

Initial Algebras Without Iteration

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Abstract

The Initial Algebra Theorem by Trnková et al. states, under mild assumptions, that an endofunctor has an initial algebra provided it has a pre-fixed point. The proof crucially depends on transfinitely iterating the functor and in fact shows that, equivalently, the (transfinite) initial-algebra chain stops. We give a constructive proof of the Initial Algebra Theorem that avoids transfinite iteration of the functor. For a given pre-fixed point A of the functor, it uses Pataraia’s theorem to obtain the least fixed point of a monotone function on the partial order formed by all subobjects of A . Thanks to properties of recursive coalgebras, this least fixed point yields an initial algebra. We obtain new results on fixed points and initial algebras in categories enriched over directed-complete partial orders, again without iteration. Using transfinite iteration we equivalently obtain convergence of the initial-algebra chain as an equivalent condition, overall yielding a streamlined version of the original proof.

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1 Introduction

Owing to the importance of initial algebras in theoretical computer science, one naturally seeks results which give the existence of initial algebras in the widest of settings. We can distinguish two different, but related ideas which are commonly used in such results. By Lambek’s Lemma, for every endofunctor $F: \mathcal{A} \rightarrow \mathcal{A}$, every initial algebra $\alpha: FA \rightarrow A$ has a structure α which is an isomorphism. So one might hope to obtain an initial algebra from a *fixed point*, viz. an F -algebra with isomorphic structure. It is sometimes much easier to find a *pre-fixed point*, an object A together with a monomorphism $m: FA \rightarrow A$. The Initial Algebra Theorem by Trnková et al. [28] states that, with inevitable but mild assumptions, any functor F which preserves monomorphisms and has a pre-fixed point also has an initial algebra. The proof uses the second prominent idea in the area: iteration, potentially into the transfinite. Indeed, transfinite iteration of F seems to be an essential feature of the proof.

The purpose of this paper is to prove the Initial Algebra Theorem in as wide a setting as possible with no use of iteration whatsoever. Moreover, the side conditions are mild: they apply, e.g. to the categories of complete metric spaces and directed-complete partial



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orders with a least element (shortly, dcpo with bottom). To situate our method in a larger context, recall that some fixed point theorems are proved with iteration, and some without. On the iterative side, we find Kleene’s Theorem: continuous functions on ω -cpo with a least element have least fixed points obtained by iteration in countably many steps; and Zermelo’s Theorem: monotone functions on chain-complete posets with a least element have least fixed points, using a transfinite iteration. On the non-iterative side, we have the Knaster-Tarski Theorem: monotone functions on complete lattices have both least and greatest fixed points, obtained by a direct definition without iteration. A relatively new result is Pataia’s Theorem: monotone functions on dcpos with bottom have a least fixed point. The latter two theorems are ordinal-free and indeed constructive.

The initial algebra for a functor F can often be constructed by iterating the functor, starting with the initial object 0 and obtaining a transfinite chain $0 \rightarrow F0 \rightarrow FF0 \cdots$ (Definition 6.2). The reason why fixed point theorems are useful for the proof of the Initial Algebra Theorem is that in every category \mathcal{A} , the collection $\mathbf{Sub}(A)$ of subobjects of a given object A is a partial order, and the iteration of F can be reflected by a particular monotone function $f: \mathbf{Sub}(A) \rightarrow \mathbf{Sub}(A)$ when $\alpha: FA \rightarrow A$ is a pre-fixed point; it takes a subobject $u: B \rightarrow A$ to $\alpha \cdot Fu$. If $\mathbf{Sub}(A)$ is sufficiently complete, then f has a least fixed point, and we show that this yields an initial algebra for F .

In order to make this step it is important for us that joins in $\mathbf{Sub}(A)$ are given by colimits in \mathcal{A} . Therefore Pataia’s Theorem is the best choice as a basis for the move from the least fixed point of f to the initial F -algebra. The reasons are that (a) it balances the weak assumption of monotonicity on f with the comparatively weak directed-completeness of the subobject lattice; and (b) its use yields an ordinal-free proof (in contrast to using Zermelo’s Theorem, for which (a) is also the case). In fact, we present many examples of categories where directed joins of subobjects are given by colimits, while this is usually not the case for arbitrary joins, rendering the Knaster-Tarski Theorem a bad choice for us.

We start our exposition in Section 2.1 with a review of Pataia’s Theorem and also its non-constructive precursor, Zermelo’s Theorem which we use later in Section 6. The second ingredient for the proof of our main result are recursive coalgebras, which we tersely review in Section 2.2. We use the fact that a recursive coalgebra which is a fixed point already is an initial algebra. Section 3 discusses the property that joins of subobjects are given by colimits. We make the technical notion of *smoothness* parametric in a class \mathcal{M} of monomorphisms (representing subobjects), and we prove our results for a smooth class \mathcal{M} .

Our main result is the new proof of the Initial Algebra Theorem in Section 4. We apply it in Section 5 to the category \mathbf{DCPO}_\perp of dcpos with bottom. The class of all embeddings is smooth. We derive a new result: if an endofunctor preserves embeddings and has a fixed point, then it has an initial algebra which coincides with the terminal coalgebra.

Finally, Section 6 rounds off our paper by providing the original Initial Algebra Theorem, which features the initial-algebra chain obtained by transfinite iteration. Although our proof has precisely the same mathematical content as the original one, it is slightly streamlined in that it appeals to Zermelo’s Theorem rather than unfolding its proof.

Related work. Independently and at the same time, Pitts and Steenkamp [22] have obtained a result on the existence of initial algebras, which makes use of *sized functors* and is formalizable in Agda. In effect, they show that a form of iteration using sized functors is sufficient to obtain initial algebras. Our work, while constructive, is not aimed at formalization, and, as previously mentioned avoids iteration.

2 Preliminaries

We assume that readers are familiar with standard notions from the theory of algebras and coalgebras for an endofunctor F . We denote an initial algebra for F , provided it exists, by

$$\iota: F(\mu F) \rightarrow \mu F.$$

Recall that Lambek's Lemma [17] states that its structure ι is an isomorphism. This means that μF is a *fixed point* of F , viz. an object $A \cong FA$.

2.1 Fixed Point Theorems

In this subsection we present preliminaries on fixed point theorems for ordered structures. The most well-known such results are, of course, what is nowadays called Kleene's fixed point theorem and the Knaster-Tarski fixed point theorem. The former is for ω -cpos, partial orders with joins of ω -chains, with a least element (*bottom*, for short). Kleene's Theorem states that every endofunction which is ω -continuous, that is preserving joins of ω -chains, on an ω -cpo has a least fixed point. The Knaster-Tarski Theorem [16, 24], makes stronger assumptions on the poset but relaxes the condition on the endofunction. In its most general form it states that a monotone endofunction f on a complete lattice P has a least and greatest fixed point. Moreover, the fixed points of f form a complete lattice again.

Here we are interested in fixed point theorems that still work for arbitrary monotone functions but make do with weaker completeness assumptions on the poset P . One such result pertains to chain-complete posets. It should be attributed to Zermelo, since the mathematical content of the result appears in his 1904 paper [31] proving the Wellordering Theorem.

An *i-chain* in a poset P for an ordinal number i is a sequence $(x_j)_{j < i}$ of elements of P with $x_j \leq x_k$ for all $j \leq k < i$. The poset P is said to be *chain-complete* if every i -chain in it has a join. In particular, P has a least element \perp (take $i = 0$).

Let $f: P \rightarrow P$ be a monotone map on the chain-complete poset P . Then we can define an ordinal-indexed sequence $f^i(\perp)$ by the following transfinite recursion:

$$f^0(\perp) = \perp, \quad f^{j+1}(\perp) = f(f^j(\perp)), \quad \text{and} \quad f^j(\perp) = \bigvee_{i < j} f^i(\perp) \quad \text{for limit ordinals } j. \quad (1)$$

It is easy to verify that this is a chain in P .

► **Theorem 2.1** (Zermelo). *Let P be a chain-complete poset. Every monotone map $f: P \rightarrow P$ has a least fixed point μf . Moreover, for some ordinal i we have $\mu f = f^i(\perp)$.*

Proof. Take i to be any ordinal larger than $|P|$, the cardinality of the set P . For this i , there must be some $j < i$ such that $f^j(\perp) = f^{j+1}(\perp)$. Indeed, this follows from Hartogs' Lemma [14], stating that for every set P there exists an ordinal i such that there is no injection $i \rightarrow P$. Thus, $f^j(\perp)$ is a fixed point of f . Let $f(x) = x$. An easy transfinite induction shows that $f^i(\perp) \leq x$ for all i . Hence, $f^j(\perp)$ is the least fixed point of f . ◀

There are also variations on Theorem 2.1, such as the result often called the Bourbaki-Witt Theorem [6, 30]; this states that every inflationary endo-map on a chain-complete poset has a fixed point above every element. (A map $f: P \rightarrow P$ is *inflationary*, if $x \leq f(x)$ for every $x \in P$.)

Theorem 2.1 is not constructive. Our proof relied on Hartogs' Lemma, which in turn builds on the standard theory of ordinals. That theory uses classical reasoning. A related point:

some prominent results depending on ordinals are known to be unavailable in constructive set theory (see [5]). For many of the end results, there is an alternative, Pataraia's Theorem [21], proved without iteration and without ordinals (see Theorem 2.4). This result is at the heart of this paper. It uses *dcpos* in lieu of chain-complete posets.

Pataraia sadly never published his result in written form. But it has appeared e.g. in work by Escardó [11], Goubault-Larrecq [13], Bauer and Lumsdane [5] based on a preprint by Dacar [9], and Taylor [27]. We present a proof based on Martin's presentation [19].

First recall that a *directed subset* of a poset P is a non-empty subset $D \subseteq P$ such that every finite subset of D has an upper bound in D . The poset P is called a *dcpo with bottom* if it has a least element and every directed subset $D \subseteq P$ has a join. Note that by Markowsky's Theorem [18], a poset is chain-complete iff it is a dcpo.

► **Remark 2.2.**

1. Observe that the set of all maps on a poset P form a poset using the point-wise order: $f \leq g$ if for every $x \in P$ we have $f(x) \leq g(x)$.
2. Hence, $f: P \rightarrow P$ is inflationary iff $\text{id}_P \leq f$, where id_P is the identity function on P .
3. Function composition is *left-monotone*: we clearly have $f \cdot h \leq g \cdot h$ whenever $f \leq g$. Right-monotonicity additionally requires that the fixed argument be a monotone map: we have $f \cdot g \leq f \cdot h$ for every $g \leq h$ whenever f is monotone.
4. A monoid $(M, \cdot, 1)$ is *partially ordered* if M carries a partial order such that multiplication is monotone: $a \leq b$ and $a' \leq b'$ implies $a \cdot a' \leq b \cdot b'$. It is *directed complete* if it is a dcpo. An element $z \in M$ is a *zero* if $z \cdot m = z = m \cdot z$ for every $m \in M$.

► **Theorem 2.3** [19, Thm. 1]. *Every directed complete monoid $(M, \cdot, 1)$ whose bottom is the unit 1 has a top element which is a zero.*

Proof. The set M itself is directed: for $m, n \in M$ we see that $m \cdot n$ is an upper bound since

$$m = m \cdot 1 \leq m \cdot n \geq n \cdot 1 = n,$$

using that 1 is the bottom and multiplication is monotone. Thus M has a top element $\top = \bigvee M$. We have $\top = \top \cdot 1 \leq \top \cdot m$ for every $m \in M$, and clearly $\top \cdot m \leq \top$. Thus, $\top \cdot m = \top$ and, similarly $m \cdot \top = \top$, whence \top is a zero. ◀

► **Theorem 2.4** (Pataraia's Theorem). *Let P be a dcpo with bottom. Then every monotone map on P has a least fixed point.*

Proof.

1. Let M the set of all monotone inflationary maps on P . This is a monoid under function composition, with the unit id_P . Furthermore, M is a dcpo with bottom. Indeed, the order is the pointwise order from Remark 2.2, the least element is id_P , and directed joins are computed pointwise in P . Function composition is monotone (in both arguments) by Remark 2.2.3. By Theorem 2.3, M therefore has a top element $t: P \rightarrow P$ which is a zero.
2. Let $f: P \rightarrow P$ be inflationary and monotone. Then $f \in M$ and therefore $f \cdot t = t$. This means that for every $x \in P$, $f(t(x)) = t(x)$, whence $t(x)$ is a fixed point of f .
3. Now let $f: P \rightarrow P$ be just monotone. Let \mathcal{S} be the collection of all subsets S of P which contain \perp , are closed under f , and under joins of directed subsets. (In more detail, we require that if $s \in S$, then $f(s) \in S$; and if $X \subseteq S$ is directed, $\bigvee X \in S$.) Clearly, \mathcal{S} is closed under arbitrary intersections. Let $T = \bigcap \mathcal{S}$.
The set of all post-fixed points $x \leq f(x)$ belongs to \mathcal{S} . Indeed, $\perp \leq f(\perp)$, and $f(x) \leq f(f(x))$ whenever $x \leq f(x)$. Moreover, a join $p = \bigvee D$ of a directed set D of post-fixed

points of f is post-fixed point: p satisfies $d \leq f(d) \leq f(p)$ for every $d \in D$ due to the monotonicity of f ; thus, $p \leq f(p)$. By the minimality of T , we therefore know that T consists of post-fixed points of f . Thus, f restricts to a function $f: T \rightarrow T$. That restriction is inflationary (and monotone, of course) and therefore has a fixed point p by item 2.

4. We show that p is a least fixed point of $f: P \rightarrow P$. Suppose that x is any fixed point. The set $L = \{y \in P : y \leq x\}$ belongs to \mathcal{S} . Therefore $T \subseteq L$, which implies $p \leq x$. ◀

► **Corollary 2.5.** *The collection of all fixed points of a monotone map on a dcpo with bottom forms a sub-dcpo.*

This is analogous to fixed points of a monotone map on a complete lattice forming a complete lattice again, see Tarski [24].

Proof. Let f be monotone on the dcpo with bottom P . Put $S = \{x \in P : x = f(x)\}$. Suppose that $D \subseteq S$ be a directed subset, and let $w = \bigvee D$ be its join in P . Then we have $x = f(x) \leq f(w)$ for every $x \in S$ since f is monotone. Therefore $w \leq f(w)$ since w is the join of S . We now see that f restricts to $W = \{y \in P, w \leq y\}$, the set of all upper bounds of D in P : for every $y \in W$ we have $w \leq f(w) \leq f(y)$, which shows that $f(y) \in W$. Moreover, W is clearly a dcpo: it has least element w , and the join of every directed set of upper bounds of D is an upper bound, too. By Theorem 2.4, the restriction of f to W has a least fixed point p , say. In other words, p is the least fixed point of f among the upper bounds of D in P , and therefore it is the desired join of D in S . ◀

Here is our statement of a principle which we shall use later as a key step in our main result. It also appears in work by Escardó [11, Thm. 2.2] and Taylor [27].

► **Corollary 2.6 (Pataria Induction Principle).** *Let P be a dcpo with bottom. If $f: P \rightarrow P$ is monotone, then μf belongs to every subset $S \subseteq P$ which contains \perp and is closed under f and under directed joins.*

This follows from the proof of Theorem 2.4: items 3 and 4 show that $\mu f \in S$.

We apply the above principle to prove the following result that we will use in Section 5. A monotone function f on a dcpo D with bottom is *continuous* if it preserves directed joins, and *strict* if $f(\perp) = \perp$.

► **Lemma 2.7.** *Let P, Q be dcpos with bottom and let $f: P \rightarrow P$ and $g: Q \rightarrow Q$ be monotone. For every strict continuous map $h: P \rightarrow Q$ such that $g \cdot h = h \cdot f$ we have $h(\mu f) = \mu g$.*

Proof. First, $h(\mu f)$ is a fixed point of g : we have $g(h(\mu f)) = h(f(\mu f)) = h(\mu f)$. Therefore $\mu g \leq h(\mu f)$. For the reverse, let $S = \{x \in P : h(x) \leq \mu g\}$. Since h is strict, we see that $\perp \in S$. Moreover, S is closed under f , for if $x \in S$ we obtain $h(f(x)) = g(h(x)) \leq g(\mu g) = \mu g$ using monotonicity of g in the second step. Finally, S is closed under directed joins: if $D \subseteq S$ is a directed set we obtain $h(\bigvee D) = \bigvee_{x \in D} h(x) \leq \bigvee_{x \in D} \mu g = \mu g$, whence $\bigvee D$ lies in S . Thus, by Corollary 2.6, $\mu f \in S$, which means that $h(\mu f) \leq \mu g$. ◀

2.2 Recursive Coalgebras

A crucial ingredient for our new proof of the Initial Algebra Theorem are recursive coalgebras. They are closely connected to well-founded coalgebras and hence to the categorical formulation of well-founded induction. In his work on categorical set theory, Osius [20] first studied the notions of well-founded and recursive coalgebras (for the power-set functor on sets and, more generally, the power-object functor on an elementary topos). He defined recursive coalgebras

6:6 Initial Algebras Without Iteration

as those coalgebras $\alpha: A \rightarrow \mathcal{P}A$ which have a unique coalgebra-to-algebra homomorphism into every algebra (see Definition 2.8).

Taylor [25–27] considered recursive coalgebras for a general endofunctor under the name ‘coalgebras obeying the recursion scheme’, and proved the General Recursion Theorem that all well-founded coalgebras are recursive for more general endofunctors; a new proof with fewer assumptions appears in recent work [2]. Recursive coalgebras were also investigated by Eppendahl [10], who called them algebra-initial coalgebras.

Capretta, Uustalu, and Vene [8] studied recursive coalgebras, and they showed how to construct new ones from given ones by using comonads. They also explained nicely how recursive coalgebras allow for the semantic treatment of recursive divide-and-conquer programs. Jeannin et al. [15] proved the general recursion theorem for polynomial functors on the category of many-sorted sets; they also provided many interesting examples of recursive coalgebras arising in programming.

In this section we will just recall the definition and a few basic results on recursive coalgebras which we will need for our proof of the initial algebra theorem.

► **Definition 2.8.** A coalgebra $\gamma: C \rightarrow FC$ is *recursive* if for every algebra $\alpha: FA \rightarrow A$ there exists a unique coalgebra-to-algebra morphism $h: C \rightarrow A$, i.e. a unique morphism h such that the square below commutes:

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ \gamma \downarrow & & \uparrow \alpha \\ FC & \xrightarrow{Fh} & FA \end{array} \quad (2)$$

Recursive coalgebras are regarded as a full subcategory of the category $\mathbf{Coalg} F$ of all coalgebras for the functor F .

► **Definition 2.9.** A *fixed-point* of an endofunctor is an object C together with an isomorphism $C \cong FC$. We consider C both as an algebra and a coalgebra for F .

► **Remark 2.10** [8, Prop. 7]. Every recursive fixed point is an initial algebra: for a coalgebra (C, γ) with γ invertible, the coalgebra-to-algebra morphisms from (C, γ) to an algebra (A, α) are the same as the algebra homomorphisms from (C, γ^{-1}) to (A, α) .

► **Proposition 2.11** [8, Prop. 6]. *If (C, γ) is a recursive coalgebra, then so is $(FC, F\gamma)$.*

Proof. Let (A, α) be an algebra and denote by $h: C \rightarrow A$ the unique coalgebra-to-algebra morphism. We will show that

$$g = (FC \xrightarrow{Fh} FA \xrightarrow{\alpha} A)$$

is the unique coalgebra-to-algebra morphism from $(FC, F\gamma)$ to (A, α) . First, diagram (2) for g commutes as can be seen on the left below:

$$\begin{array}{ccc} & \xrightarrow{g} & \\ \begin{array}{ccc} FC & \xrightarrow{Fh} & FA \xrightarrow{\alpha} A \\ F\gamma \downarrow & & \uparrow F\alpha \\ FFC & \xrightarrow{FFh} & FFA \xrightarrow{F\alpha} FA \end{array} & & \begin{array}{ccc} & \xrightarrow{k \cdot \gamma} & \\ \begin{array}{ccc} C & \xrightarrow{\gamma} & FC \xrightarrow{k} A \\ \gamma \downarrow & & \downarrow F\gamma \\ FC & \xrightarrow{F\gamma} & FFC \xrightarrow{Fk} FA \end{array} & & \end{array} \\ & \xrightarrow{Fg} & \end{array}$$

To see that g is unique, suppose that $k: FC \rightarrow A$ is a coalgebra-to-algebra morphism from $(FC, F\gamma)$ to (A, α) . Then $k \cdot \gamma: C \rightarrow A$ is one from (C, γ) to (A, α) . This is shown by the diagram on the right above. Thus, we have $h = k \cdot \gamma$, and we conclude that

$$g = \alpha \cdot Fh = \alpha \cdot Fk \cdot F\gamma = k,$$

where the last equation holds since k is a coalgebra-to-algebra morphism. ◀

► **Corollary 2.12.** *If a terminal recursive F -coalgebra exists, it is a fixed point of F .*

Indeed, the proof is the same as that for Lambek’s Lemma, using Proposition 2.11 to see that for a terminal recursive coalgebra (T, τ) , the coalgebra $(FT, F\tau)$ is recursive, too: the unique coalgebra homomorphism $h: (FT, F\tau) \rightarrow (T\tau)$ satisfies $h \cdot \tau = \text{id}_T$ since $\tau: (T, \tau) \rightarrow (FT, F\tau)$ is a coalgebra homomorphism, and finally, $\tau \cdot h = Fh \cdot F\tau = F \text{id}_T = \text{id}_{FT}$.

► **Theorem 2.13** [8, Prop. 7]. *The terminal recursive coalgebra is precisely the same as the initial algebra.*

In more detail, let $F: \mathcal{A} \rightarrow \mathcal{A}$ be an endofunctor. Then we have:

1. If (T, τ) is a terminal recursive coalgebra, then (T, τ^{-1}) is a initial algebra.
2. If $(\mu F, \iota)$ is an initial algebra, then $(\mu F, \iota^{-1})$ is a terminal recursive coalgebra.

Proof.

1. By Corollary 2.12, we know that τ is an isomorphism. By Remark 2.10, (T, τ^{-1}) is an initial algebra.
2. The coalgebra $(\mu F, \iota^{-1})$ is clearly recursive. It remains to verify its terminality. So let (C, γ) be a recursive coalgebra. There is a unique coalgebra-to-algebra morphism from (C, γ) to the algebra $(\mu F, \iota)$ to (A, α) , and this means that there is a unique coalgebra homomorphism $h: (C, \gamma) \rightarrow (\mu F, \iota^{-1})$. ◀

► **Proposition 2.14.** *Every colimit of recursive coalgebras is recursive.*

Proof. We use the fact that the colimits in $\text{Coalg } F$, the category of coalgebras for F , are formed on the level of the underlying category. Suppose that we are given a diagram of recursive coalgebras (C_i, γ_i) , $i \in I$, with a colimit cocone $c_i: (C_i, \gamma_i) \rightarrow (C, \gamma)$ in $\text{Coalg } F$. We prove that (C, γ) is recursive, too. Indeed, given an algebra (A, α) one takes for every i the unique coalgebra-to-algebra morphisms $h_i: (C_i, \gamma_i) \rightarrow (A, \alpha)$. Using unicity one sees that all h_i form a cocone of the diagram formed by all C_i in the underlying category. Therefore, there is a unique morphism $h: C \rightarrow A$ such that $h \cdot c_i = h_i$ holds for all $i \in I$. We now verify that h is the desired unique coalgebra-to-algebra morphism using the following diagram:

$$\begin{array}{ccccc}
 & & h_i & & \\
 & \swarrow & & \searrow & \\
 C_i & \xrightarrow{c_i} & C & \xrightarrow{h} & A \\
 \gamma_i \downarrow & & \downarrow \gamma & & \uparrow \alpha \\
 FC_i & \xrightarrow{Fc_i} & FC & \xrightarrow{Fh} & FA \\
 & \swarrow & Fh_i & \searrow & \\
 & & & &
 \end{array}$$

We know that the upper and lower parts, the left-hand square and the outside commute. Therefore so does the right-hand square when precomposed by every c_i . Since the colimit injections c_i form a jointly epic family, we thus see that the right-hand square commutes if and only if $h \cdot c_i = h_i$ holds for all $i \in I$. ◀

3 Smooth Monomorphisms

As we have just seen in Proposition 2.14, the collection of recursive coalgebras is closed under colimits. In order to apply an order-theoretic fixed point theorem to this collection, or to subcollections of it, we need a connection between colimits and subobjects. We make this connection by using the definition of *smooth class* of monomorphisms in a category.

For an object A of a category \mathcal{A} , a *subobject* is represented by a monomorphism $s: S \rightarrow A$. If s and $t: T \rightarrow A$ are monomorphisms, we write $s \leq t$ if s factorizes through t . If also $t \leq s$ holds, then t and s represent the same subobject; in particular S and T are then isomorphic. Generalizing a bit, let \mathcal{M} be a class of monomorphisms. An \mathcal{M} -*subobject* of A is a subobject represented by a morphism $s: S \rightarrow A$ in \mathcal{M} . If the object A has only a set of subobjects, then we write

$$\text{Sub}_{\mathcal{M}}(A)$$

for the poset of \mathcal{M} -subobjects of A .

If every object A only has a set of \mathcal{M} -subobjects, then \mathcal{A} is called \mathcal{M} -*well-powered*.

► **Definition 3.1.** Let \mathcal{M} be a class of monomorphisms closed under isomorphisms and composition.

1. We say that an object A has *smooth \mathcal{M} -subobjects* provided that $\text{Sub}_{\mathcal{M}}(A)$ is a dcpo with bottom (in particular, not a proper class) where the least element and directed joins are given by colimits of the corresponding diagrams of subobjects.
2. The class \mathcal{M} is *smooth* if every object of \mathcal{A} has smooth \mathcal{M} -subobjects.

Moreover, we say that a category has *smooth monomorphisms* if the class of all monomorphisms is smooth.

► **Remark 3.2.**

1. In more detail, let $D \subseteq \text{Sub}_{\mathcal{M}}(A)$ be a directed set of subobjects represented by $m_i: A_i \rightarrow A$ ($i \in D$). Then D has a join $m: C \rightarrow A$ in $\text{Sub}_{\mathcal{M}}(A)$. Moreover, consider the diagram of objects $(A_i)_{i \in D}$ with connecting morphisms $a_{i,j}: A_i \rightarrow A_j$ for $i \leq j$ in D given by the unique factorizations witnessing $m_i \leq m_j$:

$$\begin{array}{ccc}
 A_i & \xrightarrow{a_{i,j}} & A_j \\
 & \searrow m_i & \swarrow m_j \\
 & & A
 \end{array}$$

(Note that $a_{i,j}$ need not lie in \mathcal{M} .) Then for every $i \in D$ there exists a monomorphism $c_i: A_i \rightarrow C$ with $m \cdot c_i = m_i$, since $m_i \leq m$. The smoothness requirement is that these monomorphisms form a colimit cocone.

2. Requiring that the least subobject in $\text{Sub}_{\mathcal{M}} A$ is given by (the empty) colimit means that \mathcal{A} has an initial object 0 and the unique morphism $0 \rightarrow A$ lies in \mathcal{M} .
3. If \mathcal{M} is a smooth class, then \mathcal{A} is \mathcal{M} -well-powered.

Since the above notion of smoothness is new, we discuss examples at length now. Below we show that in a number of categories the collection of all monomorphisms is smooth, as is the collection of all strong monomorphisms (those having the diagonal fill-in property with respect to epimorphisms). We also present some counterexamples and discuss other classes \mathcal{M} .

Recall the concept of a *locally finitely presentable* (*lfp*, for short) category (e.g. [4]): it is a cocomplete category \mathcal{A} with a set of finitely presentable objects (i.e. their hom-functors

preserve filtered colimits) whose closure under filtered colimits is all of \mathcal{A} . Examples are **Set**, **Pos** (posets and monotone maps), **Gra** (graphs and homomorphisms) and all varieties of finitary algebras such as monoids, vector spaces, rings, etc.

We say that \mathcal{A} has a *simple* initial object 0 if all the morphisms with domain 0 are strong monomorphisms (equivalently, 0 has no proper quotients).

► **Example 3.3.** Both monomorphisms and strong monomorphisms are smooth in every lfp category with a simple initial object 0 [4, Cor. 1.63]. This includes **Set**, **Pos**, **Gra**, monoids and vector spaces. But not rings: in that category the initial object is \mathbb{Z} , the ring of integers, and there are non-monic ring homomorphisms with that domain (e.g. $\mathbb{Z} \rightarrow 1$).

► **Example 3.4.** Let us consider the category DCPO_\perp of dcpos with bottom and continuous maps between them, where a map is *continuous* if it is monotone and preserves directed joins.

1. In Section 5 we prove that the class of all embeddings (Definition 5.1) is smooth. (These play a major role in Smyth and Plotkin’s solution method for recursive domain equations [23].) This example is one of several motivations for our move from the class of all monomorphisms to the more general situation of a class \mathcal{M} in Definition 3.1.
2. In contrast, the class of all monomorphisms is non-smooth in DCPO_\perp . For example, consider the dcpo \mathbb{N}^\top of natural numbers with a top element \top . The subposets $C_n = \{0, \dots, n\} \cup \{\top\}$, $n \in \mathbb{N}$, form an ω -chain in DCPO_\perp . Its colimit is $\mathbb{N}^\top \cup \{\infty\}$ where $n < \infty < \top$ for all $n \in \mathbb{N}$. The cocone of inclusion maps $C_n \hookrightarrow \mathbb{N}^\top$ consists of monomorphisms. However, the factorizing morphism from $\text{colim } C_n$ to \mathbb{N}^\top is not monic, as it merges ∞ and \top .
3. The same example demonstrates that strong monomorphisms are not smooth in DCPO_\perp .

► **Example 3.5.**

1. Let us consider the category **MS** of metric spaces with distances at most 1 and *non-expanding* maps $f: (X, d_X) \rightarrow (Y, d_Y)$ (that is $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$). Although this category is not lfp, both monomorphisms and strong monomorphisms form smooth classes. The proof for strong monomorphisms is easy since the strong (equivalently, extremal) subobjects of a metric space A are represented by its subspaces (with the inherited metric). Given a directed set of subspaces $A_d \subseteq A$ ($d \in D$) their join in $\text{Sub}(A)$ is the subspace $\bigcup_{d \in D} A_d$ and this is also the colimit of the corresponding diagram in **MS**. The somewhat technical proof for monomorphisms is given in the appendix (Lemma A.1).
2. In the full subcategory **CMS** of **MS** given by all complete metric spaces monomorphisms are not smooth. This can be demonstrated as in Example 3.4.2: Let \mathbb{N}^\top be the metric space with distances $d(n, m) = |1/2^{-n} - 1/2^{-m}|$ and $d(n, \top) = 1/2^{-n}$, and consider the ω -chain of spaces C_n where $d(n, \top) = 1$ and other distances are as in \mathbb{N}^\top .
3. In contrast, strong monomorphisms are smooth in **CMS** (see Lemma A.2).

The following equivalent formulation is often used in proofs.

► **Proposition 3.6.** *An object A has smooth \mathcal{M} -subobjects if and only if for every directed diagram D of monomorphisms in \mathcal{A} (not necessarily members of \mathcal{M}), and every cocone $m_i: A_i \rightarrow A$, $i \in D$, of \mathcal{M} -monomorphisms, the following holds:*

1. *the diagram D has a colimit, and*
2. *the factorizing morphism induced by the cocone (m_i) is again an \mathcal{M} -monomorphism.*

Proof. The ‘only if’ direction is obvious. For the ‘if’ direction, suppose we are given a directed set $D \subseteq \text{Sub}(A)$ of \mathcal{M} -subobjects $m_i: A_i \rightarrow A$ for $i \in D$ as in Remark 3.2. By item 1, the ensuing directed diagram of monomorphisms $a_{i,j}: A_i \rightarrow A_j$ has a colimit $c_i: A_i \rightarrow C$,

6:10 Initial Algebras Without Iteration

$i \in D$, and we will prove that this yields the join $\bigvee_{i \in D} m_i$. By item 2, we have a unique \mathcal{M} -monomorphism $m: C \rightarrow A$ such that $m \cdot c_i = m_i$ for all $i \in D$.

Now let $s: S \rightarrow A$ be any \mathcal{M} -subobject with $m_i \leq s$ for all $i \in D$. That is, we have morphisms $s_i: A_i \rightarrow S$ with $s \cdot s_i = m_i$ for all $i \in D$. They form a cocone because for the monomorphism $a_{i,j}: A_i \rightarrow A_j$ witnessing $m_i \leq m_j$ we have

$$s \cdot s_j \cdot a_{i,j} = m_j \cdot a_{i,j} = m_i = s \cdot s_i,$$

whence $s_j \cdot a_{i,j} = s_i$ since s is monic. We therefore obtain a unique $t: C \rightarrow S$ with $t \cdot c_i = s_i$ for all $i \in D$. Consequently, we have

$$s \cdot t \cdot c_i = s \cdot s_i = m_i = m \cdot c_i \quad \text{for all } i \in D.$$

Since the colimit injections c_i form an epic family, we conclude that $s \cdot t = m$, which means that $m \leq s$ in $\text{Sub}_{\mathcal{M}}(A)$, as desired. \blacktriangleleft

► Remark 3.7.

1. Note that the conditions for \mathcal{M} to be smooth are a part of the conditions of Taylor's notion of a *locally complete class of supports* [25, Def. 6.1. & 6.3] (see also [27, Assumption 4.18]).
2. Smoothness previously appeared for joins and colimit of chains in lieu of directed sets [2]. That formulation is related to the list of conditions for a class of monomorphisms given by Trnková et al. [28]. Note that a class \mathcal{M} of monomorphisms containing the identities and closed under composition can be regarded as the subcategory of \mathcal{A} given by all morphisms in \mathcal{M} . The list of conditions in op. cit. is equivalent to stating that the inclusion functor $\mathcal{M} \hookrightarrow \mathcal{A}$ creates colimits of chains. Requiring that the inclusion creates directed colimits implies that the class \mathcal{M} is smooth. For the converse, we would need to add that for every directed diagram of \mathcal{M} -monomorphisms the colimit cocone consists of \mathcal{M} -monomorphisms.

4 The Initial Algebra Theorem

We are now ready to prove the main result of this paper.

► **Assumption 4.1.** Throughout this section we assume that \mathcal{A} is a category with a class \mathcal{M} of monomorphisms containing all isomorphisms and closed under composition. We say that $F: \mathcal{A} \rightarrow \mathcal{A}$ *preserves* \mathcal{M} if $m \in \mathcal{M}$ implies $Fm \in \mathcal{M}$.

► **Definition 4.2.** An \mathcal{M} -*pre-fixed point* of F is an algebra whose structure $m: FA \rightarrow A$ lies in \mathcal{M} . In the case where \mathcal{M} consists of all monomorphisms we speak of a pre-fixed point.

► **Theorem 4.3 (Initial Algebra Theorem).** *Let $m: FA \rightarrow A$ be an \mathcal{M} -pre-fixed point for an endofunctor preserving \mathcal{M} . If A has smooth \mathcal{M} -subobjects, then F has an initial algebra which is an \mathcal{M} -subalgebra of (A, m) .*

Proof. We have the following endomap

$$f: \text{Sub}_{\mathcal{M}}(A) \rightarrow \text{Sub}_{\mathcal{M}}(A) \quad \text{defined by} \quad f(B \xrightarrow{u} A) = (FB \xrightarrow{Fu} FA \xrightarrow{m} A). \quad (3)$$

It is clearly monotone. We are going to apply Patariaia Induction to it. We take the subset $S \subseteq \text{Sub}_{\mathcal{M}}(A)$ of all $u: B \rightarrow A$ such that $u \leq f(u)$ via some recursive coalgebra $\beta: B \rightarrow FB$. More precisely,

$$S = \{u: B \rightarrow A : u = m \cdot Fu \cdot \beta \text{ for some recursive coalgebra } \beta: B \rightarrow FB\}.$$

Note that if β exists for u , then it is unique. Moreover, $u \in S$ is a coalgebra-to-algebra morphism from (B, β) to (A, m) .

The least subobject $0 \rightarrow A$ is clearly contained in S . Further, S is closed under f since $(FB, F\beta)$ is a recursive coalgebra by Proposition 2.11: for $u \in S$ we have

$$f(u) = m \cdot Fu = m \cdot F(m \cdot Fu \cdot \beta) = m \cdot F(f(u)) \cdot F\beta.$$

We continue with the verification that S is closed under directed joins. Let $D \subseteq S$ be directed. Given $u: B_u \rightarrow A$ in D we write $\beta_u: B_u \rightarrow FB_u$ for the recursive coalgebra witnessing $u \leq f(u)$. We show that these recursive coalgebras form a (then necessarily) directed diagram. To see this, we only need to prove that every morphism $h: B_u \rightarrow B_v$ witnessing $u \leq v$ in D ; i.e. $v \cdot h = u$, is a coalgebra homomorphism. Consider the diagram below:

$$\begin{array}{ccc} B_u & \xrightarrow{\beta_u} & FB_u \\ \downarrow h & & \downarrow Fh \\ B_v & \xrightarrow{\beta_v} & FB_v \\ \downarrow v & & \downarrow Fv \\ A & \xleftarrow{m} & FA \end{array} \begin{array}{l} \\ \\ \\ \\ Fu \end{array}$$

Since the outside, the lower square and the left-hand and right-hand parts commute, we see that the upper square commutes when extended by the monomorphism $m \cdot Fv$. Thus it commutes, proving that h is a coalgebra homomorphism.

Now denote by $v: B \rightarrow A$ the join $\bigvee D$ in $\text{Sub}_{\mathcal{M}}(A)$. Since A has smooth subobjects, B is the colimit of the diagram formed by the $B_u, u \in D$, in \mathcal{A} . Since the forgetful functor $\text{Coalg } F \rightarrow \mathcal{A}$ creates colimits, we have a unique coalgebra structure $\beta: B \rightarrow FB$ such that the colimit injections are coalgebra homomorphisms; moreover (B, β) is colimit of the coalgebras $(B_u, \beta_u), u \in D$. Thus, (B, β) is recursive by Proposition 2.14. Moreover, $v: B \rightarrow A$ is the unique morphism induced by the cocone given by all $u: B_u \rightarrow A$ in D . Since every $u \in S$ is the unique coalgebra-to-algebra morphism from (B_u, β_u) to (A, m) , we know from the proof of Proposition 2.14 that v is the unique coalgebra-to-algebra morphism from (B, β) to (A, m) . Thus, v lies in S .

By Theorem 2.4, f has a least fixed point, and by Corollary 2.6, $\mu f \in S$. Denote this subobject be $u: I \rightarrow A$. Since $u \in S$, there is a recursive coalgebra $\iota: I \rightarrow FI$ such that $u = m \cdot Fu \cdot \iota$. But u and $f(u) = m \cdot Fu$ represent the same subobject of A . So ι is an isomorphism. Thus (I, ι^{-1}) is an initial algebra by Remark 2.10. \blacktriangleleft

► **Corollary 4.4.** *Let \mathcal{A} be a category with a smooth class \mathcal{M} of monomorphisms. Then the following are equivalent for every endofunctor F preserving \mathcal{M} :*

1. *an initial algebra exists,*
2. *a fixed point exists,*
3. *an \mathcal{M} -pre-fixed point exists.*

Moreover, if these hold, then μF is an \mathcal{M} -subalgebra of every \mathcal{M} -pre-fixed point of F .

Indeed, Lambek’s Lemma [17] tells us that 1 implies 2. Clearly, 2 implies 3 since \mathcal{M} contains all isomorphisms. Theorem 4.3 shows that that 3 implies 1, and it also yields our last statement.

► **Corollary 4.5.** *Let \mathcal{A} be an lfp category with a simple initial object. An endofunctor preserving monomorphisms has an initial algebra iff it has a pre-fixed point.*

► **Example 4.6.** We present examples which show, *inter alia*, that neither of the hypotheses in Theorem 4.3 can be left out. In each case \mathcal{M} is the class of all monomorphisms.

1. The assumption that $0 \rightarrow A$ is monic. Let \mathcal{A} be the variety of algebras (A, u, c) with unary operation u and a constant c . Its initial object is $(\mathbb{N}, s, 0)$ with $s(n) = n + 1$, which is not simple. We present an endofunctor having no initial algebra even though it has a fixed point and preserves monomorphisms. Let \mathcal{P}_0 be the non-empty power-set functor. We obtain an analogous endofunctor $\bar{\mathcal{P}}_0$ on \mathcal{A} defined by $\bar{\mathcal{P}}_0(A, u, c) = (\mathcal{P}_0 A, \mathcal{P}_0 u, \{c\})$. It clearly preserves monomorphisms, and the terminal object 1 is a fixed point of $\bar{\mathcal{P}}_0$ (since $\mathcal{P}_0 1 \cong 1$). This is, up to isomorphism, the only fixed point. However, it is not $\mu\bar{\mathcal{P}}_0$ because given an algebra on (A, u, c) with $u(x) \neq x$ for all $x \in A$, no $\bar{\mathcal{P}}_0$ -algebra homomorphism exists from 1 to A .
2. The assumption that $\text{Sub}(A)$ is a set in Definition 3.1.1. Let Ord be the totally ordered class of all ordinals taken as a category. In the opposite category Ord^{op} , all morphisms are monic, so every endofunctor preserves monomorphisms. For the functor F on Ord^{op} given by $F(i) = i + 1$, every object is a pre-fixed point, and there are no fixed points. For each object i , $\text{Sub}(i)$ has all the properties requested in Definition 3.1.1 except that it is a proper class.
3. Preservation of monomorphisms. Here we use the category $\text{Set} \times \text{Set}$ which satisfies all assumptions of Theorem 6.6. We define an endofunctor F by $F(X, Y) = (\emptyset, 1)$ if $X \neq \emptyset$ and $F(X, Y) = (\emptyset, \mathcal{P}Y)$ else. It is defined on morphisms as expected, using \mathcal{P} in the case where $X = \emptyset$. This functor has many pre-fixed points, e.g. $F(1, 1) = (\emptyset, 1) \rightarrow (1, 1)$. But it has no fixed points (thus no initial algebra): first, (\emptyset, Y) and $(\emptyset, \mathcal{P}Y)$ are never isomorphic, by Cantor's Theorem [7]. Second, if $X \neq \emptyset$, then there exists no morphism from (X, Y) to $F(X, Y) = (\emptyset, 1)$.

5 Initial Algebras in DCPO_⊥-enriched Categories

It follows from the seminal paper by Smyth and Plotkin [23] that every locally continuous functor F on a category \mathcal{A} enriched over ω -cpos (i.e. partial orders with a least element and joins of ω -chains) has an initial algebra $(\mu F, \iota)$ which is also a terminal coalgebra by inverting its structure. Local continuity means that the corresponding mappings $\mathcal{A}(A, B) \rightarrow \mathcal{A}(FA, FB)$ preserve (pointwise) directed joins. Here we assume the weaker property that F is locally monotone; for example, the endofunctor assigning to a dcpo its ideal completion is locally monotone, whence preserves embeddings, but not locally continuous. We apply Corollary 4.4 to derive that such an endofunctor has a pre-fixed point given by an embedding iff it has an initial algebra (being also the terminal coalgebra).

► **Definition 5.1.**

1. A category \mathcal{A} is DCPO_⊥-enriched provided that each hom-set is equipped with the structure of a dcpo with bottom, and composition preserves bottom and directed joins: for every morphism f and appropriate directed sets of morphisms g_i ($i \in D$) we have

$$f \cdot \perp = \perp, \quad \perp \cdot f = \perp, \quad f \cdot \bigvee_{i \in D} g_i = \bigvee_{i \in D} f \cdot g_i, \quad \left(\bigvee_{i \in D} g_i \right) \cdot f = \bigvee_{i \in D} g_i \cdot f. \quad (4)$$

2. A functor on \mathcal{A} is *locally monotone* if its restrictions $\mathcal{A}(A, B) \rightarrow \mathcal{A}(FA, FB)$ to the hom-sets are monotone.
3. A morphism $e: A \rightarrow B$ is called an *embedding* if there exists a morphism $\hat{e}: B \rightarrow A$ such that $\hat{e} \cdot e = \text{id}_A$ and $e \cdot \hat{e} \sqsubseteq \text{id}_B$.

It is easy to see that the morphism \hat{e} is unique for e ; it is called its *projection*.

The following result is a slight variation of a result by Smyth and Plotkin for ω -cpos [23]. We include the proof in the appendix for the convenience of the reader.

► **Theorem 5.2.** *Let D be a directed diagram of embeddings in a DCPO_\perp -enriched category. For every cocone $(c_i: D_i \rightarrow C)$ of D , the following are equivalent:*

1. *The cocone (c_i) is a colimit.*
2. *Each c_i is an embedding, the composites $c_i \cdot \widehat{c}_i$ form a directed set in $\mathcal{A}(C, C)$, and*

$$\bigsqcup_i c_i \cdot \widehat{c}_i = \text{id}_C. \tag{5}$$

► **Remark 5.3.** A DCPO_\perp -enriched category \mathcal{A} is \mathcal{M} -well-powered for the class \mathcal{M} of all embeddings. The reason is that, given an object A , a subobject represented by an embedding $e: S \rightarrow A$ is determined by the endomorphism $e \cdot \widehat{e}$ on A . Indeed, let $f: T \rightarrow A$ be an embedding with $e \cdot \widehat{e} = f \cdot \widehat{f}$. Then $e = e \cdot \widehat{e} \cdot e = f \cdot \widehat{f} \cdot e$. Therefore, $e \leq f$ in $\text{Sub}_{\mathcal{M}}(A)$. By symmetry $f \leq e$. Since $\mathcal{A}(A, A)$ is a set, \mathcal{M} -well-poweredness follows.

► **Theorem 5.4.** *Let \mathcal{A} be a DCPO_\perp -enriched category with directed colimits. Then the class of all embeddings is smooth.*

The proof is presented in Section A.3.

► **Corollary 5.5.** *Let \mathcal{A} be a DCPO_\perp -enriched category with directed colimits. For a locally monotone endofunctor F the following are equivalent:*

1. *an initial algebra exists,*
2. *a terminal coalgebra exists,*
3. *a fixed point exists.*

Moreover, if $(\mu F, \iota)$ is an initial algebra, then $(\mu F, \iota^{-1})$ is a terminal coalgebra.

Item 3 can be strengthened to state existence of a pre-fixed point carried by an embedding.

Proof. The dual category \mathcal{A}^{op} is DCPO_\perp -enriched w.r.t. the same order on hom-sets. But the embeddings in \mathcal{A}^{op} are precisely the projections in \mathcal{A} . Every locally monotone endofunctor F on \mathcal{A} clearly preserves embeddings and projections. Thus, the dual functor F^{op} on \mathcal{A}^{op} preserves embeddings. Now $1 \Leftrightarrow 3$ follows from an application of Corollary 4.4 to \mathcal{A} and F , and $2 \Leftrightarrow 3$ is an application to \mathcal{A}^{op} and F^{op} . In each case the class \mathcal{M} consists of all embeddings in \mathcal{A} and \mathcal{A}^{op} , respectively.

Finally, we prove that the initial algebra and terminal coalgebra coincide. Let $\iota: FI \rightarrow I$ be an initial algebra. Then we know that a terminal coalgebra $\tau: T \rightarrow FT$ exists. Moreover, from the last statement in Corollary 4.4 applied to F and its fixed point (T, τ^{-1}) we see that the unique F -algebra homomorphism $e: (I, \iota) \rightarrow (T, \tau^{-1})$ is an embedding. Another application of Corollary 4.4 to F^{op} and its fixed point (I, ι^{-1}) yields that the unique F^{op} -algebra homomorphism $f: \mu F^{\text{op}} = (T, \tau) \rightarrow (I, \iota^{-1})$ is an embedding in \mathcal{A}^{op} . This means that this an F -coalgebra homomorphism $f: (I, \iota^{-1}) \rightarrow (T, \tau)$ which is a projection in \mathcal{A} . By the universal properties of (I, ι) and (T, τ) , $e = f$, and this morphism is both an embedding and a projections, whence an isomorphism. ◀

The requirement of local monotonicity of F can be weakened: the theorem holds for any endofunctor F which fulfils $Ff \sqsubseteq \text{id}_{FA}$ whenever $f \sqsubseteq \text{id}_A$. Indeed, a functor satisfying that property preserves embedding-projection pairs; in categories with split idempotents the converse holds, too [3, Obs. 6.6.5].

We close this section with a proposition on locally monotone functors which gives a version of a result for ω -cpo-enriched categories proved by Freyd [12] for locally continuous functors. He used Kleene's Theorem in lieu of Pataraia's.

► **Proposition 5.6.** *Let \mathcal{A} be a DCPO_\perp -enriched category. If a locally monotone functor F has an initial algebra $(\mu F, \iota)$, then $(\mu F, \iota^{-1})$ is a terminal coalgebra.*

We shall see in the proof that it is enough to assume that composition is left-strict: $\perp \cdot f = \perp$ holds for every morphism f of \mathcal{A} (but $f \cdot \perp = \perp$ in (4) need not hold). This holds in categories typically used in semantics of programming languages, such as the category of *depos* with bottom and (non-strict) continuous maps, where composition is not (right-) strict.

Proof. Let $\iota: FI \rightarrow I$ be an initial algebra. For every coalgebra $\alpha: A \rightarrow FA$, we prove that a unique homomorphism into (I, ι^{-1}) exists.

1. Existence. The endomap g on $\mathcal{A}(A, I)$ given by $h \mapsto \iota \cdot Fh \cdot \alpha$ is monotone since F is locally monotone. Hence, it has a least fixed point $h: A \rightarrow I$ with $\iota^{-1} \cdot h = Fh \cdot \alpha$ by Pataraia's Theorem 2.4. This is a coalgebra homomorphism.
2. Uniqueness. First notice that for $\mathcal{A}(I, I)$ we have an the analogous endomap f given by $k \mapsto \iota \cdot Fk \cdot \iota^{-1}$. Since I is initial, the only fixed point of f is $k = \text{id}_I$. Thus $\text{id}_I = \mu f$. Now suppose that $h': (A, \alpha) \rightarrow (I, \iota^{-1})$ is any coalgebra homomorphism. We know that $\mathcal{A}(h', I): \mathcal{A}(I, I) \rightarrow \mathcal{A}(A, I)$ is a strict continuous map; strictness follows from left-strictness of composition: $\perp_{I, I} \cdot h' = \perp_{A, I}$. We now show that $g \cdot \mathcal{A}(h', I) = \mathcal{A}(h', I) \cdot f$. Indeed, unfolding the definitions, we have for every $k: I \rightarrow I$:

$$\begin{aligned} g \cdot \mathcal{A}(h', I)(k) &= g(k \cdot h') = \iota \cdot F(k \cdot h') \cdot \alpha = \iota \cdot Fk \cdot Fh' \cdot \alpha = \iota \cdot Fk \cdot \iota^{-1} \cdot h' \\ &= f(k) \cdot h' = \mathcal{A}(h', I)(f(k)). \end{aligned}$$

Therefore, by Lemma 2.7, $\mathcal{A}(h', I)(\mu f) = \mu g$, which means that $h' = \text{id}_I \cdot h' = h$. ◀

We leave as an open problem to find an endofunctor on DCPO_\perp which has a fixed point but not an initial algebra.

6 The Initial-Algebra Chain

The proof of Theorem 4.3, relying on Pataraia's Theorem 2.4, is constructive. However, if one admits non-constructive reasoning and ordinals, then we can add another equivalent characterization to Corollary 4.4 in terms of the convergence of the initial-algebra chain, which we now recall.

► **Remark 6.1.**

1. Recall that an ordinal i is the (linearly ordered) set of all ordinals smaller than i . As such it is also a category.
2. By an *i-chain* in a category \mathcal{C} is meant a functor $C: i \rightarrow \mathcal{C}$. It consists of objects C_j for all ordinals $j < i$ and (connecting) morphisms $c_{j, j'}: C_j \rightarrow C_{j'}$ for all pairs $j \leq j' < i$. Analogously, an *Ord-chain* in \mathcal{C} is a functor from the totally ordered class **Ord** of all ordinals to \mathcal{C} . In both cases we will speak of a (transfinite) *chain* whenever confusion is unlikely.
3. A category \mathcal{C} has *colimits of chains* if for every ordinal i a colimit of every i -chain exists in \mathcal{C} . (This does *not* include **Ord**-chains.) In particular, \mathcal{C} has an initial object since the ordinal 0 is the empty set.

► **Definition 6.2** [1]. Let \mathcal{A} be a category with colimits of chains. For an endofunctor F we define the *initial-algebra chain* $W: \text{Ord} \rightarrow \mathcal{A}$. Its objects are denoted by W_i and its

connecting morphisms by $w_{ij}: W_i \rightarrow W_j$, $i \leq j \in \text{Ord}$. They are defined by transfinite recursion as follows

$$\begin{aligned} W_0 &= 0, & W_{j+1} &= FW_j \text{ for all ordinals } j, & W_j &= \text{colim}_{i < j} W_i \text{ for all limit ordinals } j, \\ w_{0,1}: 0 &\rightarrow W_1 \text{ is unique,} & w_{j+1,k+1} &= Fw_{j,k}: FW_j \rightarrow FW_k, \\ w_{i,j} &(i < j) \text{ is the colimit cocone for limit ordinals } j \end{aligned}$$

► **Remark 6.3.**

1. There exists, up to natural isomorphism, precisely one Ord-chain satisfying the above equations. For example, $w_{\omega,\omega+1}: W_\omega \rightarrow FW_\omega$ is determined by the universal property of $W_\omega = \text{colim}_{n < \omega} W_n = \text{colim}_{n < \omega} W_{n+1}$ as the unique morphism with $w_{\omega,\omega+1} \cdot w_{n+1,\omega} = w_{n+1,\omega+1} = Fw_{n,\omega}$ for every $n < \omega$.
2. Every algebra $\alpha: FA \rightarrow A$ induces a canonical cocone $\alpha_i: W_i \rightarrow A$ ($i \in \text{Ord}$) on the initial-algebra chain; it is the unique cocone with $\alpha_{i+1} = (W_{i+1} = FW_i \xrightarrow{F\alpha_i} FA \xrightarrow{\alpha} A)$ for all ordinals i . This is easy to see using transfinite induction.

► **Definition 6.4.** We say that the initial-algebra chain of a functor F *converges in λ steps* if $w_{\lambda,\lambda+1}$ is an isomorphism, and we simply say that it *converges*, if it converges in λ steps for some ordinal λ .

If $w_{i,i+1}$ is an isomorphism, then so is $w_{i,j}$, for all $j > \lambda$. This is easy to prove by transfinite induction.

Convergence of the initial-algebra chain yields an initial algebra [1]. We obtain this as a consequence of results from Section 2.2 on recursive coalgebras:

► **Theorem 6.5.** *Let \mathcal{A} be a category with colimits of chains. If the initial-algebra chain of an endofunctor F converges in λ steps, then W_λ is the initial algebra with the algebra structure $w_{\lambda,\lambda+1}^{-1}: FW_\lambda \rightarrow W_\lambda$.*

Proof. An easy transfinite induction shows that every coalgebra $w_{i,i+1}: W_i \rightarrow FW_i$ is recursive: the coalgebra $0 \rightarrow F0$ is trivially recursive, for the isolated step use Proposition 2.11, and Proposition 2.14 yields the limit step. If $w_{\lambda,\lambda+1}$ is an isomorphism, then $(W_\lambda, w_{\lambda,\lambda+1}^{-1})$ is the initial algebra by Remark 2.10. ◀

The existence of an \mathcal{M} -pre-fixed point implies that the initial-algebra chain converges. The proof below is somewhat similar to the proof of Theorem 4.3. The difference is that one only uses the recursive coalgebras $W_i \rightarrow FW_i$ in the initial-algebra chain and applies Zermelo's Theorem 2.1 in lieu of Pataria's Theorem. For this we work again under Assumption 4.1.

► **Theorem 6.6.** *Let F preserve \mathcal{M} and $m: FA \rightarrow A$ be an \mathcal{M} -pre-fixed point. If A has smooth \mathcal{M} -subobjects, then the initial-algebra chain for F converges.*

Proof. Again, we use the monotone endomap $f: \text{Sub}_{\mathcal{M}}(A) \rightarrow \text{Sub}_{\mathcal{M}}(A)$ in (3). Theorem 2.1 applies since $\text{Sub}_{\mathcal{M}}(A)$ is a dcpo by assumption, and therefore it is a chain-complete poset. Thus, f has the least fixed point $\mu f = f^i(\perp)$ for some ordinal i . The cocone $m_j: W_j \rightarrow A$ of Remark 6.3.2 satisfies $m_j = f^j(\perp)$ for all $j \in \text{Ord}$. This is easily verified by transfinite induction. Hence, from $f(f^i(\perp)) = f^i(\perp)$ we conclude that m_i and m_{i+1} represent the same subobject of A . Since $m_i = m_{i+1} \cdot w_{i,i+1}$, it follows that $w_{i,i+1}$ is invertible, which means that the initial-algebra chain converges. ◀

We now obtain the original initial-algebra theorem by Trnková et al. [28]:

► **Corollary 6.7.** *Let \mathcal{A} be a category with colimits of chains and with a smooth class \mathcal{M} of monomorphisms. Then the following are equivalent for an endofunctor F preserving \mathcal{M} :*

1. the initial-algebra chain converges,
2. an initial algebra exists,
3. a fixed point exists,
4. an \mathcal{M} -pre-fixed point exists.

Moreover, if these hold, then μF is an \mathcal{M} -subalgebra of every \mathcal{M} -pre-fixed point of F .

Indeed, 4 implies 1 by Theorem 6.6, and 1 implies 2 is shown as in Theorem 6.5. The remaining implications are as for Corollary 4.4.

► **Remark 6.8.** Note that in lieu of assuming that \mathcal{A} has colimits of *all* chains, it suffices that the initial-algebra chain exists (i.e. the colimits in Definition 6.2 exist). This weaker condition enables more applications, e.g. the category of relations with \mathcal{M} the class of injective maps and functors F which are lifted from **Set**.

► **Remark 6.9.** For a set functor F no side condition is needed: if F has a pre-fixed point, then it has an initial algebra. This is clear if $F\emptyset = \emptyset$. If not, there is a set functor G with $G\emptyset \neq \emptyset$ which preserves monomorphisms and agrees with F on all nonempty sets and maps [29]. Since every pre-fixed point of F must be nonempty, it is also a pre-fixed point of G . Hence G has an initial algebra, which clearly is an initial algebra for F , too.

► **Corollary 6.10.** *An endofunctor on one of the categories **Set**, **Pfn**, or **K-Vec** has an initial algebra iff it has a pre-fixed point.*

Proof. For **Set**, use Remark 6.9. For **Pfn** and **K-Vec**, apply Corollary 6.7 with \mathcal{M} the class of all monomorphisms (which are split and therefore preserved by every endofunctor). ◀

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A Further Technical Details

A.1 Details for Example 3.5

► **Lemma A.1.** *Monomorphisms are smooth in MS.*

Proof. Fix a space (A, d) , and consider a directed set D of subobjects $m_i : (A_i, d_i) \rightarrow (A, d)$ ($i \in D$) with monomorphisms $a_{i,j} : A_i \rightarrow A_j$ witnessing $i \leq j$ in D . Let $B = \bigcup_{i \in D} m_i[A_i]$, and let $d' : B \rightarrow [0, 1]$ be defined as follows:

$$d'(x, y) = \inf\{d_i(x', y') : i \in D, x', y' \in A_i, m_i(x') = x \text{ and } m_i(y') = y\}. \quad (6)$$

We show that d' is a metric. It is clearly symmetric and fulfils $d'(x, x) = 0$. We verify that distinct points x, y in B have non-zero distance. For each i, x' , and y' as in (6), $d_i(x', y') \geq d(x, y)$, since m_i is non-expanding. Thus $d'(x, y) \geq d(x, y) > 0$.

Finally, we verify that d' satisfies the triangle inequality. To this end it suffices to show that for all $x, y, z \in B$ and every $\varepsilon > 0$ we have $d'(x, z) \leq d'(x, y) + d'(y, z) + \varepsilon$.

Let $x, y, z \in B$ and fix $\varepsilon > 0$. We can choose i, x', y' as in (6) such that $d_i(x', y') < d'(x, y) + \varepsilon/2$. Analogously, let j, y'', z'' be such that $d_j(y'', z'') < d'(y, z) + \varepsilon/2$. Since the collection $m_i[A_i]$ is directed, we can assume $i \leq j$ in D . Using that the connecting map $a_{i,j}$ is non-expanding we obtain $d_j(a_{i,j}(x'), a_{i,j}(y')) < d'(x, y) + \varepsilon/2$. Since m_j is injective, $a_{i,j}(y') = y''$. Let $x'' = a_{i,j}(x')$, and note that $m_j(x'') = x$. By the triangle inequality in A_j ,

$$d_j(x'', z'') \leq d_j(x'', y'') + d_j(y'', z'') < d'(x, y) + d'(y, z) + \varepsilon.$$

It follows that $d'(x, z) \leq d'(x, y) + d'(y, z) + \varepsilon$, as desired.

It is obvious that the inclusion $m : B \rightarrow A$ is non-expanding. It is also easy to check that (B, d') is the join in $\text{Sub}(A, d)$ of the directed diagram corresponding to given directed set D .

Finally, for every $i \in D$, we have the codomain restriction $m'_i : A_i \rightarrow B$ of m_i , which is non-expanding. We verify that the family of all m'_i ($i \in D$) forms a colimit cocone. It clearly is a cocone. Consider any cocone $f_i : (A_i, d_i) \rightarrow (A^*, d^*)$, $i \in D$. Clearly, the union B is the colimit in Set . Therefore, we have a unique map $f : B \rightarrow A^*$ such that $f_i = f \cdot m'_i$ for all $i \in D$. This is given by $f(x) = f_i(x)$ whenever $x \in m_i[A_i]$. We check that f is non-expanding, and this will conclude our verification. Let $x, y \in B$, and choose i, x', y' as in (6). Since f_i is non-expanding,

$$d^*(f(x), f(y)) = d^*(f_i(x'), f_i(y')) \leq d_i(x', y') \leq d'(x, y). \quad \blacktriangleleft$$

► **Lemma A.2.** *Strong monomorphisms are smooth in CMS.*

Proof. Fix a complete metric space (A, d) , and consider a directed set D of closed subspaces $A_i \hookrightarrow A$. Their join $B \hookrightarrow A$ is the closure of their union

$$B = \overline{\bigcup_{i < \lambda} A_i}.$$

We know from Lemma A.1 that the union is the colimit of the directed diagram corresponding to D in MS. Moreover, the colimit of a diagram in CMS is given by forming the Cauchy completion of the colimit of that diagram in MS. (This follows from the fact that CMS is a reflective subcategory of MS with Cauchy completions as reflections.) Since B is complete and $\bigcup_{i \in D} A_i$ is dense in it, B is the Cauchy completion of that union, whence it is desired colimit in CMS. ◀

A.2 Proof of Theorem 5.2

Proof. $1 \Rightarrow 2$: Let D have objects D_i and connecting morphisms $e_{i,j}: D_i \rightarrow D_j$. Write $\widehat{e}_{i,j}$ for the projection of $e_{i,j}$. We verify that for $i \leq j \leq k$, $\widehat{e}_{i,k} = \widehat{e}_{i,j} \cdot e_{j,k}$. In fact, $\widehat{e}_{i,j}$ is unique with $\widehat{e}_{i,j} \cdot e_{i,j} = \text{id}_{D_i}$ and $e_{i,j} \cdot \widehat{e}_{i,j} \sqsubseteq \text{id}_{D_j}$. But $\widehat{e}_{i,k} \cdot e_{j,k}$ also has these properties, since

$$\begin{aligned} (\widehat{e}_{i,k} \cdot e_{j,k}) \cdot e_{i,j} &= \widehat{e}_{i,k} \cdot e_{i,k} & \text{and} & & e_{i,j} \cdot (\widehat{e}_{i,k} \cdot e_{j,k}) &= e_{i,j} \cdot \widehat{e}_{i,j} \cdot \widehat{e}_{j,k} \cdot e_{j,k} \\ &= \text{id}_{D_i}, & & & &= e_{i,j} \cdot \widehat{e}_{i,j} \cdot \text{id}_{D_k} \\ & & & & &\sqsubseteq \text{id}_{D_k}. \end{aligned}$$

This shows that indeed $\widehat{e}_{i,k} = \widehat{e}_{i,j} \cdot e_{j,k}$ for $i \leq j \leq k$.

For each i form the subdiagram D^i of all D_j for $j \geq i$, with connecting maps $e_{j,k}$ for $i \leq j \leq k$ inherited from D . Since I is directed, the colimit of D^i is $(c_j)_{j \geq i}$. Our observation at the outset shows that we have a cocone of D^i :

$$\widehat{e}_{i,j} = (D_j \xrightarrow{e_{j,k}} D_k \xrightarrow{\widehat{e}_{i,k}} D_i).$$

Thus, there is a unique factorization $\widehat{c}_i: C \rightarrow D_i$ through the colimit cocone:

$$\widehat{e}_{i,j} = \widehat{c}_i \cdot c_j \quad \text{for } j \geq i. \quad (7)$$

In particular, for $i = j$, we see that $\widehat{c}_i \cdot c_i = \text{id}_{D_i}$. We will verify below the equation $\bigsqcup_j c_j \cdot \widehat{c}_j = \text{id}_C$, and this of course implies that $c_i \cdot \widehat{c}_i \sqsubseteq \text{id}_C$. (This justifies our use of the projection notation \widehat{c}_j and shows the first point in item 2, that c_i is an embedding.)

Next, we show that for each j , the morphisms \widehat{c}_i for $i \leq j$ form a cone of D^j :

$$\widehat{c}_i = (C \xrightarrow{\widehat{c}_j} D_j \xrightarrow{\widehat{e}_{i,j}} D_i).$$

Indeed, the colimit cocone $(c_k)_{k \geq j}$ is collectively epic, so we need only establish this after precomposing with each c_k . We apply (7) twice to obtain: $\widehat{c}_i \cdot c_k = \widehat{e}_{i,k} = \widehat{e}_{i,j} \cdot \widehat{e}_{j,k} = \widehat{e}_{i,j} \cdot \widehat{c}_j \cdot c_k$.

We are ready to argue for 2. The maps $c_i \cdot \widehat{c}_i$ form a directed subset of $\mathcal{A}(C, C)$ because for $i \leq j$, $c_i \cdot \widehat{c}_i = (c_j \cdot e_{i,j}) \cdot (\widehat{e}_{i,j} \cdot \widehat{c}_j) \sqsubseteq c_j \cdot \widehat{c}_j$. Thus, $\bigsqcup_j c_j \cdot \widehat{c}_j$ exists. We use that the family (c_i) is collectively epic, and verify that $\bigsqcup_j c_j \cdot \widehat{c}_j \cdot c_i = c_i$ for every i . Fix i , and consider the join above. Since it is over a directed set, we need only consider $\bigsqcup_{j \geq i} c_j \cdot \widehat{c}_j \cdot c_i$. In addition, for $j \geq i$ we obtain $c_j \cdot \widehat{c}_j \cdot c_i = c_j \cdot \widehat{c}_j \cdot (c_j \cdot e_{i,j}) = c_j \cdot e_{i,j} = c_i$.

$2 \Rightarrow 1$: Let $(b_i: D_i \rightarrow B)$ be a cocone. For all $i \leq j$ we have $b_i = b_j \cdot e_{i,j}$. We also have a cocone (c_i) , and so $c_i = c_j \cdot e_{i,j}$, and thus $\widehat{c}_i = \widehat{e}_{i,j} \cdot \widehat{c}_j$. From this we have

$$b_i \cdot \widehat{c}_i = (b_j \cdot e_{i,j}) \cdot (\widehat{e}_{i,j} \cdot \widehat{c}_j) \sqsubseteq b_j \cdot e_j.$$

Thus, the following join exists in $\mathcal{A}(C, B)$: $b = \bigsqcup_j b_j \cdot \widehat{c}_j$. To prove that $b_i = b \cdot c_i$ for all i , we fix one i and consider the join above with $j \geq i$:

$$b_j \cdot \widehat{c}_j \cdot c_i = (b_j \cdot \widehat{c}_j) \cdot (c_j \cdot e_{i,j}) = b_j \cdot e_{i,j} = b_i.$$

Thus $b \cdot c_i = \bigsqcup_{j \geq i} (b_j \cdot \widehat{c}_j \cdot c_i) = \bigsqcup_{j \geq i} b_i = b_i$. This shows that b is the desired factorization of (b_i) . For its uniqueness, let $b': C \rightarrow B$ be a morphism with $b' \cdot c_i = b_i$ for all i . Since $\bigsqcup_i c_i \cdot \widehat{c}_i = \text{id}_C$, we have $b' = b' \cdot (\bigsqcup_i c_i \cdot \widehat{c}_i) = \bigsqcup_i b' \cdot c_i \cdot \widehat{c}_i = \bigsqcup_i b_i \cdot \widehat{c}_i = b$. This completes the proof. \blacktriangleleft

A.3 Proof of Theorem 5.4

Proof. We use Proposition 3.6. Fix an object A in \mathcal{A} . Let \mathcal{E} be the class of embeddings, so that $\text{Sub}_{\mathcal{E}}(A)$ denotes the poset of subobjects of A represented by embeddings. Let D be a directed diagram of monomorphisms in \mathcal{A} , not necessarily embeddings, and let $m_i: A_i \rightarrow A$, $i \in D$, be a cocone of morphisms in $\text{Sub}_{\mathcal{E}}(A)$. By hypothesis, D has a colimit cocone, say $c_i: A_i \rightarrow B$. We have a unique morphism $m: B \rightarrow A$ such that for all i , $m_i = m \cdot c_i$. Our task is to show that m is an embedding, and that $m = \bigsqcup_{i \in A} m_i$ in $\text{Sub}_{\mathcal{E}}(A)$.

For $i \leq j$ in A , we have a morphism $e_{i,j}$ such that $m_i = m_j \cdot e_{i,j}$. Let us verify that $e_{i,j}$ is an embedding and that $\widehat{e}_{i,j} = \widehat{m}_i \cdot m_j$. To see this, we use the characterization of projections. First, $(\widehat{m}_i \cdot m_j) \cdot e_{i,j} = \widehat{m}_i \cdot m_i = \text{id}$. Second, we verify $e_{i,j} \cdot (\widehat{m}_i \cdot m_j) \sqsubseteq \text{id}$:

$$\begin{aligned} e_{i,j} \cdot (\widehat{m}_i \cdot m_j) &= (\widehat{m}_j \cdot m_j) \cdot e_{i,j} \cdot \widehat{m}_i \cdot m_j && \text{since } \widehat{m}_j \cdot m_j = \text{id} \\ &= \widehat{m}_j \cdot m_i \cdot \widehat{m}_i \cdot m_j && \text{since } m_i = e_{i,j} \cdot m_j \\ &\sqsubseteq \widehat{m}_j \cdot m_j && \text{since } m_i \cdot \widehat{m}_i = \text{id} \\ &= \text{id}. \end{aligned}$$

We next show that for $i \leq j$, $c_i \cdot \widehat{m}_i \sqsubseteq c_j \cdot \widehat{m}_j$. Once this is done, we put $\widehat{m} = \bigsqcup_i c_i \cdot \widehat{m}_i$ and show that it is a projection for m . We thus calculate:

$$\begin{aligned} c_i \cdot \widehat{m}_i &= c_j \cdot e_{i,j} \cdot \widehat{m}_i \\ &= c_j \cdot e_{i,j} \cdot \widehat{m}_j \cdot e_{i,j} \\ &= c_j \cdot e_{i,j} \cdot \widehat{e}_{i,j} \cdot \widehat{m}_j \\ &\sqsubseteq c_j \cdot \widehat{m}_j \end{aligned}$$

To prove that $\widehat{m} \cdot m = \text{id}$, we use that the family (c_i) is collectively epic. Thus, we show that for all i , $\widehat{m} \cdot m \cdot c_i = c_i$. We again consider $\bigsqcup_{j \geq i} c_j$ only:

$$\begin{aligned} \widehat{m} \cdot m \cdot c_i &= \left(\bigsqcup_{j \geq i} c_j \cdot \widehat{m}_j \right) \cdot m \cdot c_i \\ &= \bigsqcup_{j \geq i} c_j \cdot \widehat{m}_j \cdot m_i \\ &= \bigsqcup_{j \geq i} c_j \cdot \widehat{m}_j \cdot m_j \cdot e_{i,j} \\ &= \bigsqcup_{j \geq i} c_j \cdot e_{i,j} \\ &= \bigsqcup_{j \geq i} c_i \\ &= c_i. \end{aligned}$$

In the other direction, we show that $m \cdot \widehat{m} \sqsubseteq \text{id}$:

$$m \cdot \left(\bigsqcup_i c_i \cdot \widehat{m}_i \right) = \bigsqcup_i m \cdot c_i \cdot \widehat{m}_i = \bigsqcup_i m_i \cdot \widehat{m}_i \sqsubseteq \text{id}.$$

Our last order of business is to show that $m = \bigsqcup_{i \in A} m_i$ in $\text{Sub}_{\mathcal{E}}(A)$. Since $m \cdot c_i = m_i$, we see that $m_i \sqsubseteq m$ for all i . Let $u: U \rightarrow A$ be an embedding with $m_i \sqsubseteq u$ for all i . Thus we have morphisms u_i such that $m_i = u \cdot u_i$. The family $(u_i)_i$ is a cocone of the original diagram D , because if $i \leq j$, then

$$u_j \cdot e_{i,j} = (\widehat{u} \cdot u) \cdot u_j \cdot e_{i,j} = \widehat{u} \cdot m_j \cdot e_{i,j} = \widehat{u} \cdot m_i = \widehat{u} \cdot u \cdot u_i = u_i.$$

Since (c_i) is a colimit, there is a unique $f: M \rightarrow U$ such that $u_i = f \cdot c_i$ for all i . We aim to show that $m = u \cdot f$, so that $m \sqsubseteq u$. For this, we again use the fact that (c_i) is a collectively epic family: $m \cdot c_i = m_i = u \cdot u_i = u \cdot f \cdot c_i$. ◀