(Metric) Bisimulation Games and Real-Valued Modal Logics for Coalgebras

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Abstract

Behavioural equivalences can be characterized via bisimulations, modal logics and spoiler-defender games. In this paper we review these three perspectives in a coalgebraic setting, which allows us to generalize from the particular branching type of a transition system. We are interested in qualitative notions (classical bisimulation) as well as quantitative notions (bisimulation metrics).

Our first contribution is to introduce a spoiler-defender bisimulation game for coalgebras in the classical case. Second, we introduce such games for the metric case and furthermore define a real-valued modal coalgebraic logic, from which we can derive the strategy of the spoiler. For this logic we show a quantitative version of the Hennessy-Milner theorem.

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1 Introduction

In the characterization of behavioural equivalences one encounters the following triad: First, such equivalences can be described via bisimulation relations, where the largest bisimulation (or bisimilarity) can be characterized as a greatest fixpoint. Second, a modal logic provides us with bisimulation-invariant formulas and the aim is to prove a Hennessy-Milner theorem which says that two states are behaviourally equivalent if and only if they satisfy the same formulas [21]. A third, complementary view is given by spoiler-defender games [32]. Such games are useful both for theoretical reasons, see for instance the role of games in the Van Benthem/Rosen theorem [26], or for didactical purposes, in particular for showing that two states are not behaviourally equivalent. The game starts with two tokens on two states and the spoiler tries to make a move that cannot be imitated by the defender. If the defender is always able to match the move of the spoiler we can infer that the two initial states are behaviourally equivalent. If the states are not equivalent, a strategy for the spoiler can be derived from a distinguishing modal logic formula.

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Such games are common for standard labelled transition systems, but have been studied for other types of transition systems only to a lesser extent. For probabilistic transition systems there are game characterizations in [15, 17], where the players can make moves to sets of states, rather than take a transition to a single state. Furthermore, in [11] a general theory of games is introduced in order to characterize process equivalences of the linear/branching time spectrum.

Our aim is to extend this triad of bisimulation, logics and games in two orthogonal dimensions. First, we work in the general framework of coalgebras [28], which allows to specify and uniformly reason about systems of different branching types (e.g. non-deterministic, probabilistic or weighted), parameterized over a functor. While behavioural equivalences [31] and modal logics [29, 27] have been extensively studied in this setting, there are almost no contributions when it comes to games. We are mainly aware of the work by Baltag [7], which describes a coalgebraic game based on the bisimulation relation, which differs from the games studied in this paper and is associated with another variant of logic, namely Moss' coalgebraic logics [25]. A variant of Baltag's game was used in [24] for terminal sequence induction via games. (There are more contributions on evaluation games which describe the evaluation of a modal formula on a transition system, see for instance [18].) Our contribution generalizes the games of [15] and allows us, given a new type of system characterized by a functor on the category **Set**, satisfying some mild conditions, to automatically derive the corresponding game. The second dimension in which we generalize is to move from a qualitative to a quantitative notion of behavioural equivalence. That is, we refrain from classifying systems as either equivalent or non-equivalent, which is often too strict, but rather measure their behavioural distance. This makes sense in probabilistic systems, systems with time or real-valued output. For instance, we might obtain the result, that the running times of two systems differ by 10 seconds, which might be acceptable in some scenarios (departure of a train), but inacceptable in others (delay of a vending machine). On the other hand, two states are behaviourally equivalent in the classical sense if and only if they have distance 0. Such notions are for instance useful in the area of conformance testing [22] and differential privacy [9].

Behavioural metrics have been studied in different variants, for instance in probabilistic settings [13, 14, 8] as well as in the setting of metric transition systems [12, 16], which are non-deterministic transition systems with quantitative information. The groundwork for the treatment of coalgebras in metric spaces was laid by Turi and Rutten [33]. We showed how to characterize behavioural metrics in coalgebras by studying various possibilities to lift functors from **Set** to the category of (pseudo-)metric spaces [5, 6]. Different from [33, 35] we do not assume that the coalgebra is given a priori in the category of pseudometric spaces, that is we have to first choose a lifting of the behaviour functor in order to specify the behavioural metric. Such liftings are not unique[1] and in particular we introduced in [5, 6] the Kantorovich and the Wasserstein liftings, which generalize well-known liftings for the probabilistic case and also capture the Hausdorff metric. Here we use the Kantorovich lifting, since this lifting integrates better with coalgebraic logic. Our results are parameterized over the lifting, in particular the behavioural metrics, the game and the logics are dependent on a set Γ of evaluation functions.

In the metric setting it is natural to generalize from classical two-valued logics to real-

[1] In fact, consider the product bifunctor $F(X, Y) = X \times Y$, for which there are several liftings: we can e.g. use the maximum or the sum metric. While the maximum metric is canonically induced by the categorical product, the sum metric is also fairly natural.

valued modal logics and to state a corresponding Hennessy-Milner theorem that compares the behavioural distance of two states with the logical distance, i.e., the supremum of the differences of values, obtained by the evaluation of all formulas. Such a Hennessy-Milner theorem for probabilistic transition systems was shown in [14] and also studied in a coalgebraic setting [35, 34]. Similar results were obtained in [37] for fuzzy logics, on the way to proving a van Benthem theorem. Fuzzy logics were also studied in [30] in a general coalgebraic setting, but without stating a Hennessy-Milner theorem.

Here we present a real-valued coalgebraic modal logic and give a Hennessy-Milner theorem for the general coalgebraic setting as a new contribution. Our proof strategy follows the one for the probabilistic case in [35]. We need several concepts from real analysis, such as non-expansiveness and total boundedness in order to show that the behavioural distance (characterized via a fixpoint) and the logical distance coincide.

Furthermore we give a game characterization of this behavioural metric in a game where we aim to show that $d(x, y) \leq \varepsilon$, i.e., the behavioural distance of two states x, y is bounded by ε. Furthermore, we work out the strategies for the defender and spoiler: While the strategy of the defender is based on the knowledge of the behavioural metric, the strategy of the spoiler can be derived from a logical formula that distinguishes both states.

Again, work on games is scarce: [15] presents a game which characterizes behavioural distances, but pairs it with a classical logic.

The paper is organized as follows: we will first treat the classical case in Section 2, followed by the metric case in Section 3. The development in the metric case is more complex, but in several respects mimics the classical case. Hence, in order to emphasize the similarities, we will use the same structure within both sections: we start with foundations, followed by the introduction of modal logics and the proof of the Hennessy-Milner theorem. Then we will introduce the game with a proof of its soundness and completeness. Finally we will show how the strategy for the spoiler can be derived from a logical formula. In the end we wrap everything up in the conclusion (Section 4). The proofs can be found in the full version of this paper [23].

2 Logics and Games for the Classical Case

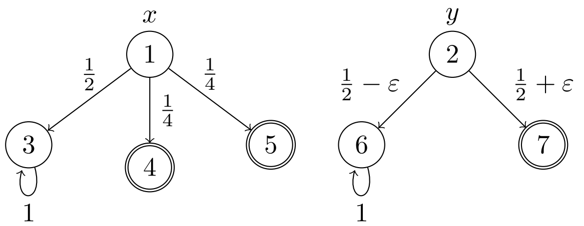
2.1 Foundations for the Classical Case

We fix an endofunctor $F\colon \mathbf{Set} \to \mathbf{Set}$, intuitively describing the branching type of the transition system under consideration. A *coalgebra*, describing a transition system of this branching type is given by a function $\alpha\colon X \to FX$ [28]. Two states $x, y \in X$ are considered to be *behaviourally equivalent* ($x \sim y$) if there exists a coalgebra homomorphism f from α to some coalgebra $\beta\colon Y \to FY$ (i.e., a function $f\colon X \to Y$ with $\beta \circ f = Ff \circ \alpha$) such that $f(x) = f(y)$.

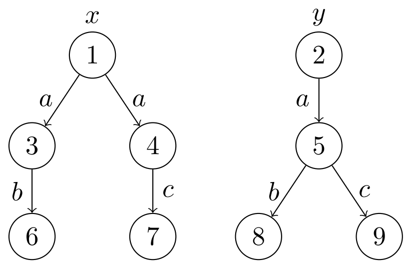
▶ **Example 1.** *We consider the (finitely or countably supported) probability distribution functor $\mathcal{D}$ with $\mathcal{D}X = \{p\colon X \to [0,1] \mid \sum_{x\in X} p(x) = 1\}$ (where the p are either finitely or countable supported). Furthermore let $1 = \{\bullet\}$ be a singleton set.*

Now coalgebras of the form $\alpha\colon X \to \mathcal{D}X + 1$ can be seen as probabilistic transition systems where each state x is either terminating ($\alpha(x) = \bullet$) or is associated with a probability distribution on its successor states. Note that one could easily integrate labels as well, but we will work with this version for simplicity.

Figure 1a displays an example coalgebra (where $0 \leq \varepsilon \leq \frac{1}{2}$). Note that whenever $\varepsilon = 0$, we have $x \sim y$, since there is a coalgebra homomorphism from the entire state space into the right-hand side component of the transition system. If $\varepsilon > 0$ we have $1 \not\sim 2$.



(a) Probabilistic transition system for the functor $FX = \mathcal{D}X + 1$



(b) Non-deterministic transition system for the functor $FX = \mathcal{P}(A \times X)$

Figure 1

Furthermore we need the lifting of a preorder under a functor F. For this we use the lifting introduced in [4] which guarantees that the lifted relation is again a preorder whenever F preserves weak pullbacks: Let $\leq$ be a preorder on Y, i.e. $\leq \subseteq Y \times Y$. We define the preorder $\leq^F \subseteq FY \times FY$ with $t_1, t_2 \in FY$ as follows: $t_1 \leq^F t_2$ whenever some $t \in F(\leq)$ exists such that $F\pi_i(t) = t_i$, where $\pi_i \colon \leq \to Y$ with $i \in \{1, 2\}$ are the usual projections.

▶ **Lemma 2.** *Let $(Y, \leq)$ be an ordered set and let $p_1, p_2 \colon X \to Y$ be two functions. Then $p_1 \leq p_2$ implies $Fp_1 \leq^F Fp_2$. (Both inequalities are to be read as pointwise ordering.)*

▶ **Example 3.** *We are in particular interested in lifting the order $0 \leq 1$ on $2 = \{0, 1\}$. In the case of the distribution functor $\mathcal{D}$ we have $\mathcal{D}2 \cong [0, 1]$ and $\leq^{\mathcal{D}}$ corresponds to the order on the reals. For the powerset functor $\mathcal{P}$ we obtain the order $\{0\} \leq^{\mathcal{P}} \{0, 1\} \leq^{\mathcal{P}} \{1\}$ where $\emptyset$ is only related to itself.*

2.2 Modal Logics for the Classical Case

We will first review coalgebraic modal logics, following mainly [27, 29], using slightly different, but equivalent notions. In particular we will introduce a logic where a predicate lifting is given by an evaluation map of the form $\lambda \colon F2 \to 2$, rather than by a natural transformation. In particular, each predicate $p \colon X \to 2$ is lifted to a predicate $\lambda \circ Fp \colon FX \to 2$. We do this to obtain a uniform presentation of the material. Of course, both views are equivalent, as spelled out in [29].

Given a cardinal κ and a set Λ of evaluation maps $\lambda \colon F2 \to 2$, we define a coalgebraic modal logic $\mathcal{L}^\kappa(\Lambda)$ via the grammar:

$$\varphi ::= \bigwedge \Phi \mid \neg\varphi \mid [\lambda]\varphi \quad \text{where } \Phi \subseteq \mathcal{L}^\kappa(\Lambda) \text{ with } \mathit{card}(\Phi) < \kappa \text{ and } \lambda \in \Lambda.$$

The last case describes the prefixing of a formula φ with a modality $[\lambda]$.

Given a coalgebra $\alpha \colon X \to FX$ and a formula φ, the semantics of such a formula is given by a map $[\![\varphi]\!]_\alpha \colon X \to 2$, where conjunction and negation are interpreted as usual and $[\![[\lambda]\varphi]\!]_\alpha = \lambda \circ F[\![\varphi]\!]_\alpha \circ \alpha$. For simplicity we will often write $[\![\varphi]\!]$ instead of $[\![\varphi]\!]_\alpha$. Furthermore for $x \in X$ we write $x \models \varphi$ whenever $[\![\varphi]\!]_\alpha(x) = 1$.

In [27] Pattinson has isolated the property of a separating set of predicate liftings to ensure that logical and behavioural equivalence coincide, i.e., the Hennessy-Milner property holds. It means that every $t \in FX$ is uniquely determined by the set $\{(\lambda, p) \mid \lambda \in \Lambda, p \colon X \to 2, \lambda(Fp(t)) = 1\}$.

▶ **Definition 4.** *A set Λ of evaluation maps is separating for a functor $F\colon \textbf{Set} \to \textbf{Set}$ whenever for all sets X and $t_1, t_2 \in FX$ with $t_1 \neq t_2$ there exists $\lambda \in \Lambda$ and $p\colon X \to 2$ such that $\lambda(Fp(t_1)) \neq \lambda(Fp(t_2))$.*

The Hennessy-Milner theorem for coalgebraic modal logics can be stated as follows. The result has already been presented in [27, 29, 20]

▶ **Proposition 5** ([29]). *The logic $\mathcal{L}^\kappa(\Lambda)$ is sound, that is given a coalgebra $\alpha\colon X \to FX$ and $x, y \in X$, $x \sim y$ implies that $[\![\varphi]\!]_\alpha(x) = [\![\varphi]\!]_\alpha(y)$ for all formulas φ.*

Whenever F is κ-accessible[2] and Λ is separating for F, the logic is also expressive: whenever $[\![\varphi]\!]_\alpha(x) = [\![\varphi]\!]_\alpha(y)$ for all formulas φ we have that $x \sim y$.

In [29] it has been shown that a functor F has a separating set of predicate liftings iff $(Fp\colon FX \to F2)_{p\colon X\to 2}$ is jointly injective. We extend this characterization to monotone predicate liftings, respectively evaluation maps, i.e., order-preserving maps $\lambda\colon (F2, \leq^F) \to (2, \leq)$ where $\leq$ is the order $0 \leq 1$. This result will play a role in Section 2.3.

▶ **Proposition 6.** *F has a separating set of monotone evaluation maps iff $\leq^F$ is anti-symmetric and $(Fp\colon FX \to F2)_{p\colon X\to 2}$ is jointly injective.*

Note that an evaluation map is monotone if and only if its induced predicate lifting is monotone (see [23]).

2.3 Games for the Classical Case

We will now present the rules for the behavioural equivalence game. At the beginning of a game, there are two states x, y available for selection. The aim of the spoiler (S) is to prove that $x \not\sim y$, the defender (D) attempts to show the opposite.

- **Initial situation:** Given a coalgebra $\alpha\colon X \to FX$, we start with $x, y \in X$.
- **Step 1:** S chooses $s \in \{x, y\}$ and a predicate $p_1\colon X \to 2$.
- **Step 2:** D takes $t \in \{x, y\}\backslash\{s\}$ and has to answer with a predicate $p_2\colon X \to 2$, which satisfies $Fp_1(\alpha(s)) \leq^F Fp_2(\alpha(t))$.
- **Step 3:** S chooses p_i with $i \in \{1, 2\}$ and some state $x' \in X$ with $p_i(x') = 1$.
- **Step 4:** D chooses some state $y' \in X$ with $p_j(y') = 1$ where $i \neq j$.

After one round the game continues in Step 1 with states x' and y'. D wins the game if the game continues forever or if S has no move at Step 3. In the other cases, i.e. D has no move at Step 2 or Step 4, S wins.

In a sense such a game seems to mimic the evaluation of a modal formula, but note that the chosen predicates do not have to be bisimulation-invariant, as opposed to modal formulas.

▶ **Theorem 7.** *Assume that F preserves weak pullbacks and has a separating set of monotone evaluation maps. Then $x \sim y$ iff D has winning strategy for the initial situation (x, y).*

Part of the proof of Theorem 7 is to establish a winning strategy for D whenever $x \sim y$. We will quickly sketch this strategy: In Step 1 S plays p_1 which represents a set of states.

[2] A functor $F\colon \textbf{Set} \to \textbf{Set}$ is κ-accessible if for all sets X and all $x \in FX$ there exists $Z \subseteq X$, $|Z| < \kappa$ such that $x \in FZ \subseteq FX$ [1]. (Note that we use the fact that **Set**-functors preserve injections $f\colon A \to B$ whenever $A \neq \emptyset$.) For details and a more categorical treatment see [2].

One good strategy for D in Step 2 is to close this set under behavioural equivalence, i.e., to add all states which are behaviourally equivalent to a state in p_1, thus obtaining p_2. It can be shown that $Fp_1(\alpha(s)) \leq^F Fp_2(\alpha(t))$ always holds for this choice. Now, if S chooses x', p_1 in Step 3, D simply takes x' as well. On the other hand, if S chooses x', p_2, then either x' is already present in p_1 or a state y' with $x' \sim y'$. D simply chooses y' and the game continues.

▶ **Example 8.** *Now consider an example coalgebra for the functor $FX = \mathcal{P}(A \times X)$, where $\mathcal{P}$ is the powerset functor (see Figure 1b). Obviously $x \not\sim y$, so S must have a winning strategy. Somewhat different from the usual bisimulation game, here the two players play subsets of the state space, instead of single states. Otherwise the game proceeds similarly.*

Assume that S chooses $s = 1$ and defines a predicate p_1, which corresponds to the set $\{3\}$. Then $Fp_1(\alpha(s))$ is $\{(a, 0), (a, 1)\}$ (one a-successor of s – 3 – satisfies the predicate, the other – 4 – does not). Now D must take $t = 2$ and has to choose a predicate p_2 where at least $p_2(5) = 1$. In this case $Fp_2(\alpha(t))$ is $\{(a, 1)\}$ and $\{1\}$ is larger than $\{0, 1\}$ in the lifted order (see Example 3). However, now S can pick 5, which leaves only 3 to D.

In the next step, S can choose $s = 5$ and a predicate p_1, which corresponds to $\{9\}$. Hence $Fp_1(\alpha(s))$ is $\{(b, 0), (c, 1)\}$, but it is impossible for D to match this, since $(c, 1)$ is never contained in $Fp_2(\alpha(t))$ for $t = 3$.

We can see from this game why it is necessary to use an inequality $\leq^F$ instead of an equality. If there were no b, c-transitions (just a-transitions), $1 \sim 2$ would hold. And then, as explained above, D cannot match the move of S exactly, but only by choosing a larger value.

This game is inspired by the game for labelled Markov processes in [15] and the connection is explained in more detail in [23].

Note that in the probabilistic version of the game, it can again be easily seen that an inequality is necessary in Step 2: if, in the system in Figure 1a (where $\varepsilon = 0$), S chooses $s = 1$ and p_1 corresponds to $\{4\}$, then D can only answer by going to 7, which results in a strictly larger value. That is, we must allow D to do "more" than S.

Game variant:

By looking at the proof of Theorem 7 it can be easily seen that the game works as well if we replace the condition $Fp_1(\alpha(s)) \leq^F Fp_2(\alpha(t))$ in Step 2 by $\lambda(Fp_1(\alpha(s))) \leq \lambda(Fp_2(\alpha(t)))$ for all $\lambda \in \Lambda$, provided that Λ is a separating set of monotone evaluation maps. This variant is in some ways less desirable, since we have to find such a set Λ (instead of simply requiring that it exists), on the other hand in this case the proof does not require weak pullback preservation, since we do not any more require transitivity of $\leq^F$. This variant of the game is conceptually quite close to the Λ-bisimulations studied in [19]. In our notation, a relation $S \subseteq X \times X$ is a Λ-bisimulation, if whenever $x\,S\,y$, then for all $\lambda \in \Lambda$, $p\colon X \to 2$, $\lambda(Fp(\alpha(x))) \leq \lambda(Fq(\alpha(y)))$, where q corresponds to the image of p under S (and the same condition holds for S^{-1}). Λ-bisimulation is sound and complete for behavioural equivalence if F admits a separating set of monotone predicate liftings, which coincides with our condition.

2.4 Spoiler Strategy for the Classical Case

In bisimulation games the winning strategy for D can be derived from the bisimulation or, in our case, from the map f that witnesses the behavioural equivalence of two states x, y (see the remark after Theorem 7). Here we will show that the winning strategy for S can be derived from a modal formula φ which distinguishes x, y, i.e., $x \models \varphi$ and $y \not\models \varphi$. We assume that all modalities are monotone (cf. Proposition 6).

The spoiler strategy is defined over the structure of φ:

- $\varphi = \bigwedge \Phi$: in this case S picks a formula $\psi \in \Phi$ with $y \not\models \psi$.
- $\varphi = \neg\psi$: in this case S takes ψ and reverses the roles of x, y.
- $\varphi = [\lambda]\psi$: in this case S chooses x and $p_1 = [\![\psi]\!]$ in Step 1. After D has chosen y and some predicate p_2 in Step 2, we can observe that $p_2 \not\leq [\![\psi]\!]$ (see proof of Theorem 9 in [23]). Now in Step 3 S chooses p_2 and a state y' with $p_2(y') = 1$ and $y' \not\models \psi$. Then D must choose $[\![\psi]\!]$ and a state x' with $x' \models \psi$ in Step 4 and the game continues with x', y' and ψ.

It can be shown that this strategy is successful for the spoiler.

▶ **Theorem 9.** *Assume that $\alpha\colon X \to FX$ is a coalgebra and F satisfies the requirements of Theorem 7. Let φ be a formula that contains only monotone modalities and let $x \models \varphi$ and $y \not\models \varphi$. Then the spoiler strategy described above is winning for S.*

3 Logics and Games for the Metric Case

We will now consider the metric version of behavioural equivalence, including logics and games. Analogous to Section 2 we will first introduce behavioural distance, which will in this case be defined via the Kantorovich lifting and is parameterized over a set Γ of evaluation maps. Then we introduce a modal logic inspired by [34] and show a quantitative coalgebraic analogue of the Hennessy-Milner theorem. This leads us to the definition of a game for the metric case, where we prove that the distance induced by the game coincides with the behavioural distance. We will conclude by explaining how the strategy for the spoiler can be derived from a logical formula distinguishing two states.

3.1 Foundations for the Metric Case

Note that this subsection contains several results which are new with respect to [6], in particular the extension of the Kantorovich lifting to several evaluation maps and Propositions 18, 19, 20, 21 and 24.

In the following we assume that $\top$ is an element of $\mathbb{R}_0$, it denotes the upper bound of our distances.

We first define the standard notions of pseudometric space and non-expansive functions.

▶ **Definition 10** (Pseudometric, pseudometric space). *Let X be a set and $d\colon X \times X \to [0, \top]$ a real-valued function, we call d a pseudometric if it satisfies*

1. $d(x, x) = 0$ *(d is a metric if in addition $d(x, y) = 0$ implies $x = y$.)*
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$. If d satisfies only Condition 1 and 3, it is a directed pseudometric.

A (directed) pseudometric space is a pair (X, d) where X is a set and d is a (directed) pseudometric on X.

▶ **Example 11.** *We will consider the following (directed) metrics on $[0, \top]$: the Euclidean distance $d_e : [0, \top] \times [0, \top] \to [0, \top]$ with $d_e(a, b) = |a - b|$ and truncated subtraction with $d_\ominus(a, b) = a \ominus b = \max\{a - b, 0\}$. Note that $d_e(a, b) = \max\{d_\ominus(a, b), d_\ominus(b, a)\}$.*

Maps between pseudometric spaces are given by non-expansive functions, which guarantee that mapping two elements either preserves or decreases their distance.

▶ **Definition 12** (Non-expansive function). *Let (X, d_X), (Y, d_Y) be pseudometric spaces. A function $f\colon X \to Y$ is called non-expansive if $d_X(x, y) \geq d_Y(f(x), f(y))$ for all $x, y \in X$. In this case we write $f\colon (X, d_X) \xrightarrow{1} (Y, d_Y)$. By **PMet** (**DPMet**) we denote the category of (directed) pseudometric spaces and non-expansive functions.*

On some occasions we need to transform an arbitrary function into a non-expansive function, which can be done as follows.

▶ **Lemma 13.** *Let d be a pseudometric on X and let $f\colon X \to [0, \top]$ be any function. Then the function $h\colon (X, d) \to ([0, \top], d_e)$ defined via $h(z) = \sup\{f(u) - d(u, z) \mid u \in X\}$ is non-expansive and satisfies $f \leq h$.*

Analogously the function $g\colon (X, d) \to ([0, \top], d_e)$ defined via $g(z) = \inf\{f(u) + d(u, z) \mid u \in X\}$ is non-expansive and satisfies $g \leq f$.

We will now define the Kantorovich lifting for **Set**-functors, introduced in [5]. Given a functor F we lift it to a functor $\bar{F}\colon \mathbf{PMet} \to \mathbf{PMet}$ such that $UF = \bar{F}U$, where U is the forgetful functor, discarding the pseudometric. The Kantorovich lifting is parameterized over a set Γ of evaluation maps $\gamma\colon F[0, \top] \to [0, \top]$, the analogue to the evaluation maps for modalities in the classical case. This is an extension of the lifting in [5] where we considered only a single evaluation map. The new version allows to capture additional examples, without going via the somewhat cumbersome multifunctor lifting described in [5].

▶ **Definition 14** (Kantorovich lifting). *Let F be an endofunctor on **Set** and let Γ be a set of evaluation maps $\gamma\colon F[0, \top] \to [0, \top]$. For every pseudometric space (X, d) the Kantorovich pseudometric on FX is the function $d^{\uparrow\Gamma}\colon FX \times FX \to [0, \top]$, where for $t_1, t_2 \in FX$:*

$$d^{\uparrow\Gamma}(t_1, t_2) := \sup\{d_e(\gamma(Ff(t_1)), \gamma(Ff(t_2))) \mid f\colon (X, d) \xrightarrow{1} ([0, \top], d_e), \gamma \in \Gamma\}$$

We define $\bar{F}_\Gamma(X, d) = (FX, d^{\uparrow\Gamma})$ on objects, while $\bar{F}_\Gamma$ is the identity on arrows.

We will abbreviate $\tilde{F}_\gamma f = \gamma \circ Ff$. Note that $\tilde{F}_\gamma$ is a functor on the slice category $\mathbf{Set}/[0, \top]$, which lifts real-valued predicates $p\colon X \to [0, \top]$ to real-valued predicates $\tilde{F}_\gamma p\colon FX \to [0, \top]$.

It still has to be shown that $\bar{F}$ is well-defined. The proofs are a straighforward adaptation of the proofs in [5].

▶ **Lemma 15.** *The Kantorovich lifting for pseudometrics (Definition 14) is well-defined, in particular it preserves pseudometrics and maps non-expansive functions to non-expansive functions.*

As a sanity check we observe that all evaluation maps $\gamma \in \Gamma$ are non-expansive for the Kantorovich lifting of d_e. In fact, the Kantorovich lifting is the least lifting that makes the evaluation maps non-expansive. This also means that a non-expansive function $f\colon (X, d) \xrightarrow{1} ([0, 1], d_e)$ is always mapped to a non-expansive $\tilde{F}_\gamma f\colon (FX, d^{\uparrow\Gamma}) \xrightarrow{1} ([0, 1], d_e)$.

For the following definitions we need the supremum metric on functions.

▶ **Definition 16** (Supremum metric). *Let (Y, d) be a pseudometric space. Then the set of all functions $f\colon X \to Y$ is equipped with a pseudometric d^∞, the supremum metric, defined as $d^\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$ for $f, g\colon X \to Y$.*

We consider the following restrictions on evaluation maps respectively predicate liftings, which are needed in order to prove the results.

▶ **Definition 17** (Properties of evaluation maps). *Let $\gamma\colon F[0, \top] \to [0, \top]$ be an evaluation map for a functor $F\colon Set \to Set$.*

- *The predicate lifting $\tilde{F}_\gamma$ induced by γ is non-expansive wrt. the supremum metric whenever $d_e^\infty(\tilde{F}_\gamma f, \tilde{F}_\gamma g) \le d_e^\infty(f,g)$ for all $f,g\colon X \to [0,\top]$ and the same holds if we replace d_e by $d_\ominus$.*
- *The predicate lifting $\tilde{F}_\gamma$ is contractive wrt. the supremum metric whenever $d_e^\infty(\tilde{F}_\gamma f, \tilde{F}_\gamma g) \le c \cdot d_e^\infty(f,g)$ for some c with $0 < c < 1$.*
- *The predicate lifting $\tilde{F}_\gamma$ is ω-continuous, whenever for an ascending chain of functions f_i (with $f_i \le f_{i+1}$) we have that $\tilde{F}_\gamma(\sup_{i<\omega} f_i) = \sup_{i<\omega}(\tilde{F}_\gamma f_i)$.*

It can be shown that the first property is equivalent to a property of the lifted functor, called local non-expansiveness, studied in [33].

▶ **Proposition 18** (Local non-expansiveness). *Let Γ be a set of evaluation maps and let $\bar{F}$ be the Kantorovich lifting of a functor F via Γ. It holds that*

$$(d_Y^F)^\infty(\bar{F}f, \bar{F}g) \le (d_Y)^\infty(f,g)$$

for all non-expansive functions $f,g\colon (X,d_X) \to (Y,d_Y)$ (where $\bar{F}(Y,d_Y) = (FY, d_Y^F)$) if and only if

$$d_e^\infty(\tilde{F}_\gamma f, \tilde{F}_\gamma g) \le d_e^\infty(f,g)$$

for all non-expansive functions $f,g\colon (X,d_X) \to ([0,\top], d_e)$ and all $\gamma \in \Gamma$.

Assumption: *In the following we will always assume the first property in Definition 17 for every evaluation map γ, i.e., the predicate lifting $\tilde{F}_\gamma$ is non-expansive wrt. the supremum metric.*

Under this assumption it can be shown that the Kantorovich lifting itself is non-expansive (respectively contractive).

▶ **Proposition 19.** *Let Γ be a set of evaluation maps and let $d_1, d_2\colon X \times X \to [0,\top]$ be two pseudometrics. Then $d_e^\infty(d_1^{\uparrow\Gamma}, d_2^{\uparrow\Gamma}) \le d_e^\infty(d_1,d_2)$, that is, the Kantorovich lifting of metrics is non-expansive for the supremum metric.*

If, in addition, every predicate lifting $\tilde{F}_\gamma$ for $\gamma \in \Gamma$ is contractive (cf. Definition 17), we have that $d_e^\infty(d_1^{\uparrow\Gamma}, d_2^{\uparrow\Gamma}) \le c \cdot d_e^\infty(d_1,d_2)$ for some c with $0 < c < 1$, that is, the Kantorovich lifting of metrics is contractive.

We will now see that for the functors studied in this paper, we have evaluation maps that satisfy the required conditions.

▶ **Proposition 20.** *The following evaluation maps induce predicate liftings which are non-expansive wrt. the supremum metric and ω-continuous.*

- *The evaluation map $\gamma_\mathcal{P}$ for the (finite or general) powerset functor $\mathcal{P}$ with $\gamma\colon \mathcal{P}[0,\top] \to [0,\top]$ where $\gamma_\mathcal{P}(R) = \sup R$.*
- *The evaluation map $\gamma_\mathcal{D}$ for the (finitely or countably supported) probability distribution functor $\mathcal{D}$ (for its definition see Example 1) with $\gamma_\mathcal{D}\colon \mathcal{D}[0,1] \to [0,1]$ where $\gamma_\mathcal{D}(p) = \sum_{r\in[0,1]} r \cdot p(r)$. Note that $\gamma_\mathcal{D}$ corresponds to the expectation of the identity random variable.*
- *The evaluation map $\gamma_\mathcal{M}$ for the constant functor $\mathcal{M}X = [0,\top]$ with $\gamma_\mathcal{M}\colon [0,\top] \to [0,\top]$ and $\gamma_\mathcal{M}(r) = r$.*
- *The evaluation map $\gamma_\mathcal{S}$ for the constant functor $\mathcal{S}X = 1 = \{\bullet\}$ with $\gamma_\mathcal{S}\colon 1 \to [0,\top]$ and $\gamma_\mathcal{S}(r) = \top$.*

As shown in [5] the evaluation map $\gamma_{\mathcal{P}}$ induces the Hausdorff lifting[3] on metrics and the evaluation map $\gamma_{\mathcal{D}}$ the classical Kantorovich lifting[4] for probability distributions [36].

Contractivity can be typically obtained by using a predicate lifting which is non-expansive and multiplying with a discount factor $0 < c < 1$, for instance by using $\gamma_{\mathcal{P}}(R) = c \cdot \sup R$ in the first item of Proposition 20 above.

It can be shown that the properties of evaluation maps are preserved under various forms of composition.

▶ **Proposition 21** (Composition of evaluation maps). *The following constructions on evaluation maps preserve non-expansiveness for the supremum metric and ω-continuity for the induced predicate liftings. Let $\gamma_F\colon F[0,\top] \to [0,\top]$, $\gamma_G\colon G[0,\top] \to [0,\top]$ be evaluation maps for functors F, G.*

- $\gamma\colon F[0,\top] \times G[0,\top] \to [0,\top]$ *with* $\gamma = \gamma_F \circ \pi_1$, *as an evaluation map for* $F \times G$.
- $\gamma\colon F[0,\top] + G[0,\top] \to [0,\top]$ *with* $\gamma(t) = \gamma_F(t)$ *if* $t \in F[0,\top]$ *and* $\gamma(t) = 0$ *otherwise, as an evaluation map for* $F + G$.
- $\gamma\colon FG[0,\top] \to [0,\top]$ *with* $\gamma = \gamma_F \circ F\gamma_G$, *as an evaluation map for* FG.

Now we can define behavioural distance on a coalgebra, using the Kantorovich lifting. Note that the behavioural distance is parameterized over Γ, since, if we are given a coalgebra in **Set**, the notion of behaviour in the metric case is dependent on the chosen functor lifting.

▶ **Definition 22** (Behavioural distance). *Let the coalgebra $\alpha\colon X \to FX$ and a set of evaluation maps Γ for F be given. We define the pseudometric $d_\alpha\colon X \times X \to [0,\top]$ as the smallest fixpoint of $d_\alpha = d_\alpha^{\uparrow\Gamma} \circ (\alpha \times \alpha)$.*

Note that every lifting of metrics is necessarily monotone (since it turns the identity into a non-expansive function, cf. [5]). Since in addition the space of pseudometrics forms a complete lattice (where sup is taken pointwise), the smallest fixpoint exists by Knaster-Tarski.

It has been shown in [6] that whenever the Kantorovich lifting preserves metrics (which is the case for our examples) and the final chain converges, then d_α characterizes behavioural equivalence, i.e., $d_\alpha(x,y) = 0$ iff $x \sim y$.

▶ **Example 23.** *Using the building blocks introduced above we consider the following coalgebras with their corresponding behavioural metrics, generalizing notions from the literature. In both cases we are interested in the smallest fixpoint.*

- Metric transition systems [12]: $FX = [0,\top] \times \mathcal{P}X$ *with two evaluation maps* $\gamma_i\colon [0,\top] \times \mathcal{P}[0,\top] \to [0,\top]$, $i \in \{1,2\}$ *with* $\gamma_1(r,R) = r$, $\gamma_2(r,R) = \sup R$.
  This gives us the following fixpoint equation, where d^H is the Hausdorff lifting of a metric d. Let $\alpha(x) = (r_x, S_x)$, $\alpha(y) = (r_y, S_y)$, then
  $$d(x,y) = \max\{|r_x - r_y|, d^H(S_x, S_y)\}$$
- Probabilistic transition systems: $GX = \mathcal{D}X + 1$ *with two evaluation maps* $\bar{\gamma}_{\mathcal{D}}, \gamma_\bullet\colon \mathcal{D}[0,1] + 1 \to [0,1]$, $i \in \{1,2\}$ *with* $\bar{\gamma}_{\mathcal{D}}(p) = \gamma_{\mathcal{D}}(p)$, $\gamma_\bullet(p) = 0$ *where* $p \in \mathcal{D}[0,1]$, $\bar{\gamma}_{\mathcal{D}}(\bullet) = 0$, $\gamma_\bullet(\bullet) = 1$.

[3] Given a metric d on X, the Hausdorff lifting of d is the metric d^H with $d^H(X_1,X_2) = \max\{\sup_{x_1\in X_1}\inf_{x_2\in X_2} d(x_1,x_2), \sup_{x_2\in X_2}\inf_{x_1\in X_1} d(x_1,x_2)\}$ for $X_1, X_2 \subseteq X$.

[4] Given a metric d on X, the (probabilistic) Kantorovich lifting of d is the metric d^K with $d^K(p_1,p_2) = \sup\{|\sum_{x\in X} f(x)\cdot(p_1(x) - p_2(x))| \mid f\colon (X,d) \xrightarrow{1} ([0,1], d_e)\}$ where $p_1, p_2\colon X \to [0,1]$ are probability distributions.

This gives us the following fixpoint equation, where d^K is the (probabilistic) Kantorovich lifting of a metric d. Let $T = \{x \mid \alpha(x) = \bullet\}$ and let $p_x = \alpha(x) \neq \bullet$.

$$d(x,y) = \begin{cases} 1 & \text{if } x \in T, y \notin T \text{ or } x \notin T, y \in T \\ 0 & \text{if } x, y \in T \\ d^K(p_x, p_y) & \text{otherwise} \end{cases}$$

Some of the results on (real-valued) modal logics in Section 3.2 will require that the fixpoint iteration terminates in ω steps. This is related to the fact that the original Hennessy-Milner theorem requires finite branching.

▶ **Proposition 24.** *Let Γ be a set of evaluation maps and let $\alpha\colon X \to FX$ be a coalgebra. We define an ascending sequence of metrics $d_i\colon X \times X \to [0, \top]$ as follows: d_0 is the constant 0-function and $d_{i+1} = d_i^{\uparrow\Gamma} \circ \alpha \times \alpha$. Furthermore $d_\omega = \sup_{i<\omega} d_i$.*

- *If for all evaluation maps $\gamma \in \Gamma$ the induced predicate liftings are ω-continuous (see Definition 17) and F is ω-accessible, the fixpoint d_α equals d_ω.*
- *If for all evaluation maps $\gamma \in \Gamma$ the induced predicate liftings are contractive wrt. the supremum metric (see Definition 17), the fixpoint d_α equals d_ω.*

Hence, if we are working with the finite powerset functor or the finitely supported distribution functor, the first case applies, whereas in the case of contractiveness, these restrictions are unnecessary (compare this with the result of [33] which guarantees the existence of a final coalgebra for a class of locally contractive functors).

3.2 Modal Logics for the Metric Case

We now define a coalgebraic modal logic $\mathcal{M}(\Gamma)$, which is inspired by [35]. Assume also that Γ is a set of evaluation maps.

The formulas of the logic are defined together with their semantics $[\![\varphi]\!]_\alpha$ and their modal depth $md(\varphi)$ in Table 1. Given a coalgebra $\alpha\colon X \to FX$ and a formula φ, the semantics of such a formula is given by a real-valued predicate $[\![\varphi]\!]_\alpha\colon X \to [0, \top]$, defined inductively, where $\gamma \in \Gamma$, $q \in \mathbb{Q} \cap [0, \top]$. Again we will occasionally omit the subscript α.

φ:	$\top$	$[\gamma]\psi$	$\min(\psi, \psi')$	$\neg\psi$	$\psi \ominus q$
$[\![\varphi]\!]_\alpha$:	$\top$	$\gamma \circ F[\![\psi]\!]_\alpha \circ \alpha$	$\min\{[\![\psi]\!]_\alpha, [\![\psi']\!]_\alpha\}$	$\top - [\![\psi]\!]_\alpha$	$[\![\psi]\!]_\alpha \ominus q$
$md(\varphi)$:	0	$md(\psi) + 1$	$\max\{md(\psi), md(\psi')\}$	$md(\psi)$	$md(\psi)$

Table 1 Overview of the modal logic formulas, their semantics $[\![\varphi]\!]_\alpha$ and modal depths $md(\varphi)$.

Note that, given a state x and a logical formula φ, we do not just obtain a true value (true, false) dependent on whether x satisfies the formula or not. Instead we obtain a value in the interval $[0, 1]$ that measures the degree or weight according to which x satisfies φ.

▶ **Definition 25** (Logical distance)**.** *Let $\alpha\colon X \to FX$ be a coalgebra and let $x, y \in X$. We define the logical distance of x, y as*

$$d^L_\alpha(x,y) = \sup\{d_e([\![\varphi]\!]_\alpha(x), [\![\varphi]\!]_\alpha(y)) \mid \varphi \in \mathcal{M}(\Gamma)\}.$$

We also define the logical distance up to modal depth i.

$$d^L_i(x,y) = \sup\{d_e([\![\varphi]\!]_\alpha(x), [\![\varphi]\!]_\alpha(y)) \mid \varphi \in \mathcal{M}(\Gamma), md(\varphi) \leq i\}.$$

▶ **Example 26.** *We are considering probabilistic transition systems with evaluation maps as defined in Example 23.*

The formula $\varphi = [\bar{\gamma}_{\mathcal{D}}][\gamma_\bullet]\top$ distinguishes the states x, y in Figure 1a. The formula $\psi = [\gamma_\bullet]\top$ evaluates to a predicate $[\![\psi]\!]$ that assigns 1 to terminating states and 0 to non-terminating states. Now x makes a transition to a terminating state with probability $\frac{1}{2}$, which means that $[\![\varphi]\!](x) = \bar{\gamma}_{\mathcal{D}}(\mathcal{D}[\![\psi]\!](\alpha(x))) = \frac{1}{2}$. Similarly $[\![\varphi]\!](y) = \frac{1}{2} + \varepsilon$. Hence $d^L_\alpha(x, y) \geq d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) = \varepsilon$. (In fact, the logical distance equals ε.)

We will now show that the logical distance d^L_α and the behavioural distance d_α coincide, i.e. a quantitative version of the Hennessy-Milner theorem, by generalizing the proof from [35]. Note that in some respects we simplify wrt. [35] by not working in **Meas**, the category of measurable spaces, but in a discrete setting. On the other hand, we generalize by considering arbitrary **Set**-endofunctors.

▶ **Theorem 27.** *Let d_i be the sequence of pseudometrics from Proposition 24. Then:*
1. *For every $i \in \mathbb{N}_0$ $d^L_i \leq d_i$.*
2. *For every φ with $md(\varphi) \leq i$ we have non-expansiveness: $[\![\varphi]\!]\colon (X, d_i) \to ([0, \top], d_e)$.*
3. *$d^L_\alpha \leq d_\alpha$.*

Note that from Theorem 27 it also follows that for every formula φ the function $[\![\varphi]\!]$ is non-expansive. Non-expansiveness is analogous to bisimulation-invariance that holds for formulas in a classical logic. In particular, in the classical case if $x \sim y$, then $[\![\varphi]\!](x) = [\![\varphi]\!](y)$ for every φ, in other words $[\![\varphi]\!]$ is non-expansive for the discrete metric d.

The other inequality ($d^L_\alpha \geq d_\alpha$) is more difficult to prove: we will first show that each d_i is totally bounded and then show that each non-expansive function can be approximated at each pair of points by a modal formula. Since modal formulas are closed under min and max, this enables us to use a variant of a lemma from [3] to prove that the formulas form a dense subset of all non-expansive functions. In order to achieve the approximation, we need all operators of the logic.

We first have to recall some definitions from real-valued analysis.

▶ **Definition 28** (Total boundedness)**.** *A pseudometric space (X, d) is totally bounded iff for every $\varepsilon > 0$ there exist finitely many elements $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n \mathcal{B}_\varepsilon(x_i)$ where $\mathcal{B}_\varepsilon(x_i) = \{z \in X \mid d(z, x_i) \leq \varepsilon\}$ denotes the ε-ball around x_i.*

Our first result is to show that the lifting preserves total boundedness, by adapting a proof from [37] from a specific functor to arbitrary functors.

▶ **Proposition 29.** *Let (X, d) be a totally bounded pseudometric space, then $(FX, d^{\uparrow\Gamma})$ is totally bounded as well.*

Using this result it can be shown that every pseudometric in the ascending chain from Proposition 24 (apart from d_ω) is totally bounded.

▶ **Proposition 30.** *Let d_i be the sequence of pseudometrics from Proposition 24. Then every (X, d_i) is a totally bounded pseudometric space.*

Since total boundedness is not preserved by taking a supremum, d_ω is not necessarily totally bounded and we can not iterate the argument. This is one of the reasons for requiring that the fixpoint is reached in ω steps in Theorem 32 below.

In the next step we show that the formulas are dense in the non-expansive functions.

▶ **Proposition 31.** *$\{[\![\varphi]\!]\colon X \to [0, 1] \mid md(\varphi) \leq i\}$ is dense (wrt. the supremum metric) in $\{f\colon (X, d^L_i) \xrightarrow{1} ([0, \top], d_e)\}$.*

Finally we can show under which conditions the inequality $d_\alpha \le d^L_\alpha$ holds.

▶ **Theorem 32.** *If the fixpoint d_α is reached in ω steps, it holds that $d_\alpha \le d^L_\alpha$.*

3.3 Games for the Metric Case

We will now present the two-player game characterizing the behavioural distance between two states. The roles of S and D are similar to those in the first game, where D wants to defend the statement that the distance of two states $x, y \in X$ in a coalgebra α is bounded by $\varepsilon \in [0, \top]$, i.e., $d_\alpha(x, y) \le \varepsilon$. S wants to disprove this claim.

- **Initial situation:** Given a coalgebra $\alpha\colon X \to FX$, we start with (x, y, ε) where $x, y \in X$ and $\varepsilon \in [0, \top]$.
- **Step 1:** S chooses $s \in \{x, y\}$ and a real-valued predicate $p_1\colon X \to [0, \top]$.
- **Step 2:** D takes $t \in \{x, y\}\backslash\{s\}$ and has to answer with a predicate $p_2\colon X \to [0, \top]$, which satisfies $d_\ominus(\tilde{F}_\gamma p_1(\alpha(s)), \tilde{F}_\gamma p_2(\alpha(t))) \le \varepsilon$ for all $\gamma \in \Gamma$.
- **Step 3:** S chooses p_i with $i \in \{1, 2\}$ and some state $x' \in X$.
- **Step 4:** D chooses some state $y' \in X$ with $p_i(x') \le p_j(y')$ where $j \ne i$
- **Next round:** (x', y', ε') with $\varepsilon' = p_j(y') - p_i(x')$.

After one round the game continues with the initial step, but now D tries to show that $d_\alpha(x', y') \le \varepsilon'$. D wins if the game continues forever. In the other case, e.g., D has no move at Step 2 or Step 4, S wins.

The game distance of two states is defined as follows.

▶ **Definition 33** (Game distance)**.** *Let $\alpha\colon X \to FX$ be a coalgebra and let $x, y \in X$. We define the game distance of x, y as*

$$d^G_\alpha(x, y) = \inf\{\varepsilon \mid D \textit{ has a winning strategy for } (x, y, \varepsilon)\}.$$

We now prove that the behavioural distance and the game distance coincide. We first show that d^G_α is indeed a pseudometric.

▶ **Proposition 34.** *The game distance d^G_α is a pseudometric.*

Next we show that the game distance is always bounded by the behavioural distance.

▶ **Theorem 35.** *It holds that $d^G_\alpha \le d_\alpha$.*

While the general proof of this theorem is given in [23], the strategy for D can be straightforwardly explained whenever X is finite. In particular we want to show that whenever $d_\alpha(x, y) \le \varepsilon$, then D has a winnning strategy for (x, y, ε). Assume that S chooses $s \in \{x, y\}$ with $p_1\colon X \to [0, \top]$. In this case D chooses p_2 with $p_2(z) = \sup\{p_1(u) - d_\alpha(u, z) \mid u \in X\}$ in Step 2. From Lemma 13 we know that $p_1 \le p_2$ and p_2 is non-expansive. It can be shown that this choice satisfies $d_\ominus(\tilde{F}_\gamma p_1(\alpha(s)), \tilde{F}_\gamma p_2(\alpha(t))) \le \varepsilon$ for all $\gamma \in \Gamma$. Now S chooses i and $x' \in X$ in Step 3. Then either $i = 1$ and D can choose $y' = x'$ in Step 4 and the game continues with x', y' and $\varepsilon' = p_2(y') - p_1(x') \ge 0$. Or $i = 2$ and D can choose y' such that $p_1(y') - d_\alpha(y', x') = p_2(x')$ (the supremum is reached since X is finite). This means that $p_1(y') \ge p_2(x')$ and $\varepsilon' = p_1(y') - p_2(x') = d_\alpha(x', y')$. In both cases, the game can continue.

▶ **Example 36.** *Imagine the initial game situation (x, y, ε) for our example in Figure 1a and S chooses x with predicate $p_1(4) = 1$ and zero for all remaining states. Now D follows the strategy above and plays a predicate p_2 with $p_2(4) = p_2(5) = p_2(7) = 1$ and zero for all other* *states. Since* $5, 7$ *are at distance* 0 *to* 4*, they are now mapped to* 1 *as well. Since in particular* 4 *and* 7 *are mapped to* 1*, we obtain* $d_\ominus(\tilde{\mathcal{D}}_{\bar{\gamma}_\mathcal{D}} p_1(\alpha(x)), \tilde{\mathcal{D}}_{\bar{\gamma}_\mathcal{D}} p_2(\alpha(y))) = d_\ominus(\frac{1}{4}, \frac{1}{2} + \varepsilon) = 0 \leq \varepsilon$ *(we obtain the same value for* $\gamma_\bullet$*). Note again that D must be allowed to do "more" than S. Now the winning strategy for D is obvious: if S picks a terminating state* x' *and* p_i*, D can also pick a terminating state* y' *and* p_j *with* $p_j(y') - p_i(x') = 0$ *(similarly for non-terminating states). We then end up in* $(x', y', 0)$ *where* x', y' *are behaviourally equivalent.*

If S had instead chosen y *a prediate* p_1 *with* $p_1(7) = 1$ *and zero for all other states, D would choose the same predicate* p_2 *with* $d_\ominus(\tilde{\mathcal{D}}_{\bar{\gamma}_\mathcal{D}} p_1(\alpha(y)), \tilde{\mathcal{D}}_{\bar{\gamma}_\mathcal{D}} p_2(\alpha(x))) = d_\ominus(\frac{1}{2} + \varepsilon, \frac{1}{2}) = \varepsilon$.

We now demonstrate that in the case of infinite branching, the construction of the winning strategy for the D is not as simple as described before.

▶ **Example 37.** *Consider the coalgebra* $\alpha\colon X \to \mathcal{D}X + 1$ *in Figure 2 on the state space* $X = \{y, y_0, x, x_1, x_2, \dots\}$*, where the probability of going from* x *to* x_i *is* $\alpha(x)(x_i) = \frac{1}{2^i}$.

For both states x, y *the probability to terminate is* 1 *and hence* $x \sim y$*. Now imagine that S selects* x *and the real-valued predicate* p_1 *with* $p_1(x_i) = 1 - \frac{1}{2^i}$ *and* $p_1(x) = 0$*. If we would construct the predicate for D as above, via* $p_2(z) = \sup\{p_1(u) - d_\alpha(u, z) \mid u \in X\}$*, this would yield* $p_2(y_0) = 1$ *since the distance of all terminating states is* 0.

Then S chooses $x' = y_0$ *and* p_2 *in Step 3 and D has no available state* y' *with which to answer in Step 4. If* $y' = x_i$*, then* $p_1(x_i) = 1 - \frac{1}{2^i} < 1 = p_2(x')$*, otherwise* $p_1(y') = 0 < 1$.

In fact, D has no winning strategy for $\varepsilon = 0$*, but we can show that there is a winning strategy for every* $\varepsilon > 0$ *(since D can play a predicate that is below* p_2*, but at distance* ε*). Since the game distance is defined as the infinum over all such* ε*'s it still holds that* $d^G_\alpha(x, y) = 0$.

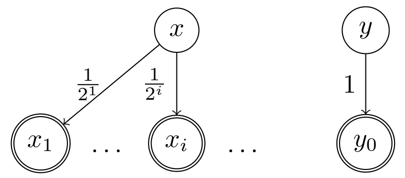


Figure 2 Probabilistic transition system for the functor $FX = \mathcal{D}X + 1$, where X is infinite.

Finally, we can show the other inequality.

▶ **Theorem 38.** *It holds that* $d_\alpha \leq d^G_\alpha$.

3.4 Spoiler Strategy for the Metric Case

The strategy for S for (x, y, ε) can be derived from a modal formula φ with $d_\ominus([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$. If $\varepsilon < d_\alpha(x, y) = \sup\{d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) \mid \varphi\}$, such a formula must exist (since we can use negation to switch x, y if necessary). The spoiler strategy is defined over the structure of φ:

- $\varphi = \top$: this case can not occur.
- $\varphi = [\gamma]\psi$: S chooses x, $p_1 = [\![\psi]\!]$ at Step 1. After D has chosen y, p_2 at Step 2, we can observe that $p_2 \not\leq [\![\psi]\!]$ (see proof of Theorem 39 in [23]). Now in Step 3 S chooses p_2 and x' such $p_2(x') > [\![\psi]\!](x')$. Now D needs to choose y' such that $[\![\psi]\!](y') \geq p_2(x')$ in Step 4 and $\varepsilon' = [\![\psi]\!](y') - p_2(x') < [\![\psi]\!](y') - [\![\psi]\!](x') = d_e([\![\psi]\!](x'), [\![\psi]\!](y'))$ and so the game continues in the situation (x', y', ε') with the formula ψ.
- $\varphi = \min(\psi, \psi')$: In this case either $d_e([\![\psi]\!](x), [\![\psi]\!](y)) > \varepsilon$ or $d_e([\![\psi']\!](x), [\![\psi']\!](y)) > \varepsilon$ and S picks ψ or ψ' accordingly.
- $\varphi = \neg\psi$: In this case S takes ψ, since $d_e([\![\psi]\!](x), [\![\psi]\!](y)) = d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$.
- $\varphi = \psi \ominus q$: In this case $d_e([\![\psi]\!](x), [\![\psi]\!](y)) \geq d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$ and hence S takes ψ.

It can be shown that this strategy is indeed correct.

▶ **Theorem 39.** *Assume that $\alpha\colon X \to FX$ is a coalgebra. Let φ be a formula with $d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$. Then the spoiler strategy described above is winning for S in the situation (x, y, ε).*

Note that Theorem 38 is not a direct corollary of this theorem, since here we require that a formula φ with $d_e([\![\varphi]\!](x), [\![\varphi]\!](y)) > \varepsilon$ exists, which is not necessarily true in scenarios where the fixpoint iteration does not terminate in ω steps.

▶ **Example 40.** *We will show how S can construct a winning strategy for $(x, y, \frac{\varepsilon}{2})$ based on the formula $\varphi = [\bar{\gamma}_{\mathcal{D}}][\gamma_\bullet]\top$ from Example 26. The transition system is shown in Figure 1a.*

It holds that $d_\ominus([\![\varphi]\!](y), [\![\varphi]\!](x)) = \varepsilon > \frac{\varepsilon}{2}$. S plays y and $p_1 = [\![[\gamma_\bullet]\top]\!]$ which, due to the definition of $\gamma_\bullet$ equals 1 on terminating states and zero on non-terminating states. Now $\bar{\gamma}_{\mathcal{D}}(\mathcal{D}p_1(\alpha(y))) = \frac{1}{2} + \varepsilon$, so D must play in such a way that $\bar{\gamma}_{\mathcal{D}}(\mathcal{D}p_2(\alpha(x))) \geq \frac{1}{2} + \frac{\varepsilon}{2}$. This can only be achieved by setting $p_2(3) = \varepsilon$ (or to a larger value). Now S chooses p_2, $x' = 3$ and D can only take p_1 and either 4, 5 or 7 as y'. In each case we obtain $\varepsilon' = p_1(y') - p_2(x') = 1 - \varepsilon < 1 = d_e(0, 1) = d_e([\![[\gamma_\bullet]\top]\!](x'), [\![[\gamma_\bullet]\top]\!](y'))$.

The spoiler continues to follow his strategy and plays x', $p_1 = [\![\top]\!]$ in the next step, which is successful, since y' is a terminating state and x' is not.

4 Conclusion

Comparison to related work can be found in the introduction and throughout the text.

We will conclude by discussing some open points and questions: Section 3, which treats the metric cases, follows the outline of Section 2, which treats the classical case, with some variations. An important difference is the fact that the metric case is parameterized over a set Γ of evaluation maps. Note that we actually mimic the variant of the game discussed at the end of Section 2.3, where we fix evaluation maps, but omit the requirement of weak pullback preservation. The requirement of monotonicity is replaced by local non-expansiveness in the metric case. The fact that monotonicity for partial orders generalizes to non-expansiveness for directed metrics has already been discussed in [33]. The variant of the classical game that uses the lifted order $\leq^F$ is more reminiscent of the Wasserstein lifting for metrics, which has been introduced in [5] and compared to the Kantorovich lifting. It is future work to define a variant of the metric game that corresponds to the Wasserstein lifting (or other liftings) of metrics.

Another open question is to prove the Hennessy-Milner theorem for the real-valued logic in the case where the fixpoint is not reached in ω steps. The original variant of the Hennessy-Milner-theorem only holds for finitely-branching transition systems, but this result can be generalized if we allow infinite conjunctions (cf. the logic in Section 3.2). A natural question is whether the same solution is applicable to the metric case, by replacing the min- by an inf-operator (of restricted cardinality κ, as in Section 2.2). However, for this it seems necessary to generalize the notion of total boundedness to a new variant where we do not require that the set of "anchors" $\{x_1, \ldots, x_n\}$ of Definition 28 is finite, but bounded by κ.

A related question is the following: does the Kantorovich lifting preserve completeness of metrics? (A metric space (X, d) is complete if every Cauchy sequence converges in X.) Furthermore we would like to add ∞ as a possible distance value, as in [5]. However, this can not be integrated so easily, for instance it is unclear how to define negation.

Finally, in the quantitative case it could be interesting to know whether we can use existing efficient algorithms (for the probabilistic case), for instance in order to generate the strategy of the spoiler (see e.g. [10]).

References

1 J. Adámek, H.P. Gumm, and V. Trnková. Presentation of Set functors: A coalgebraic perspective. *Journal of Logic and Computation*, 20(5), 2010.

2 J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

3 R.B. Ash. *Real Analysis and Probability*. Academic Press, 1972.

4 A. Balan and A. Kurz. Finitary functors: From **Set** to **Preord** and **Poset**. In *Proc. of CALCO '11*, pages 85–99. Springer, 2011. LNCS 6859.

5 P. Baldan, F. Bonchi, H. Kerstan, and B. König. Behavioral metrics via functor lifting. In *Proc. of FSTTCS '14*, volume 29 of *LIPIcs*, 2014.

6 P. Baldan, F. Bonchi, H. Kerstan, and B. König. Coalgebraic behavioral metrics. *Logical Methods in Computer Science*, to appear. Selected Papers of the 6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015).

7 A. Baltag. Truth-as-simulation: Towards a coalgebraic perspective on logic and games. Technical Report SEN-R9923, Centrum voor Wiskunde en Informatica (CWI), November 1999.

8 V. Castiglioni, D. Gebler, and S. Tini. Logical characterization of bisimulation metrics. In *Proc. of QAPL '16*, 2016. EPTCS 227.

9 K. Chatzikokolakis, D. Gebler, C. Palamidessi, and L. Xu. Generalized bisimulation metrics. In *Proc. of CONCUR '14*. Springer, 2014. LNCS/ARCoSS 8704.

10 Di Chen, Franck van Breugel, and James Worrell. On the complexity of computing probabilistic bisimilarity. In *Proc. of FOSSACS '12*, pages 437–451. Springer, 2012. LNCS/ARCoSS 7213.

11 X. Chen and Y. Deng. Game characterizations of process equivalences. In *Proc. of APLAS '08*, pages 107–121. Springer, 2008. LNCS 5356.

12 L. de Alfaro, M. Faella, and M. Stoelinga. Linear and branching system metrics. *IEEE Trans. Softw. Eng.*, 35(2):258–273, March 2009.

13 J. Desharnais. *Labelled Markov processes*. PhD thesis, McGill University, Montreal, November 1999.

14 J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden. Metrics for labelled Markov processes. *Theoretical Computer Science*, 318:323–354, 2004.

15 J. Desharnais, F. Laviolette, and M. Tracol. Approximate analysis of probabilistic processes: Logic, simulation and games. In *Proc. of QEST '08*, pages 264–273. IEEE, 2008.

16 U. Fahrenberg, A. Legay, and C. Thrane. The quantitative linear-time–branching-time spectrum. In *Proc. of FSTTCS '11*, volume 13 of *LIPIcs*, pages 103–114, 2011.

17 N. Fijalkow, B. Klin, and P. Panangaden. Expressiveness of probabilistic modal logics. In *Proc. of ICALP '17*, volume 80 of *LIPIcs*, pages 105:1–12. Schloss Dagstuhl – Leibniz Center for Informatics, 2017.

18 G. Fontaine, R.A. Leal, and Y. Venema. Automata for coalgebras: An approach using predicate liftings. In *Proc. of ICALP '10*, pages 381–392. Springer, 2010. LNCS 6198.

19 D. Gorín and L. Schröder. Simulations and bisimulations for coalgebraic modal logics. In *Proc. of CALCO '13*, pages 253–266. Springer, 2013. LNCS 8089.

20 H. Peter Gumm. Universal coalgebras and their logics. *AJSE-Mathematics*, 34(1D):105–130, 2009.

21 M. Hennessy and R. Milner. On observing nondeterminism and concurrency. In *Proc. of ICALP '80*, pages 299–309. Springer, 1980. LNCS 85.

22 N. Khakpour and M.R. Mousavi. Notions of conformance testing for cyber-physical systems: Overview and roadmap (invited paper). In *Proc. of CONCUR '15*, volume 42. LIPIcs, 2015.

23 B. König and C. Mika-Michalski. (Metric) bisimulation games and real-valued modal logics for coalgebras, 2018. arXiv:1705.10165. URL: `https://arxiv.org/abs/1705.10165`.

24 C. Kupke. Terminal sequence induction via games (international tbilisi symposium on language, logic and computation). In *Prof. of TbiLLC '07*, pages 257–271. Springer, 2009. LNAI 5422.

25 L.S. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96(1–3):277–317, 1999.

26 M. Otto. Elementary proof of the van Benthem-Rosen characterisation theorem. Technical Report 2342, Department of Mathematics, Technische Universität Darmstadt, 2004.

27 D. Pattinson. Coalgebraic modal logic: soundness, completeness and decidability of local consequence. *Theoretical Computer Science*, 309(1):177 – 193, 2003.

28 J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249:3–80, 2000.

29 L. Schröder. Expressivity of coalgebraic modal logic: The limits and beyond. *Theoretical Computer Science*, 390(2):230 – 247, 2008.

30 L. Schröder and D. Pattinson. Description logics and fuzzy probability. In *Proc. of IJCAI '11*, volume 2, pages 1075–1080. AAAI Press, 2011.

31 S. Staton. Relating coalgebraic notions of bisimulation. In *Proc. of CALCO '09*, pages 191–205. Springer, 2009. LNCS 5728.

32 C. Stirling. Bisimulation, modal logic and model checking games. *Logic Journal of the IGPL*, 7(1):103 – 124, 1999.

33 D. Turi and J. Rutten. On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. *Mathematical Structures in Computer Science*, 8:481–540, 1998.

34 F. van Breugel and J. Worrell. Approximating and computing behavioural distances in probabilistic transition systems. *Theoretical Computer Science*, 360:373–385, 2005.

35 F. van Breugel and J. Worrell. A behavioural pseudometric for probabilistic transition systems. *Theoretical Computer Science*, 331(1):115 – 142, 2005.

36 C. Villani. *Optimal Transport – Old and New*, volume 338 of *A Series of Comprehensive Studies in Mathematics*. Springer, 2009.

37 P. Wild, L. Schröder, D. Pattinson, and B. König. A van Benthem theorem for fuzzy modal logic. In *Proc. of LICS '18*, 2018. to appear, arXiv:1802.00478.