^S$. Then the morphisms f and s form a compatible family, so the universal property of the pushout yields an $s' : P \to X$ in $\widehat{\mathscr{D}}^S$ with $s' \cdot e' = s$ and $s' \cdot h = f$. Moreover, by part (a) the object P carries a $\widehat{\mathbf{T}}$-algebra structure (P, ϱ) making p a $\widehat{\mathbf{T}}$-homomorphism. Since $p = h \cdot \hat{e}$, this implies that $h : A \twoheadrightarrow \hat{U}(P, \varrho)$ is an object in $\mathscr{S}$, so $f = s' \cdot h$ is the desired factorization of f.

(c) Since the forgetful functor from $\mathbf{Alg}\,\widehat{\mathbf{T}}$ to $\widehat{\mathscr{D}}^S$ creates limits, see A.3, it follows from (b) that there is a unique $\widehat{\mathbf{T}}$-algebra structure $\alpha : \hat{T}A \to A$ on A making $(h : (A, \alpha) \twoheadrightarrow (A', \alpha'))$ a cofiltered limit cone in $\mathbf{Alg}\,\widehat{\mathbf{T}}$. Thus (A, α) is profinite. To see that $\hat{e} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow (A, \alpha)$ is a $\widehat{\mathbf{T}}$-homomorphism, consider the diagram below:

$$\begin{tikzcd}
\hat{T}\hat{T}\Sigma \arrow[r, "\hat{\mu}_\Sigma"] \arrow[d, two heads, "\hat{T}\hat{e}"'] & \hat{T}\Sigma \arrow[d, two heads, "\hat{e}"] \\
\hat{T}A \arrow[r, dashed, "\alpha"] \arrow[d, two heads, "\hat{T}h"'] & A \arrow[d, two heads, "h"] \\
\hat{T}A' \arrow[r, "\alpha'"'] & A'
\end{tikzcd}$$

The lower square commutes for all $h : A \to \hat{U}(A', \alpha')$ in $\mathscr{S}$ by the definition of α, and the outside commutes because by the definition of $\mathscr{S}$ the morphism $h \cdot \hat{e}$ is a $\widehat{\mathbf{T}}$-homomorphism. Thus also the upper square commutes, as it commutes when composed with the limit projections h in $\widehat{\mathscr{D}}^S$. □

E.2 Proof of Proposition 4.3

Before we come to the proof we develop a number of auxiliary results.

Remark E.9. To prove Proposition 4.3 we first explain how to translate a local pseudovariety into a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra and vice versa.

1. To each local pseudovariety $\mathscr{P}$ of Σ-generated $\mathbf{T}$-algebras we associate a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $\varphi^{\mathscr{P}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^{\mathscr{P}}$ as follows. Viewed as full subcategory of the comma category $(\mathbf{T}\Sigma \downarrow \mathbf{Alg}_f\,\mathbf{T})$, the category $\mathscr{P}$ is cofiltered because $\mathscr{P}$ is closed under subdirect products. Let $P_\Sigma^{\mathscr{P}}$ be the cofiltered limit of the diagram
$$\mathscr{P} \to \mathbf{Alg}\,\widehat{\mathbf{T}}, \quad (e : \mathbf{T}\Sigma \twoheadrightarrow A) \mapsto A,$$
and denote the limit projections by $e^*_{\mathscr{P}} : P_\Sigma^{\mathscr{P}} \twoheadrightarrow A$. They are surjective by Lemma B.2. Thus $P_\Sigma^{\mathscr{P}}$ is a profinite $\widehat{\mathbf{T}}$-algebra. Moreover, the $\widehat{\mathbf{T}}$-homomorphisms $e^+ : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow A$ (where e ranges over all elements of $\mathscr{P}$) form a compatible family over the above diagram, so there exists a unique $\widehat{\mathbf{T}}$-homomorphism $\varphi^{\mathscr{P}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_\Sigma^{\mathscr{P}}$ with $e^+ = e^*_{\mathscr{P}} \cdot \varphi^{\mathscr{P}}$ for all $e \in \mathscr{P}$. Note

that $\varphi^{\mathscr{P}}$ is surjective by Lemma B.1. This yields the desired Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $\varphi^{\mathscr{P}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_{\Sigma}^{\mathscr{P}}$.

2. Conversely, given a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $\varphi : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_{\Sigma}$, define $\mathscr{P}^{\varphi}$ to be the class of all finite Σ-generated $\mathbf{T}$-algebras of the form

$$e = (\mathbf{T}\Sigma \xrightarrow{\iota_\Sigma} V\widehat{\mathbf{T}}\Sigma \xrightarrow{V\varphi} VP_{\Sigma} \xrightarrow{Ve'} A),$$

where $e' : P_{\Sigma} \twoheadrightarrow A$ is a surjective $\widehat{\mathbf{T}}$-homomorphism with $A \in \mathbf{Alg}_f\,\widehat{\mathbf{T}}$. Note that any such morphism e is indeed a surjective $\mathbf{T}$-homomorphism by Remark C.6.5 and C.6.6. It is easy to see that $\mathscr{P}^{\varphi}$ forms a local pseudovariety of Σ-generated $\mathbf{T}$-algebras.

Lemma E.10. *For any local pseudovariety $\mathscr{P}$ of Σ-generated $\mathbf{T}$-algebras we have $\mathscr{P} = \mathscr{P}^{\varphi}$ where $\varphi := \varphi^{\mathscr{P}}$.*

Proof. $\mathscr{P} \subseteq \mathscr{P}^{\varphi}$: Let $(e : \mathbf{T}\Sigma \twoheadrightarrow A) \in \mathscr{P}$. Then for the corresponding limit projection $e^{*}_{\mathscr{P}} : P_{\Sigma}^{\mathscr{P}} \twoheadrightarrow A$ we have

$$e = (\ T\Sigma \xrightarrow{\iota_\Sigma} V\hat{T}\Sigma \xrightarrow{V\varphi} VP_{\Sigma}^{\mathscr{P}} \xrightarrow{Ve^{*}_{\mathscr{P}}} A\ )$$

since $e^{+} = e^{*}_{\mathscr{P}} \cdot \varphi^{\mathscr{P}}$ by the definition of $\varphi = \varphi^{\mathscr{P}}$ and since $e^{+} \cdot \iota_{\Sigma} = e$ by Remark 2.11.2. Therefore $e \in \mathscr{P}^{\varphi}$ by the definition of $\mathscr{P}^{\varphi}$.

$\mathscr{P}^{\varphi} \subseteq \mathscr{P}$: Let $(e : \mathbf{T}\Sigma \twoheadrightarrow A) \in \mathscr{P}^{\varphi}$. Thus there exists a surjective $\widehat{\mathbf{T}}$-homomorphism $e' : P_{\Sigma}^{\mathscr{P}} \twoheadrightarrow A$ with $A \in \mathbf{Alg}_f\,\widehat{\mathbf{T}}$ and

$$e = (\ T\Sigma \xrightarrow{\iota_\Sigma} V\hat{T}\Sigma \xrightarrow{V\varphi} VP_{\Sigma}^{\mathscr{P}} \xrightarrow{Ve'} A\ ).$$

Since A is finitely copresentable in $\mathbf{Alg}\,\widehat{\mathbf{T}}$, see Remark C.6.4, the $\widehat{\mathbf{T}}$-homomorphism e' factors through the the limit cone defining $P_{\Sigma}^{\mathscr{P}}$; that is, there exists an $h : \mathbf{T}\Sigma \twoheadrightarrow A'$ in $\mathscr{P}$ and a $\widehat{\mathbf{T}}$-homomorphism $s : A' \twoheadrightarrow A$ with $e' = s \cdot h^{*}_{\mathscr{P}}$. Since e' is surjective, so is s. Then the commutative diagram below shows that e is a quotient of $h \in \mathscr{P}$, and thus lies in $\mathscr{P}$ because the latter is closed under quotients.

$$\begin{tikzcd}
T\Sigma \arrow[r, "\iota_\Sigma"] \arrow[rrrr, bend left=50, "e"] \arrow[rrr, two heads, bend right=60, "h"'] & V\hat{T}\Sigma \arrow[r, two heads, "V\varphi^{\mathscr{P}}"] \arrow[rr, two heads, bend right=40, "h^{+}"'] & VP_{\Sigma}^{\mathscr{P}} \arrow[r, two heads, "Vh^{*}_{\mathscr{P}}"] \arrow[rr, two heads, bend left=40, "Ve'"] & A' \arrow[r, two heads, "s"] & A
\end{tikzcd}$$

□

Lemma E.11. *For each Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $\varphi : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow P_{\Sigma}$ we have an isomorphism $\varphi \cong \varphi^{\mathscr{P}}$ where $\mathscr{P} := \mathscr{P}^{\varphi}$.*

More precisely, the lemma states that φ and $\varphi^{\mathscr{P}}$ are isomorphic quotients of $\widehat{\mathbf{T}}\Sigma$, i.e. there exists an isomorphism $j : P_\Sigma \xrightarrow{\cong} P_\Sigma^{\mathscr{P}}$ with $\varphi^{\mathscr{P}} = j \cdot \varphi$.

Proof. Let $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}})$ be the full subcategory of $(P_\Sigma \downarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}})$ on all surjective $\widehat{\mathbf{T}}$-homomorphisms $e' : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow A$ with $A \in \mathbf{Alg}_f\,\widehat{\mathbf{T}}$. Consider the functor

$$F : (P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}}) \to \mathscr{P}$$

that maps $e' : P_\Sigma \twoheadrightarrow A$ to the Σ-generated finite $\mathbf{T}$-algebra

$$F(e') = (\ T\Sigma \xrightarrow{\iota_\Sigma} V\hat{T}\Sigma \xrightarrow{V\varphi} \mathrel{\mkern-14mu}\twoheadrightarrow VP_\Sigma \xrightarrow{Ve'} \mathrel{\mkern-14mu}\twoheadrightarrow A\ ).$$

and acts as identity on morphisms. Note that $F(e') \in \mathscr{P}$ by the definition of $\mathscr{P} = \mathscr{P}^\varphi$, so F is well-defined. We claim that F is an isomorphism. Indeed, F is injective on objects because φ is surjective and ι_Σ is dense. The surjectivity on objects is the definition of $\mathscr{P}$. The bijectivity on morphisms is clear.

Next observe that F commutes with the projection functors π and π':

$$\begin{tikzcd}
(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}}) \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathscr{P} \arrow[dl, "\pi'"] \\
& \mathbf{Alg}\,\widehat{\mathbf{T}} &
\end{tikzcd}$$

The limit of π is P_Σ by Corollary E.4, and the limit of π' is $P_\Sigma^{\mathscr{P}}$ by the definition of $P_\Sigma^{(\mathscr{P})}$. Since F is an isomorphism (in particular, a final functor) and limits are unique up to isomorphism, there is an isomorphism $j : P_\Sigma \xrightarrow{\cong} P_\Sigma^{\mathscr{P}}$ with $e' = F(e')^*_{\mathscr{P}} \cdot j$ for all $e' : P_\Sigma \twoheadrightarrow A$ in $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\,\widehat{\mathbf{T}})$. Thus in the diagram below the outside and all inner parts except, perhaps, for the upper inner triangle commute:

$$\begin{tikzcd}
\hat{T}\Sigma \arrow[d, "\iota_\Sigma"] \arrow[ddd, bend right=60, "F(e')"'] & \\
V\hat{T}\Sigma \arrow[d, two heads, "V\varphi"'] \arrow[dr, two heads, "V\varphi^{\mathscr{P}}"] \arrow[dd, bend left=80, two heads, "VF(e')^+"] & \\
VP_\Sigma \arrow[r, "Vj"] \arrow[d, two heads, "Ve'"'] & VP_\Sigma^{\mathscr{P}} \arrow[dl, two heads, "VF(e')^*_{\mathscr{P}}"'] \\
A &
\end{tikzcd}$$

It follows that this triangle also commutes, as it commutes when precomposed with the dense map ι_Σ and postcomposed with the limit projections $VF(e')^*_{\mathscr{P}}$. $\square$

Proof (Proposition 4.3). It suffices to show that the posets of all $\varSigma$-generated profinite $\widehat{\mathbf{T}}$-algebras and all local peudovarieties of $\varSigma$-generated $\mathbf{T}$-algebras are isomorphic. By Lemma E.10 and E.11 the maps $\mathscr{P} \mapsto \varphi^{\mathscr{P}}$ and $\varphi \mapsto \mathscr{P}^{\varphi}$ are mutually inverse and thus give a bijection between the two posets. It only remains to prove that both maps are order-preserving. Given local pseudovarieties $\varphi \leq \varphi'$, we clearly have $\mathscr{P}^{\varphi} \subseteq \mathscr{P}^{\varphi'}$ because every quotient of $P_\varSigma$ is also a quotient of $P'_\varSigma$. Given local pseudovarieties $\mathscr{P} \subseteq \mathscr{P}'$, the morphisms $e^*_{\mathscr{P}'} : P_\varSigma^{\mathscr{P}'} \twoheadrightarrow A$, where e ranges over all $e : \mathbf{T}\mathbf{\Sigma} \twoheadrightarrow A$ in $\mathscr{P}$, form a compatible family over the diagram defining $P_\varSigma^{\mathscr{P}}$. Indeed, for each morphism $h : e \to e'$ in $\mathscr{P} \subseteq \mathscr{P}'$ (cf. Remark E.9.1) $h \cdot e^*_{\mathscr{P}'} = (e')^*_{\mathscr{P}'}$ holds for the limit projections. Hence there exists a unique morphism $q : P_\varSigma^{\mathscr{P}'} \to P_\varSigma^{\mathscr{P}}$ with $e^*_{\mathscr{P}'} = e^*_{\mathscr{P}} \cdot q$ for all $e \in \mathscr{P}$. It follows that $q \cdot \varphi^{\mathscr{P}'} = \varphi^{\mathscr{P}}$, because this holds when postcomposed with the limit projections $e^*_{\mathscr{P}}$.

$$\begin{tikzcd}
\hat{T}\mathbf{\Sigma} \arrow[d, two heads, "\varphi^{\mathscr{P}'}"'] \arrow[dr, two heads, "\varphi_\varSigma^{\mathscr{P}}"] & \\
P_\varSigma^{\mathscr{P}'} \arrow[r, two heads, "q"] \arrow[d, two heads, "e^*_{\mathscr{P}'}"'] & P_\varSigma^{\mathscr{P}} \arrow[dl, two heads, "e^*_{\mathscr{P}}"] \\
A &
\end{tikzcd}$$

Indeed, the outside of the above diagram commutes because both sides yield e when we apply V and precompose with the dense map $\iota_{\mathbf{\Sigma}}$. Therefore we have $\varphi^{\mathscr{P}} \leq \varphi^{\mathscr{P}'}$ as desired. □

E.3 Details for Remark 4.4

Remark E.12. The *homomorphism theorem* states that given $e\colon A \twoheadrightarrow B$ and $f\colon A \to C$ in $\widehat{\mathscr{D}}^S$ with e surjective, there exists a morphism g in $\widehat{\mathscr{D}}^S$ with $g \cdot e = f$ iff, for all sorts s and $a, a' \in |A|_s$, $e(a) = e(a')$ implies $f(a) = f(a')$ (resp. $e(a) \leq e(a')$ implies $f(a) \leq f(a')$ if $\mathscr{D}$ is variety of ordered algebras). Indeed, there clearly is a $\mathscr{D}^S$*-morphism* g with this property, and it is continuous because A, B, C are compact Hausdorff spaces. Moreover, if A, B, C are $\widehat{\mathbf{T}}$-algebras and e and f are $\widehat{\mathbf{T}}$-homomorphisms, so is g. This follows from the fact that $\hat{T}$ preserves epimorphisms, see Remark C.6.7.

We consider the case where $\mathscr{D}$ is variety of ordered algebras; for the unordered case, replace all inequations by equations.

For a set E of profinite inequations over $\varSigma$, let $\mathscr{P}[E]$ denote the class of all $\varSigma$-generated $\mathbf{T}$-algebras satisfying all inequations in E. Conversely, for a class $\mathscr{P}$ of $\varSigma$-generated finite $\mathbf{T}$-algebras, let $E[\mathscr{P}]$ be set of all profinite inequations over $\varSigma$ satisfied by all algebras in $\mathscr{P}$. The claim is that $\mathscr{P}$ forms a local pseudovariety iff $\mathscr{P} = \mathscr{P}[E]$ for some E.

The “if” direction is a straightforward verification. For the “only if” direction, suppose that $\mathscr{P}$ is a local pseudovariety of $\varSigma$-generated $\mathbf{T}$-algebras, and let

$\varphi^{\mathscr{P}} : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow P_{\Sigma}^{\mathscr{P}}$ be the correponding Σ-generated profinite $\widehat{\mathbf{T}}$-algebra, see Remark E.9.1. From the definition of $\varphi^{\mathscr{P}}$ it immediately follows that a profinite inequation $u \leq v$ lies in $E[P]$ iff $\varphi^{\mathscr{P}}(u) \leq \varphi^{\mathscr{P}}(v)$. We claim that $\mathscr{P} = \mathscr{P}[E[\mathscr{P}]]$. The inclusion $\subseteq$ is trivial. To prove $\supseteq$, let $e : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ be an element of $\mathscr{P}[E[\mathscr{P}]]$, i.e. e satisfies every equation that every algebra in $\mathscr{P}$ satisfies. By the homomorphism theorem, see Remark E.12, there exists a (surjective) $\widehat{\mathbf{T}}$-homomorphism $h : P_{\Sigma}^{\mathscr{P}} \twoheadrightarrow A$ with $e^+ = h \cdot \varphi^{\mathscr{P}}$. Indeed, every pair u, v with $\varphi^{\mathscr{P}}(u) \leq \varphi^{\mathscr{P}}(v)$ forms a profinite inequation $u \leq v$ satisfied by $\mathscr{P}$. Thus $u \leq v$ is satisfied by e, i.e. $e^+(u) \leq e^+(v)$.

We conclude that A lies in $\mathscr{P}^{(\varphi^{\mathscr{P}})}$ by the definition of $\mathscr{P}^{(-)}$, and thus in $\mathscr{P}$ by Lemma E.10). ☐

E.4 Details for Example 4.8.2

Let $\mathbb{A} = \{(\Sigma, \emptyset) \colon \Sigma \in \mathbf{Set}_f\}$. We prove that a finite ω-semigroup $A = (A_+, A_\omega)$ is $\mathbb{A}$-generated iff it is *complete*, i.e. every element $a \in A_\omega$ can be expressed as an infinite product $a = \pi(a_0, a_1, \ldots)$ for some $a_i \in A_+$. For the "only if" direction, suppose that A is $\mathbb{A}$-generated, i.e. there exists a surjective ω-semigroup morphism $e : (\Sigma^+, \Sigma^\omega) \twoheadrightarrow (A_+, A_\omega)$ for some $\Sigma \in \mathbf{Set}_f$. For each $a \in A_\omega$, choose $s_0 s_1 \ldots \in \Sigma^\omega$ with $a = e(s_0 s_1 \ldots)$. Then

$$a = e(s_0 s_1 \ldots) = e(\pi(s_0, s_1, \ldots)) = \pi(e(s_0), e(s_1), \ldots),$$

which shows that A is complete. For the "if" direction, suppose that A is complete. Let $\Sigma := A_+ \in \mathbf{Set}_f$, and extend the map $(id, \emptyset) : (\Sigma, \emptyset) \to (A_+, A_\omega)$ to an ω-semigroup morphism $e : (\Sigma^+, \Sigma^\omega) \to (A_+, A_\omega)$, using that $(\Sigma^+, \Sigma^\omega)$ is the free ω-semigroup on $(\Sigma, \emptyset)$. Clearly the component $e : \Sigma^+ \to A_+$ is surjective because $e(a) = a$ for all $a \in A_+$. To show that also the component $e : \Sigma^\omega \to A_\omega$ is surjective, let $a \in A_\omega$ and choose elements $a_i \in A_+$ with $a = \pi(a_0, a_1, \ldots)$, using the completeness of A. It follows that

$$a = \pi(a_0, a_1, \ldots) = \pi(e(a_0), e(a_1), \ldots) = e(\pi(a_0, a_1, \ldots)).$$

Thus e is surjective, which proves that A is $\mathbb{A}$-generated. ☐

E.5 Details for Remark 4.9

See Lemma C.7. ☐

E.6 Proof of Proposition 4.11

The proof of Proposition 4.11, establishing the equivalence of profinite theories and pseudovarieties, is achieved through a sequence of lemmas. First an auxiliary result:

Lemma E.13. *Given $\Sigma, \Delta \in \mathbf{Set}_f^S$, a surjective $\widehat{\mathbf{T}}$-homomorphism $e : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$ with $A \in \mathbf{Alg}_f\, \widehat{\mathbf{T}}$ and a $\widehat{\mathbf{T}}$-homomorphism $h : \widehat{\mathbf{T}}\mathbb{\Delta} \to A$, there exists a $\mathbf{T}$-homomorphism $g : \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$ with $h = e \cdot \hat{g}$.*

Proof. Let $e' = Ve \cdot \iota_{\mathbb{\Sigma}} : \mathbf{T}\mathbb{\Delta} \twoheadrightarrow A$ and $h' = Vh \cdot \iota_{\mathbb{\Sigma}} : \mathbf{T}\mathbb{\Sigma} \to A$ be the restrictions of e and h to $\mathbf{T}$-homomorphisms, see Remark C.6.5. Since the free object $\mathbb{\Delta}$ is projective in the S-sorted variety $\mathscr{D}^S$, and e' is surjective by Remark C.6.6, there exists a morphism $g' : \mathbb{\Delta} \to T\mathbb{\Sigma}$ in $\mathscr{D}^S$ with $h' \cdot \eta_{\mathbb{\Delta}} = e' \cdot g'$. Since $\mathbf{T}\mathbb{\Delta}$ is the free $\mathbf{T}$-algebra on $\mathbb{\Delta}$, see A.2, we can extend g' to a $\mathbf{T}$-homomorphism $g : \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$. Then $\hat{g}$ has the desired property: the lower triangle in the diagram below commutes, because it does when precomposed by the dense map $\iota_{\mathbb{\Delta}}$ and the unit $\eta_{\mathbb{\Delta}}$.

$$\begin{tikzcd}
\mathbb{\Delta} \arrow[d, "\eta_{\mathbb{\Delta}}"'] \arrow[dr, "g'"] & & \\
T\mathbb{\Delta} \arrow[r, "g"] \arrow[d, "\iota_{\mathbb{\Delta}}"'] & T\mathbb{\Sigma} \arrow[d, "\iota_{\mathbb{\Sigma}}"] \arrow[dr, two heads, "e'"] & \\
V\hat{T}\mathbb{\Delta} \arrow[r, "V\hat{g}"'] \arrow[rr, bend right, "Vh"'] & V\hat{T}\mathbb{\Sigma} \arrow[r, two heads, "Ve"'] & A
\end{tikzcd}$$

□

Lemma E.14. *Let φ be a profinite theory, and (A, α) be an $\mathbb{A}$-generated finite $\mathbf{T}$-algebra. The following statements are equivalent:*

(i) There exists a surjective $\widehat{\mathbf{T}}$-homomorphism $e : P_{\Sigma} \twoheadrightarrow (A, \alpha^{+})$ for some $\Sigma \in \mathbb{A}$.

(ii) Every $\widehat{\mathbf{T}}$-homomorphism $h : \widehat{\mathbf{T}}\mathbb{\Delta} \to (A, \alpha^{+})$ with $\Delta \in \mathbb{A}$ factors through φ_{Δ}:

$$\begin{tikzcd}
\widehat{\mathbf{T}}\mathbb{\Delta} \arrow[d, two heads, "\varphi_{\Delta}"'] \arrow[dr, "h"] & \\
P_{\Delta} \arrow[r, dashed, "h'"'] & A
\end{tikzcd}$$

Proof. (i)⇒(ii) Given a $\widehat{\mathbf{T}}$-homomorphism $h : \widehat{\mathbf{T}}\mathbb{\Delta} \to A$, choose a $\mathbf{T}$-homomorphism $g : \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$ with $h = e \cdot \varphi_{\Sigma} \cdot \hat{g}$, see Lemma E.13. Since φ is a profinite theory, there exists a $\widehat{\mathbf{T}}$-homomorphism $g_P : P_{\Delta} \to P_{\Sigma}$ with $g_P \cdot \varphi_{\Delta} = \varphi_{\Sigma} \cdot \hat{g}$. Thus $h = (e \cdot g_P) \cdot \varphi_{\Delta}$ is the desired factorization of h through φ_{Δ}.

$$\begin{tikzcd}
\widehat{\mathbf{T}}\mathbb{\Delta} \arrow[r, "\hat{g}"] \arrow[d, two heads, "\varphi_{\Delta}"'] \arrow[rrd, bend left, "h"] & \widehat{\mathbf{T}}\mathbb{\Sigma} \arrow[d, two heads, "\varphi_{\Sigma}"] & \\
P_{\Delta} \arrow[r, "g_P"'] & P_{\Sigma} \arrow[r, two heads, "e"'] & A
\end{tikzcd}$$

(ii)⇒(i) Since A is $\mathbb{A}$-generated, there exists a surjective $\mathbf{T}$-homomorphism $e : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ for some $\Sigma \in \mathbb{A}$. By hypothesis the surjective $\widehat{\mathbf{T}}$-homomorphism $e^{+} : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$ factors through φ_{Σ}. Thus A is a quotient of P_{Σ}. □

Lemma E.15. *Let φ be a profinite theory. Then the class $\mathscr{V}_\varphi$ of all $\mathbb{A}$-generated finite $\mathbf{T}$-algebras (A, α) satisfying the equivalent properties of Lemma E.14 forms a pseudovariety.*

Proof. Let A_i $(i \in I)$ be finitely many objects in $\mathscr{V}_\varphi$. Form the product $\pi_i \colon \prod_i A_i \to A_i$ and let $m \colon A \rightarrowtail \prod_i A_i$ be an $\mathbb{A}$-generated subalgebra. We show that any $\widehat{\mathbf{T}}$-homomorphism $h \colon \widehat{\mathbf{T}}\mathbb{\Delta} \to A$ factors through φ_Δ. For each i, there exists a $\widehat{\mathbf{T}}$-homomorphism $h_i \colon P_\Delta \to A_i$ with $h_i \cdot \varphi_\Delta = \pi_i \cdot m \cdot h$, because $A_i \in \mathscr{V}_\varphi$. Then h factors through φ_Δ via the diagonal fill-in property, which shows that $A \in \mathscr{V}_\varphi$.

$$\begin{tikzcd}
\mathbf{T}\mathbb{\Delta} \arrow[r, two heads, "\varphi_\Delta"] \arrow[d, "h"'] & P_\Delta \arrow[dl, dashed] \arrow[d, "\langle h_i \rangle"] \arrow[dr, "h_i"] & \\
A \arrow[r, tail, "m"'] & \prod_i A_i \arrow[r, "\pi_i"'] & A_i
\end{tikzcd}$$

The closure of $\mathscr{V}_\varphi$ under quotients follows from Lemma E.14(i). ☐

The reverse passage from pseudovarieties to profinite theories requires some preparation.

Lemma E.16. *Let $\mathscr{V}$ be a class of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras closed under $\mathbb{A}$-generated subalgebras of finite products. Then for each $\Sigma \in \mathbb{A}$ the comma categories $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ and $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$ of all (surjective) $\mathbf{T}$-homomorphisms $h : \mathbf{T}\mathbb{\Sigma} \to A$ with $A \in \mathscr{V}$ are cofiltered.*

Proof. We only show that $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ is cofiltered; the argument for $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$ is analogous. To this end we verify the criterion of A.8.

(i) $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ is nonempty: let $h : \mathbf{T}\mathbb{\Sigma} \to 1$ be the unique $\mathbf{T}$-homomorphism into the terminal $\mathbf{T}$-algebra, and consider its factorization

$$h \;=\; (\mathbf{T}\mathbb{\Sigma} \xrightarrow{e} \!\!\!\!\rightarrow A \rightarrowtail\!\xrightarrow{m} 1)$$

Then A is an $\mathbb{A}$-generated subalgebra of 1 (the empty product) and thus lies in $\mathscr{V}$. Hence $e \in (\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$.

(ii) Given $h_i : \mathbf{T}\mathbb{\Sigma} \to A_i$ $(i = 0, 1)$ in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$, form the product $\pi_i : A_0 \times A_1 \to A_i$ in $\mathbf{Alg}\,\mathbf{T}$ and factorize the $\mathbf{T}$-homomorphism $\langle h_0, h_1 \rangle : \mathbf{T}\mathbb{\Sigma} \to A_0 \times A_1$ as $\langle h_0, h_1 \rangle = m \cdot e$ with e surjective and m injective.

$$\begin{tikzcd}
 & \mathbf{T}\mathbb{\Sigma} \arrow[dl, "h_0"'] \arrow[dr, "h_1"] \arrow[d, two heads, "e"] \arrow[dd, bend right, "\langle h_0,h_1\rangle"'] & \\
A_0 & A \arrow[d, tail, "m"] & A_1 \\
 & A_0 \times A_1 \arrow[ul, "\pi_0"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Then $A \in \mathscr{V}$, being an $\mathbb{A}$-generated subalgebra of the product $A_0 \times A_1$, and we have the morphisms $\pi_i \cdot m : e \to h_i$ in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$.

(iii) Given $h_i : \mathbf{T}\mathbb{\Sigma} \to A_i$ $(i = 0, 1)$ in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ and two morphisms $g, g' : h_0 \to h_1$, form the equalizer $q : E \rightarrowtail A_0$ of g and g' in $\mathbf{Alg}\,\mathbf{T}$. Since $g \cdot h_0 = g' \cdot h_0$, the universal property of q gives a unique $\mathbf{T}$-homomorphism $h : \mathbf{T}\mathbb{\Sigma} \to E$ with $h_0 = q \cdot h$. Let $h = e \cdot m$ be the (surjective, injective) factorization of h, see the diagram below:

$$\begin{tikzcd}
 & & \mathbf{T}\mathbb{\Sigma} \arrow[lld, two heads, "e"'] \arrow[lldd, dashed, "h"] \arrow[dd, "h_0"] \arrow[rdd, "h_1"] & \\
E' \arrow[d, tail, "m"'] & & & \\
E \arrow[rr, tail, "q"'] & & A_0 \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & A_1
\end{tikzcd}$$

Then E' lies in $\mathscr{V}$, being an $\mathbb{A}$-generated subalgebra of $A_0 \in \mathscr{V}$. It follows that $q \cdot m : e \to h_0$ is a morphism in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ merging g and g', i.e. $g \cdot (q \cdot m) = g' \cdot (q \cdot m)$. ☐

Remark E.17. We review a construction given in [12]. As in the above lemma, let $\mathscr{V}$ be a class of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras closed under $\mathbb{A}$-generated subalgebras of finite products.

1. In analogy to the profinite monad $\widehat{\mathbf{T}}$, see Theorem 2.9 and Remark C.6.1/2, one can construct the *pro-$\mathscr{V}$ monad* $\widehat{\mathbf{T}}_{\mathscr{V}} = (\hat{T}_{\mathscr{V}}, \hat{\eta}^{\mathscr{V}}, \hat{\mu}^{\mathscr{V}})$ of $\mathbf{T}$. This is the codensity monad of the forgetful functor

$$\mathscr{V} \hookrightarrow \mathbf{Alg}_f\,\mathbf{T} \to \mathscr{D}_f^S \xrightarrow{\cong} \widehat{\mathscr{D}}_f^S \hookrightarrow \widehat{\mathscr{D}}^S.$$

By the limit formula for right Kan extensions, the object $\hat{T}_{\mathscr{V}} D$ for $D \in \mathscr{D}_f^S$ is the limit of the diagram

$$(\mathbf{T}D \downarrow \mathscr{V}) \to \widehat{\mathscr{D}}^S, \quad (h : \mathbf{T}D \to (A, \alpha)) \mapsto A.$$

We denote the limit projections by

$$h_{\mathscr{V}}^{+} : \hat{T}_{\mathscr{V}} D \to A. \tag{E.3}$$

If $D = \mathbb{\Sigma}$ with $\Sigma \in \mathbb{A}$, the above limit is cofiltered by Lemma E.16, and one can restrict to the cofiltered subdiagram $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V}) \to \widehat{\mathscr{D}}^S$ (cf. Remark 2.11.1).

2. For each $(A, \alpha) \in \mathscr{V}$ we have the $\mathbf{T}$-homomorphism $\alpha \colon \mathbf{T}A \twoheadrightarrow (A, \alpha)$, and thus the limit projection $\alpha_{\mathscr{V}}^{+} : \hat{T}_{\mathscr{V}} A \to A$. Then the following squares commute for all $\mathbf{T}$-homomorphisms $h : \mathbf{T}D \to (A, \alpha)$:

$$\begin{tikzcd}
D \arrow[r, "\hat{\eta}_D^{\mathscr{V}}"] \arrow[rd, "h\eta_D"'] & \hat{T}_{\mathscr{V}} D \arrow[d, "h_{\mathscr{V}}^{+}"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{T}_{\mathscr{V}} \hat{T}_{\mathscr{V}} D \arrow[r, "\hat{\mu}_D^{\mathscr{V}}"] \arrow[d, "\hat{T}_{\mathscr{V}} h_{\mathscr{V}}^{+}"'] & \hat{T}_{\mathscr{V}} D \arrow[d, "h_{\mathscr{V}}^{+}"] \\
\hat{T}_{\mathscr{V}} A \arrow[r, "\alpha_{\mathscr{V}}^{+}"'] & A
\end{tikzcd} \tag{E.4}$$

3. The universal property of right Kan extensions gives a monad morphism $\varphi^{\mathscr{V}} \colon \widehat{\mathbf{T}} \to \widehat{\mathbf{T}}_{\mathscr{V}}$ (see A.1). For $D \in \mathscr{D}_f^S$ the morphisms h^+ form a compatible family over the diagram defining $\hat{T}_{\mathscr{V}} D$, and the component $\varphi_D^{\mathscr{V}}$ is the unique morphism in $\widehat{\mathscr{D}}^S$ making the triangle below commute for all $\mathbf{T}$-homomorphisms $h \colon \mathbf{T}D \to A$ with $A \in \mathscr{V}$:

$$\begin{tikzcd}
\hat{T}D \arrow[r, "\varphi_D^{\mathscr{V}}"] \arrow[d, "h^+"'] & \hat{T}_{\mathscr{V}} D \arrow[dl, "h_{\mathscr{V}}^+"] \\
A &
\end{tikzcd} \tag{E.5}$$

Note that if h^+ is surjective, then so is $h_{\mathscr{V}}^+$. Moreover, by Remark E.17.1 and Lemma B.1, each component $\varphi_{\Sigma}^{\mathscr{V}}$ with $\Sigma \in \mathbb{A}$ is surjective. Note further that, since $\varphi^{\mathscr{V}}$ is a monad morphism, $\varphi_{\Sigma}^{\mathscr{V}}$ is a $\widehat{\mathbf{T}}$-homomorphism

$$\varphi_{\Sigma}^{\mathscr{V}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow (\hat{T}_{\mathscr{V}}\Sigma, \hat{\mu}_{\Sigma}^{\mathscr{V}} \cdot \varphi_{\hat{T}_{\mathscr{V}}\Sigma}^{\mathscr{V}}).$$

Lemma E.18. *Let $\mathscr{V}$ be a pseudovariety of $\mathbf{T}$-algebras. Then the family*

$$\varphi_{\mathscr{V}} = (\varphi_{\Sigma}^{\mathscr{V}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow \hat{T}_{\mathscr{V}}\Sigma)_{\Sigma \in \mathbb{A}}$$

forms a profinite theory.

Proof. (1) For all $\mathbf{T}$-homomorphisms $h : \mathbf{T}\Sigma \to (A, \alpha)$ with $\Sigma \in \mathbb{A}$ and $(A, \alpha) \in \mathscr{V}$ we have the following commutative diagram.

$$\begin{tikzcd}
\hat{T}\hat{T}_{\mathscr{V}}\Sigma \arrow[r, "\varphi_{\hat{T}_{\mathscr{V}}\Sigma}^{\mathscr{V}}"] \arrow[d, "\hat{T}h_{\mathscr{V}}^+"'] & \hat{T}_{\mathscr{V}}\hat{T}_{\mathscr{V}}\Sigma \arrow[r, "\hat{\mu}_{\Sigma}^{\mathscr{V}}"] \arrow[d, "\hat{T}_{\mathscr{V}}h_{\mathscr{V}}^+"'] & \hat{T}_{\mathscr{V}}\Sigma \arrow[d, "h_{\mathscr{V}}^+"] \\
\hat{T}A \arrow[r, "\varphi_A^{\mathscr{V}}"] \arrow[rr, bend right, "\alpha^+"'] & \hat{T}_{\mathscr{V}}A \arrow[r, "\alpha_{\mathscr{V}}^+"] & A
\end{tikzcd}$$

Indeed, the right-hand square commutes by Remark E.17.2, the left-hand square commutes by the naturality of $\varphi^{\mathscr{V}}$, and for the lower triangle see Remark E.17.3. Since the forgetful functor from $\mathbf{Alg}\,\widehat{\mathbf{T}}$ to $\widehat{\mathscr{D}}^S$ reflects limits, see A.3, this shows that the $\widehat{\mathbf{T}}$-homomorphisms

$$h_{\mathscr{V}}^+ : (\hat{T}_{\mathscr{V}}\Sigma, \hat{\mu}_{\Sigma}^{\mathscr{V}} \cdot \varphi_{\hat{T}_{\mathscr{V}}\Sigma}^{\mathscr{V}}) \to (A, \alpha^+),$$

form a cofiltered limit cone in $\mathbf{Alg}\,\widehat{\mathbf{T}}$. Hence the $\widehat{\mathbf{T}}$-algebra $(\hat{T}_{\mathscr{V}}\Sigma, \hat{\mu}_{\Sigma}^{\mathscr{V}} \cdot \varphi_{\hat{T}_{\mathscr{V}}\Sigma}^{\mathscr{V}})$ is profinite.

(2) Given a $\mathbf{T}$-homomorphism $g : \mathbf{T}\Delta \to \mathbf{T}\Sigma$ with $\Sigma, \Delta \in \mathbb{A}$, the morphisms $(hg)_{\mathscr{V}}^+$ (where h ranges over all $\mathbf{T}$-homomorphisms $h : \mathbf{T}\Sigma \to A$ with $A \in \mathscr{V}$) form a compatible family over the diagram defining $\hat{T}_{\mathscr{V}}\Sigma$. Thus there exists a

unique $g' : \hat{T}_{\mathscr{V}}\mathbb{\Delta} \to \hat{T}_{\mathscr{V}}\mathbb{\Sigma}$ with $(hg)^+_{\mathscr{V}} = h^+_{\mathscr{V}} \cdot g'$ for all h. It follows that the upper square in the following diagram commutes, as it commutes when postcomposed with the limit projections $h^+_{\mathscr{V}}$ (the outside commutes due to Lemma C.7).

$$\begin{tikzcd}
\widehat{\mathbf{T}}\mathbb{\Delta} \arrow[rr, "\hat{g}"] \arrow[d, two heads, "\varphi^{\mathscr{V}}_{\mathbb{\Delta}}"'] \arrow[ddr, bend right=60, "(hg)^+"'] & & \widehat{\mathbf{T}}\mathbb{\Sigma} \arrow[d, two heads, "\varphi^{\mathscr{V}}_{\mathbb{\Sigma}}"] \arrow[ddl, bend left=60, "h^+"] \\
\hat{T}_{\mathscr{V}}\mathbb{\Delta} \arrow[rr, "g'"'] \arrow[dr, "(hg)^+_{\mathscr{V}}"'] & & \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \arrow[dl, "h^+_{\mathscr{V}}"] \\
& A &
\end{tikzcd}$$

∎

The following two lemmas demonstrate that the constructions $\varphi \mapsto \mathscr{V}_\varphi$ and $\mathscr{V} \mapsto \varphi_{\mathscr{V}}$ of Lemma E.15 and E.18 are mutually inverse.

Lemma E.19. *For any pseudovariety $\mathscr{V}$ of $\mathbf{T}$-algebras we have $\mathscr{V} = \mathscr{V}_\varphi$ where $\varphi := \varphi_{\mathscr{V}}$.*

Proof. $\mathscr{V} \subseteq \mathscr{V}_\varphi$: Let $A \in \mathscr{V}$. Since A is $\mathbb{A}$-generated, there exists a surjective $\mathbf{T}$-homomorphism $e : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ with $\Sigma \in \mathbb{A}$. Then we have the surjective $\widehat{\mathbf{T}}$-homomorphism $e^+_{\mathscr{V}} : \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \twoheadrightarrow A$ and therefore $A \in \mathscr{V}_\varphi$.

$\mathscr{V}_\varphi \subseteq \mathscr{V}$: Let $A \in \mathscr{V}_\varphi$. Since A is $\mathbb{A}$-generated, there exists a surjective $\mathbf{T}$-homomorphism $e : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ with $\Sigma \in \mathbb{A}$. Thus we have the surjective $\widehat{\mathbf{T}}$-homomorphism $e^+ : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$, see Remark E.17.3. By the definition of $\mathscr{V}_\varphi$ there exists a (surjective) $\widehat{\mathbf{T}}$-homomorphism $e' : \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \twoheadrightarrow A$ with $e^+ = e' \cdot \varphi^{\mathscr{V}}_{\mathbb{\Sigma}}$, see Lemma E.14(ii). Since the finite $\widehat{\mathbf{T}}$-algebra A is finitely copresentable in $\mathbf{Alg}\,\widehat{\mathbf{T}}$, see Remark 2.11.4, the homomorphism e' factors through the limit cone defining $\hat{T}_{\mathscr{V}}\mathbb{\Sigma}$; that is, there exist $\mathbf{T}$-homomorphisms $h : \mathbf{T}\mathbb{\Sigma} \to B$ and $e'' : B \to A$ with $B \in \mathscr{V}$ and $e' = e'' \cdot h^+_{\mathscr{V}}$.

$$\begin{tikzcd}
\hat{T}\mathbb{\Sigma} \arrow[r, two heads, "\varphi^{\mathscr{V}}_{\mathbb{\Sigma}}"] \arrow[d, two heads, "e^+"'] & \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \arrow[d, "h^+_{\mathscr{V}}"] \arrow[dl, two heads, "e'"] \\
A & B \arrow[l, two heads, "e''"]
\end{tikzcd}$$

Since e' is surjective, so is e''. Hence the closure of $\mathscr{V}$ under quotients implies that $A \in \mathscr{V}$. ∎

Lemma E.20. *For any profinite theory $\varphi = (\varphi_\Sigma : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow P_\Sigma)$ we have $\varphi \cong \varphi_{\mathscr{V}}$ where $\mathscr{V} := \mathscr{V}_\varphi$.*

More precisely, for each $\Sigma \in \mathbb{A}$ there is an isomorphism $j_\Sigma : \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \xrightarrow{\cong} P_\Sigma$ with $\varphi_\Sigma = j_\Sigma \cdot \varphi^{\mathscr{V}}_\Sigma$.

Proof. (1) Every surjective $\widehat{\mathbf{T}}$-homomorphism $e : P_\Sigma \twoheadrightarrow A$ with $A \in \mathbf{Alg}_f\,\widehat{\mathbf{T}}$ yields the surjective $\mathbf{T}$-homomorphism

$$e' = (T\mathbb{\Sigma} \xrightarrow{\iota_{\mathbb{\Sigma}}} V\hat{T}\mathbb{\Sigma} \xrightarrow{V\varphi_\Sigma} VP_\Sigma \xrightarrow{Ve} A),$$

see Remark E.9.2. Moreover, $A \in \mathscr{V}$ by the definition of $\mathscr{V}$, see Lemma E.14. Thus the map $e \mapsto e'$ defines a functor (acting as identity on morphisms)

$$F : (P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}}) \to (\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V}),$$

where $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ and $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$ are the full subcategories of the comma categories $(P_\Sigma \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ and $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ on surjective homomorphisms.

(2) We claim that F is final, see A.7. Since $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ is cofiltered, this requires to show that (i) for any object e' in $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$ there exists a morphism $F(e) \to e'$ in $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$ for some $e \in (P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$, and (ii) any two parallel morphisms $F(e) \rightrightarrows e'$ are merged by some morphism in $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$.

For (i), let $e' : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ be an object of $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$. Then we have the surjective $\widehat{\mathbf{T}}$-homomorphism $(e')^+ : \widehat{\mathbf{T}}\mathbb{\Sigma} \twoheadrightarrow A$. Since $A \in \mathscr{V}$, there exists a surjective $\widehat{\mathbf{T}}$-homomorphism $e : P_\Sigma \twoheadrightarrow A$ with $(e')^+ = e \cdot \varphi_\Sigma$. Then $F(e) = e'$, as shown by the commutative diagram below.

$$\begin{tikzcd}
T\mathbb{\Sigma} \arrow[r, "\iota_{\mathbb{\Sigma}}"] \arrow[d, two heads, "e'"'] & V\hat{T}\mathbb{\Sigma} \arrow[dl, two heads, "V(e')^+" description] \arrow[d, two heads, "V\varphi_\Sigma"] \\
A & VP_\Sigma \arrow[l, two heads, "Ve"]
\end{tikzcd}$$

Thus we have the desired connecting arrow $id : F(e) \to e'$ in $(\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V})$.

The property (ii) is trivially satisfied: since $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$ consists only of surjections, there is at most one morphism $F(e) \to e'$ for each e in $(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$. This shows the finality of F.

(3) F commutes with the projection functors π and π':

$$\begin{tikzcd}
(P_\Sigma \twoheaddownarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}}) \arrow[rr, "F"] \arrow[dr, "\pi"'] & & (\mathbf{T}\mathbb{\Sigma} \twoheaddownarrow \mathscr{V}) \arrow[dl, "\pi'"] \\
& \mathbf{Alg}\, \widehat{\mathbf{T}} &
\end{tikzcd}$$

The limit of π is P_Σ by Corollary E.4, and the limit of π' is $\hat{T}_{\mathscr{V}}\mathbb{\Sigma}$ by Remark E.17.1. Thus the finality of F and the uniqueness of limits implies the existence of an isomorphism $j_\Sigma : \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \xrightarrow{\cong} P_\Sigma$ with $F(e)^+_{\mathscr{V}} = e \cdot j_\Sigma$ for all $e : P_\Sigma \twoheadrightarrow A$ in

$(P_\Sigma \downarrow \mathbf{Alg}_f\, \widehat{\mathbf{T}})$. Now consider the diagram below:

$$\begin{tikzcd}
& \hat{T}\mathbb{\Sigma} \arrow[d, "\iota_{\mathbb{\Sigma}}"] \arrow[ddd, bend left=60, two heads, "F(e)"] & \\
& V\hat{T}\mathbb{\Sigma} \arrow[d, two heads, "V\varphi^{\mathscr{V}}_{\mathbb{\Sigma}}"'] \arrow[dr, two heads, "V\varphi_\Sigma"] \arrow[dd, bend right=80, two heads, "V(F(e)^+)"'] & \\
& V\hat{T}_{\mathscr{V}}\mathbb{\Sigma} \arrow[r, "Vj_\Sigma"] \arrow[d, two heads, "V(F(e)^+_{\mathscr{V}})"'] & VP_\Sigma \arrow[dl, two heads, "Ve"'] \\
& A &
\end{tikzcd}$$

The outside commutes by Remark 2.11.2, and all inner parts except for the central triangle commute by the definition of j_Σ, the definition of F and Remark E.17.3. It follows that the central triangle also commutes, as it commutes when precomposed with the dense map $\iota_{\mathbb{\Sigma}}$ and postcomposed with the limit projections Ve. $\square$

Proof (Proposition 4.11). By Lemmas E.19 and E.20 the maps $\mathscr{V} \mapsto \varphi_{\mathscr{V}}$ and $\varphi \mapsto \mathscr{V}_\varphi$ give mutually inverse object maps between the two posets. It remains to show that both constructions are order-preserving.

(1) Given profinite theories $\varphi \leq \varphi'$, we have $\mathscr{V}_\varphi \subseteq \mathscr{V}_{\varphi'}$ since, for each $\Sigma \in \mathbb{A}$, any quotient of P_Σ is also a quotient of P'_Σ.

(2) Let $\mathscr{V} \subseteq \mathscr{V}'$ be pseudovarieties. For each $\Sigma \in \mathbb{A}$ the morphisms $e^+_{\mathscr{V}'} : \hat{T}_{\mathscr{V}'}\mathbb{\Sigma} \twoheadrightarrow A$, where e ranges over surjective $\mathbf{T}$-homomorphisms $e : \mathbf{T}\mathbb{\Sigma} \twoheadrightarrow A$ with $A \in \mathscr{V}$, form a compatible family over the diagram defining $\hat{T}_{\mathscr{V}}\mathbb{\Sigma}$. Indeed, since $\mathscr{V} \subseteq \mathscr{V}'$, each morphism $h : e \to e'$ in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V})$ is also a morphism in $(\mathbf{T}\mathbb{\Sigma} \downarrow \mathscr{V}')$, and therefore $h \cdot e^+_{\mathscr{V}'} = (e')^+_{\mathscr{V}'}$ hold for the limit projections. Hence there exists a unique morphism $q : \hat{T}_{\mathscr{V}'}\mathbb{\Sigma} \twoheadrightarrow \hat{T}_{\mathscr{V}}\mathbb{\Sigma}$ with $e^+_{\mathscr{V}'} = e^+_{\mathscr{V}} \cdot q$ for all e. It follows that $q \cdot \varphi^{\mathscr{V}'}_{\mathbb{\Sigma}} = \varphi^{\mathscr{V}}_{\mathbb{\Sigma}}$, because this holds when postcomposed with the limit projections $e^+_{\mathscr{V}}$.

$$\begin{tikzcd}
\hat{T}\mathbb{\Sigma} \arrow[d, two heads, "\varphi^{\mathscr{V}'}_{\mathbb{\Sigma}}"'] \arrow[dr, two heads, "\varphi^{\mathscr{V}}_{\mathbb{\Sigma}}"] & \\
\hat{T}_{\mathscr{V}'}\mathbb{\Sigma} \arrow[r, two heads, "q"] \arrow[d, two heads, "e^+_{\mathscr{V}'}"'] & \hat{T}_{\mathscr{V}}\mathbb{\Sigma} \arrow[dl, two heads, "e^+_{\mathscr{V}}"] \\
A &
\end{tikzcd}$$

The outside of the above diagram commutes because both sides yield e^+ by (E.5). Therefore $\varphi_{\mathscr{V}} \leq \varphi_{\mathscr{V}'}$. $\square$

E.7 Details for Remark 4.12

We consider the case where $\mathscr{D}$ is variety of ordered algebras; for the unordered case, replace all inequations by equations.

For a class E of profinite inequations over (possibly different) alphabets in $\mathbb{A}$, let $\mathscr{V}[E]$ denote the class of all $\mathbb{A}$-generated finite $\mathbf{T}$-algebras satisfying all inequations in E. Conversely, for a class $\mathscr{V}$ of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras let $E[\mathscr{V}]$ be the class of all profinite inequations over alphabets $\varSigma \in \mathbb{A}$ satisfied by all algebras in $\mathscr{V}$. We claim that $\mathscr{V}$ forms a pseudovariety iff $\mathscr{V} = \mathscr{V}[E]$ for some E.

The "if" direction is an easy verification. For the "only if" direction, let $\mathscr{V}$ be a pseudovariety of $\mathbf{T}$-algebras, and let $\varphi_{\mathscr{V}} = (\varphi_{\Sigma}^{\mathscr{V}} : \widehat{\mathbf{T}}\Sigma \twoheadrightarrow \hat{T}_{\mathscr{V}}\Sigma)_{\varSigma\in\mathbb{A}}$ be the corresponding profinite theory, see Lemma E.18. We claim that $\mathscr{V} = \mathscr{V}[E[\mathscr{V}]]$. The inclusion $\subseteq$ is trivial. To prove $\supseteq$, let $A \in \mathscr{V}[E[\mathscr{V}]]$, i.e. A satisfies every profinite inequation over $\mathbb{A}$ that all algebras in $\mathscr{V}$ satisfy. Since A is $\mathbb{A}$-generated, there is a surjective $\mathbf{T}$-homomorphism $e : T\Sigma \twoheadrightarrow A$ with $\varSigma \in \mathbb{A}$. By the definition of $\varphi_{\mathscr{V}}$, any profinite inequation $u \leq v$ over $\varSigma$ with $\varphi_{\Sigma}^{\mathscr{V}}(u) \leq \varphi_{\Sigma}^{\mathscr{V}}(v)$ is satisfied by every algebra in $\mathscr{V}$ and thus also by A. Hence $e^{+}(u) \leq e^{+}(v)$. The homomorphism theorem then shows that e^{+} factors through $\varphi_{\varSigma}^{\mathscr{V}}$. In particular, A is a quotient of $\hat{T}_{\mathscr{V}}\Sigma$, which implies that A lies in $\mathscr{V}$ by Lemma E.19. □

F Details for Section 5

F.1 Proof of Proposition 5.4

Let $L\colon T\Sigma \to O_{\mathscr{D}}$ be a recognizable language. By Theorem 3.3 there exists a morphism $\hat{L}\colon \hat{T}\Sigma \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$ with $L = V\hat{L} \cdot \iota_{\Sigma}$.

(a) Let $u\colon (T\Sigma)_s \to (T\Sigma)_{s'}$ in $\mathbb{U}_{\varSigma}$, and take its continuous extension $\hat{u}$, see Lemma E.6. Then we have the commutative diagrams below (where $t \neq s$ in the right-hand diagram).

$$\begin{tikzcd}
(T\Sigma)_s \arrow[r, "u"] \arrow[d, "\iota_{\Sigma}"'] \arrow[rr, bend left, "u^{-1}L"] & (T\Sigma)_{s'} \arrow[r, "L"] \arrow[d, "\iota_{\Sigma}"] & O_{\mathscr{D}} \\
V(\hat{T}\Sigma)_s \arrow[r, "V\hat{u}"'] & V(\hat{T}\Sigma)_{s'} \arrow[ur, "V\hat{L}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(T\Sigma)_t \arrow[r, "\iota_{\Sigma}"'] \arrow[rr, bend left, "u^{-1}L"] & V(\hat{T}\Sigma)_t \arrow[r, "V\perp"'] & O_{\mathscr{D}}
\end{tikzcd}$$

This shows that $u^{-1}L$ corresponds to the morphism $\hat{T}\Sigma \to O_{\mathscr{D}}$ in $\widehat{\mathscr{D}}^S$ being $\hat{L}\cdot\hat{u}$ in sort s and $\perp$ in every other sort. By Theorem 3.3, $u^{-1}L$ is recognizable.

(b) Let $g\colon \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$ be a $\mathbf{T}$-homomorphism and $\hat{g}$ its continuous extension, see Lemma C.7. Then we have the commutative diagrams below.

$$\begin{tikzcd}
T\mathbb{\Delta} \arrow[r, "g"] \arrow[d, "\iota_{\mathbb{\Delta}}"'] & T\mathbb{\Sigma} \arrow[r, "L"] \arrow[d, "\iota_{\mathbb{\Sigma}}"] & O_{\mathscr{D}} \\
V\hat{T}\mathbb{\Delta} \arrow[r, "V\hat{g}"'] & V\hat{T}\mathbb{\Sigma} \arrow[ur, "V\hat{L}"'] &
\end{tikzcd}$$

This shows that $g^{-1}L = L \cdot g$ corresponds to $\hat{L} \cdot \hat{g} : \hat{T}\mathbb{\Sigma} \to O_{\mathscr{D}}$, whence $g^{-1}L$ is recognizable by Theorem 3.3. □

F.2 Details for Remark 5.5

Remark F.1. 1. For each $\Sigma \in \mathbf{Set}_f^S$ and each sort s we have

$$\begin{aligned}
|P(\hat{T}\mathbb{\Sigma})_s| &\cong \mathscr{C}(\mathbb{1}, P(\hat{T}\mathbb{\Sigma})_s) \\
&\cong \widehat{\mathscr{D}}((\hat{T}\mathbb{\Sigma})_s, O_{\mathscr{D}}) \\
&\cong \{\, (T\mathbb{\Sigma})_s \xrightarrow{L_s} O_{\mathscr{D}} : L \in \mathsf{Reg}(\Sigma) \,\}.
\end{aligned}$$

The last bijection is given by $\hat{f} \mapsto V\hat{f} \cdot \iota_{\mathbb{\Sigma}}$. Indeed, observe that for each recognizable language $L\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$, the language $L'\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ with $L'_s = L_s$ and $L_t = V\bot \cdot \iota_{\mathbb{\Sigma}}$ for $t \neq s$ is also recognizable (by the same $\mathbf{T}$-homomorphism, using the naturality of $\bot$). From this and Theorem 3.3 the bijection immediately follows.

Thus, from now on we assume that $P(\hat{T}\mathbb{\Sigma})_s$ is carried by the set $\{\, L_s : L \in \mathsf{Rec}(\Sigma) \,\}$. With this identification, the isomorphism $\mathsf{Rec}(\Sigma) \cong \prod_s P(\hat{T}\mathbb{\Sigma})_s$ of Remark 3.4 maps a recognizable language $L\colon T\mathbb{\Sigma} \to O_{\mathscr{D}}$ to the tuple $((T\mathbb{\Sigma})_s \xrightarrow{L_s} O_{\mathscr{D}})_{s \in S}$ of its components.

2. For any subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ and any sort s, let $m_s\colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\mathbb{\Sigma})_s$ be the subobject of $P(\hat{T}\mathbb{\Sigma})_s$ making the following diagram commute:

$$\begin{tikzcd}
W_\Sigma \arrow[r, tail, "\subseteq"] \arrow[dd, two heads, "e_s"'] & \mathsf{Rec}(\Sigma) \arrow[d, "\cong"] \\
& \prod_s P(\hat{T}\mathbb{\Sigma})_s \arrow[d, "\pi_s"] \\
(W'_\Sigma)_s \arrow[r, tail, "m_s"'] & P(\hat{T}\mathbb{\Sigma})_s
\end{tikzcd}$$

By point 1 above, $(W'_\Sigma)_s$ is (up to isomorphism) carried by

$$\{\, (T\mathbb{\Sigma})_s \xrightarrow{L_s} O_{\mathscr{D}}) : L \in W_\Sigma \,\}.$$

For $e = \langle e_s \rangle_{s\in S}$ we get the following commutative square:

$$\begin{tikzcd}
W_\Sigma \arrow[r, tail, "\subseteq"] \arrow[d, "e"'] & \mathsf{Rec}(\Sigma) \arrow[d, "\cong"] \\
\prod_s (W'_\Sigma)_s \arrow[r, tail, "\prod_s m_s"'] & \prod_s P(\hat{T}\mathbb{\Sigma})_s
\end{tikzcd}$$

Clearly e is monic. The subobject W_Σ is called *admissible* if e is also surjective (i.e. an isomorphism), cf. Remark 5.5. This means precisely that W_Σ is closed under *diagonals*: for any S-indexed family L^s $(s \in S)$ of languages in W_Σ, the *diagonal language* $L^*\colon T\mathbb{\Sigma} \to O_\mathscr{D}$ with $L^*_s = L^s_s$ lies in W_Σ.

3. Every subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ contains the “empty language”, i.e. the language with $V\bot \cdot \iota_\mathbb{\Sigma}\colon (T\mathbb{\Sigma})_t \to O_\mathscr{D}$ in each sort t. Indeed, by the definition of $\bot$ (the dual of the natural transformation choosing a constant, see Remark 5.1) this language is precisely the constant in $\mathsf{Rec}(\Sigma) \cong \prod_s P(\hat{T}\mathbb{\Sigma})_s$, and every subobject of $\mathsf{Rec}(\Sigma)$ contains the constant.

Suppose that $\mathbb{U}_\Sigma$ contains all identity morphisms, and let $\mathscr{C}$ be one of the varieties of Example 2.3. We claim that any subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ closed under derivatives is admissible, i.e. closed under diagonals, see Remark F.1.2. Thus suppose that L^s $(s \in S)$ is an S-indexed family in W_Σ. Since W_Σ is closed under derivatives and $\mathbb{U}_\Sigma$ contains all identity morphisms, the language $(id_{(T\mathbb{\Sigma})_s})^{-1}L^s$ lies in W_Σ for each s. Recall that $\bot$ has been chosen as the zero map, see Remark 5.1. Therefore this derivative agrees with L^s in sort s, and is empty in all other sorts. Finally, observe that for $\mathscr{C} = \mathbf{BA}$, $\mathbf{DL}_{01}$, $\mathbf{JSL}_0$, the set W_Σ is closed under union, since $\mathsf{Rec}(\Sigma) \rightarrowtail \prod_s O_\mathscr{C}^{|T\mathbb{\Sigma}|_s}$ by Remark 3.4. Thus the diagonal language $L^* = \bigcup_s (id_{(T\mathbb{\Sigma})_s})^{-1}L^s$ lies in W_Σ. Analogously for $\mathscr{C} = \mathbf{Vec}_K$ where W_Σ, being a subspace of $\mathsf{Rec}(\Sigma)$, is closed under taking sums of languages. ☐

F.3 Proof of Theorem 5.7

Lemma F.2. *Let $\mathbb{U}_\Sigma$ be a unary presentation of $\mathbf{T}$ over Σ, and let $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ be an admissible subobject of $\mathsf{Rec}(\Sigma)$, represented by subobjects $m_t\colon (W'_\Sigma)_t \rightarrowtail P(\hat{T}\mathbb{\Sigma})_t$ $(t \in S)$. Let $u\colon (T\mathbb{\Sigma})_s \to (T\mathbb{\Sigma})_{s'}$ in $\mathbb{U}_\Sigma$ and $\hat{u} : (\hat{T}\mathbb{\Sigma})_s \to (\hat{T}\mathbb{\Sigma})_{s'}$ its continuous extension, see Lemma E.6. Then the following statements are equivalent:*

(i) $u^{-1}L \in W_\Sigma$ for all $L \in W_\Sigma$.
(ii) There exists a morphism u' making the following square commute:

$$\begin{tikzcd}
(W'_\Sigma)_{s'} \arrow[r, "u'"] \arrow[d, tail, "m_{s'}"'] & (W'_\Sigma)_s \arrow[d, tail, "m_s"] \\
P(\hat{T}\mathbb{\Sigma})_{s'} \arrow[r, "P\hat{u}"'] & P(\hat{T}\mathbb{\Sigma})_s.
\end{tikzcd} \tag{F.1}$$

In particular, W_Σ is a local variety (w.r.t. $\mathbb{U}_\Sigma$) iff a morphism u' with (F.1) *exists for every $u \in \mathbb{U}_\Sigma$.*

Proof. Recall from Remark F.1 that $P(\hat{T}\mathbb{\Sigma})_t$ is, up to isomorphism, carried by the set $\{\, L_t : L \in \mathsf{Rec}(\Sigma)\,\}$, and $(W'_\Sigma)_t$ by the subset $\{\, L_t : L \in W_\Sigma\,\}$. From the definition of $\hat{u}$ it follows that $P\hat{u}$ takes an element $L_{s'}$ of $P(\hat{T}\mathbb{\Sigma})_{s'}$ to $L_{s'} \cdot u$. Thus (ii) is equivalent to the statement that $L_{s'} \cdot u \in (W'_\Sigma)_s$ for all $L \in W_\Sigma$. From this observation the implication (i)⇒(ii) follows immediately, since $(u^{-1}L)_s = L_{s'} \cdot u$.

Conversely, suppose that (ii) holds, and let $L \in W_\Sigma$. By the above argument, we have $L_{s'} \cdot u \in (W'_\Sigma)_s$. Moreover, by Remark F.1.3 the "empty language" with $V\bot \cdot \iota_{\mathbb{\Sigma}}$ in each sort lies in W_Σ. The admissibility of W_Σ (i.e. closure under diagonals, see Remark F.1) thus implies that the language with $L_{s'} \cdot u$ in sort s and $V\bot \cdot \iota_{\mathbb{\Sigma}}$ in all sorts $t \neq s$ lies in W_Σ. But this is precisely the derivative $u^{-1}L$, which proves (ii)⇒(i). □

Lemma F.3. *For $\Sigma, \Delta \in \mathbf{Set}_f^S$ let $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ and $W_\Delta \subseteq \mathsf{Rec}(\Delta)$ be admissible subobjects, represented by $m_s^\Sigma \colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\mathbb{\Sigma})_s$ and $m_s^\Delta \colon (W'_\Delta)_s \rightarrowtail P(\hat{T}\mathbb{\Delta})_s$ ($s \in S$), respectively. Then for any $\mathbf{T}$-homomorphism $g : \mathbf{T}\mathbb{\Delta} \to \mathbf{T}\mathbb{\Sigma}$, the following statements are equivalent:*

(i) $g^{-1}L \in W_\Delta$ for all $L \in W_\Sigma$.

(ii) There is a morphism $g' \colon W'_\Sigma \to W'_\Delta$ in $\mathscr{C}^S$ making the following square commute for any sort s, where $\hat{g} : \widehat{\mathbf{T}}\mathbb{\Delta} \to \widehat{\mathbf{T}}\mathbb{\Sigma}$ is the continuous extension of g (see Lemma C.7).

$$\begin{tikzcd}
(W'_\Sigma)_s \arrow[r, "g'_s"] \arrow[d, tail, "m_s^\Sigma"'] & (W'_\Delta)_s \arrow[d, tail, "m_s^\Delta"] \\
P(\hat{T}\mathbb{\Sigma})_s \arrow[r, "P\hat{g}_s"'] & P(\hat{T}\mathbb{\Delta})_s
\end{tikzcd} \tag{F.2}$$

Proof. Again we use that $P(\hat{T}\mathbb{\Sigma})_s$ can assumed to be carried by the set $\{\, L_s : L \in \mathsf{Rec}(\Sigma)\,\}$, and $(W'_\Sigma)_s$ is the subset $\{L_s : L \in W_\Sigma\}$. Analogously for $P(\hat{T}\mathbb{\Delta})_s$ and $(W'_\Delta)_s$. From the definition of $\hat{g}$ it follows that $P\hat{g}_s$ takes an element L_s of $P(\hat{T}\mathbb{\Sigma})_s$ to $L_s \cdot g_s$. Thus (ii) is equivalent to the statement that $L_s \cdot g_s \in (W'_\Delta)_s$ for all $L \in W_\Sigma$ and all sorts s. From this the implication of (i)⇒(ii) follows immediately, since $(g^{-1}L)_s = L_s \cdot g_s$. Conversely, suppose that (ii) holds, and let $L \in W_\Sigma$. By the above argument, we have $L_s \cdot g_s \in (W'_\Delta)_s$ for all s. By admissability of W_Δ, this implies that $g^{-1}L = L \cdot g$ lies in W_Δ, i.e. (ii)⇒(i) holds. □

Proof (Theorem 5.7). We first prove the local variety theorem. Let $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ be an admissible subobject, represented by a subobject

$$m = \left(\; (W'_\Sigma)_s \xrightarrow{m_s} P(\hat{T}\mathbb{\Sigma})_s \;\right)_{s \in S}$$

in $\mathscr{C}^S$. From Lemma F.2 and E.8, it follows that W_Σ forms a local variety of languages iff the dual quotient

$$\left((\hat{T}\mathbb{\Sigma})_s \xrightarrow{\cong} P^{-1}P(\hat{T}\mathbb{\Sigma})_s \xrightarrow{P^{-1}m_s} P^{-1}(W'_\Sigma)_s \right)_{s\in S}$$

in $\widehat{\mathscr{D}}^S$ carries a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra. Then Proposition 4.3 gives the isomorphism between local varieties of languages over Σ and local pseudovarieties of Σ-generated $\mathbf{T}$-algebras.

For the non-local variety theorem, observe further that by Lemma F.3, a family $(W_\Sigma \subseteq \mathsf{Rec}(\Sigma))_{\Sigma\in\mathbb{A}}$ of local varieties forms a variety of languages (i.e., is closed under preimages) iff the dual family of Σ-generated profinite $\widehat{\mathbf{T}}$-algebras forms a profinite theory. Then Proposition 4.11 gives the isomorphism between varieties of languages and pseudovarieties of $\mathbf{T}$-algebras. ☐

F.4 Details for Remark 5.8

Let C be a family associating to each pair $(\Sigma, \Delta) \in \mathbb{A}^2$ a set $\mathsf{C}(\Delta, \Sigma)$ of $\mathbf{T}$-homomorphisms from $\mathbf{T}\mathbb{\Delta}$ to $\mathbf{T}\mathbb{\Sigma}$. A C-*variety of languages* is given as in Definition 5.6.2, but with g restricted to elements of C. Similarly, a *profinite* C-*theory* is given as in Definition 4.10, but with g again restricted to C. This leads to the following theorem, which for the monad $\mathbf{T} = \mathbf{T}_*$ on **Set** (see Example 2.4.1) is due to Straubing [34].

Theorem F.4 (Straubing Theorem for T-algebras). *The lattice of* C*-varieties of languages is isomorphic to the lattice of profinite* C*-theories.*

Proof. This follows via duality from Lemma F.3 and Lemma E.8, in complete analogy to the proof of Theorem 5.7. ☐

G Details for Section 6

We provide some details for the case of finite words. Let $\mathscr{D}$ be a commutative variety of algebras or ordered algebras. Then $(\mathscr{D}, \otimes, \mathbb{1})$ is a symmetric monoidal closed category w.r.t. the usual tensor product $\otimes$ (representing bimorphisms), see [7]. Moreover, $\mathscr{D}$-monoids, as introduced in Section 6(a), correspond precisely to the monoid objects in $(\mathscr{D}, \otimes, \mathbb{1})$. A *morphism* of $\mathscr{D}$-monoids is a morphism in $\mathscr{D}$ that preserves the monoid structure.

Consider the following diagram of left and right adjoints, where **Mon** and **Mon**$(\mathscr{D})$ are the categories of monoids and $\mathscr{D}$-monoids and U, U' are the forgetful functors. Both the outer and the inner square commute.

$$\begin{tikzcd}
\mathscr{D} \arrow[r, shift right=1.5, "F'"'] \arrow[r, phantom, "\top"] \arrow[d, shift right=1.5, "|-|"'] \arrow[d, phantom, "\vdash"] & \mathbf{Mon}(\mathscr{D}) \arrow[l, shift right=1.5, "U'"'] \arrow[d, shift left=1.5, "U"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Set} \arrow[u, shift right=1.5, "\Psi"'] \arrow[r, shift left=1.5, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbf{Mon} \arrow[u, shift left=1.5, "F"] \arrow[l, shift left=1.5, "|-|"]
\end{tikzcd}$$

The left adjoint F sends a monoid $M = (M, \cdot, e)$ to the $\mathscr{D}$-monoid $FM = (\Psi|M|, \bullet, e)$, where $\bullet : \Psi|M| \times \Psi|M| \to \Psi|M|$ is the unique bimorphism in $\mathscr{D}$ that extends the multiplication $\cdot : |M| \times |M| \to |M|$, and it sends a monoid morphism $h : M \to M'$ to $\Psi|h| : \Psi|M| \to \Psi|M'|$. This implies that the free $\mathscr{D}$-monoid on a free object $\mathbb{\Sigma}$ in $\mathscr{D}$ is given by

$$F'\mathbb{\Sigma} = F'\Psi\Sigma = F\Sigma^* = (\Psi\Sigma^*, \bullet, \varepsilon),$$

where ε is the empty word and $\bullet$ extends the concatentation of words.

Let $\mathbf{T}_M$ be the monad on $\mathscr{D}$ associated to the adjunction $F' \dashv U'$, i.e. constructing free $\mathscr{D}$-monoids on objects of $\mathscr{D}$. Clearly U' is monadic, and thus $\mathbf{Alg}\,(\mathbf{T}_M) \cong \mathbf{Mon}(\mathscr{D})$. A language $L : \mathbf{T}_M\mathbb{\Sigma} = \Psi\Sigma^* \to O_{\mathscr{D}}$ in $\mathscr{D}$ corresponds (via the adjunction $\Psi \dashv |-| : \mathscr{D} \to \mathbf{Set}$) to a function $L' : \Sigma^* \to |O_{\mathscr{D}}|$.

Lemma G.1. *L is $\mathbf{T}_M$-recognizable iff L' is regular, i.e. computed by some finite Moore automaton with output set $|O_{\mathscr{D}}|$.*

Proof. For $\mathscr{D} = \mathbf{Set}$ with $O_{\mathbf{Set}} = \{0, 1\}$, this is the well-known equivalence of regular and monoid-recognizable languages, see e.g. [25]. Now let $\mathscr{D}$ be any commutative variety. If L is recognizable, there exists a $\mathscr{D}$-monoid morphism $h : \Psi\Sigma^* \to D$, where D is finite, and a morphism $p : D \to O_{\mathscr{D}}$ in $\mathscr{D}$ with $L = p \cdot h$. Then h restricts to a monoid morphism

$$h' = (\Sigma^* \rightarrowtail U\Psi\Sigma^* \xrightarrow{Uh} UD)$$

that recognizes L' via $|p|$. Thus L' is regular.

Conversely, suppose that L' is regular. Then L' is monoid-recognizable (in **Set**), so there exists a monoid morphism $h : \Sigma^* \to M$, where M is a finite monoid, and a function $p : M \to |O_{\mathscr{D}}|$ such that $L' = p \cdot h$. Let $p' : \Psi M \to O_{\mathscr{D}}$ in $\mathscr{D}$ be the adjoint transpose of p (via the adjunction $\Psi \dashv |-| : \mathscr{D} \to \mathbf{Set}$). Then $\Psi h : \Psi\Sigma^* \to \Psi M$ is a $\mathscr{D}$-monoid morphism that recognizes L via p', where ΨM is finite since $\mathscr{D}$ is assumed to be a locally finite variety (see Assumptions 2.1). □